Exploring the learning and teaching of multiplicative reasoning through measures: A design-based research project.

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DECLARATIONS

This thesis is the result of my own investigation, except where otherwise stated. Other sources are acknowledged by reference. A reference list is appended. The referencing style used is Cite them Rite Harvard format.

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ABSTRACT

Exploring the learning and teaching of multiplicative reasoning through measures: A design-based research project.

This study explores the learning and teaching of multiplicative reasoning through measurement contexts with Year 2 learners, aged 6 and 7, in a primary school in South Wales, through the design and evaluation of learning tasks. Typically, in the UK, the multiplicative relationship is introduced as an extension of counting, in which counting in composite units (units with a value greater than one) is developed using visual and concrete resources with discrete quantities. A contrasting approach, developed in the 1960s in Russia, by Davydov and Elkonin, involves the introduction of the concept of number, and later the multiplicative relationship, through contexts involving measures with continuous quantities. It is not common for a measures approach to be used to introduce the multiplicative relationship when counting, and number operations as an extension of counting, is the predominant approach within a curriculum. This study explores the teaching and learning of the multiplicative relationship through measures contexts, in a situation where learners have typically been introduced to number through discrete quantities.

A design-based research (design research) approach was adopted. Using research informed design principles, and applying a socio-constructivist theoretical framework, tasks were designed, implemented, evaluated and developed through two cycles of research, with Year 2 learners. Observation and interviews, with learners (n=21) and practitioners (n=5), were used to inform task development and evaluation. The tasks with the learners, led by the researcher, were audio recorded, transcribed and coded. Through analysis of the data from both cycles, themes were constructed. Data collected support the assertion that measures tasks offer rich opportunities for multiplicative reasoning. A key theme from analysis is the construction of the equality relationship, and this study offers new insight into how this might be perceived in measures contexts.

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CHAPTER 1: INTRODUCTION

1.1 RATIONALE FOR STUDY

The term 'multiplicative relationship' is used to refer to a relationship that may exist between quantities which includes interconnected ideas, processes, and relationships such as multiplication, division, and fractions. Though evident in early research into cognitive development (e.g., Piaget, 1952), interest in the multiplicative relationship revived in the 1980s (Confrey and Harel, 1994), and since then international research has considered the development of reasoning with the multiplicative relationship. Indeed, understanding of, and reasoning with, the multiplicative relationship can be concluded to be a key predictor of progress in mathematics (Siemon, Breed and Virgona, 2008; Sieglar *et al*., 2012; Nunes *et al*., 2012).

Meyer and Land (2006, p.3) introduce the idea of a 'threshold concept' in student learning, where such a concept, when understood, is seen to be 'transformative'. As Meyer and Land (2006, p.3) note, this transformation might take time and can prove to be 'troublesome'. I believe that reasoning with the multiplicative relationship can be seen as a threshold concept in mathematics. As discussed in Chapter 2 of this thesis, understanding the multiplicative relationship can be problematic, and typically develops over many years of schooling. Furthermore, multiplicative reasoning is an essential threshold for ensuring progression in mathematics. For example, through a large-scale study of learners' responses to assessment tasks in their middle years of schooling (ages 10 to 14) in Australia, Siemon, Breed and Virgona (2008, p.6) conclude that incomplete understanding of the multiplicative relationship 'almost guarantees failure in relation to developing deep understanding of fractions, decimals, per cent, ratio and algebra', thus implying a causal relationship. Through analyses of UK and US longitudinal national test data, Sieglar *et al*. (2012, p.1) conclude that, even when accounting for factors such as general intellect, working memory, family income and education, students' understanding of fractions and division (and thus understanding the multiplicative relationship) 'uniquely predicts' success in algebra and overall mathematics achievement. Similarly, Nunes *et al*. (2012) analysed longitudinal UK national test data in English, mathematics and science, and found, even when accounting for intellect, working memory and age, that mathematical reasoning, including multiplicative reasoning, more so than arithmetic, impacted on mathematical and scientific achievement. Indeed, Nunes *et al*. (2012) argue that, from the early years of primary school, mathematical reasoning, such as reasoning with the multiplicative relationship, should be given greater priority over calculation, and this should be maintained to the end of secondary school.

Through experience as a primary teacher and as primary teacher educator, with a specialist interest in mathematics, I have witnessed learners and student teachers struggling to apply and reason with the multiplicative relationship, even though multiplication and division facts may be known, and processes such as multiplication and division algorithms may be established. This led to my interest in this area of mathematics learning and teaching; I sought to understand more about how the learning of the multiplicative relationship might be supported and developed, through facilitating the learning of pupils and teachers.

Zwanch and Wilkins (2021) note that there are different perspectives evident in the study and exploration of students' multiplicative reasoning. Notably, through the literature, discussed in Chapter 2, two dichotomous approaches to the introduction of the multiplicative relationship can be identified. The first approach involves the introduction of the multiplicative relationship as an extension of counting; for example, Steffe (1994, p.7) notes that the extension of counting in ones to counting in composite units (units with a value greater than one) is 'crucial in learning multiplication and division'. As Coles (2017, p.206) notes, the evolution of concepts through counting, the 'counting world', is the 'predominant narrative' in mathematics education. A contrasting and less familiar approach, developed in the 1960s in Russia, by Davydov and Elkonin (e.g., Davydov, 1990; Davydov, 1992) involves the introduction of the concept of number, and later the multiplicative relationship, through contexts involving measures. Coles (2017, p.206) summarises this ‘measurement world’ as predominantly involving a focus on relationships between quantities. Indeed, Coles and Sinclair (2022, p.19) question why a focus on relationships before focusing on number, as incorporated into Davydov’s curriculum (e.g., Davydov, 1990), is not considered in every curriculum, although they note that making such changes would require ‘extensive training for teachers’ and would require ‘a significant sustained effort’ to make such shifts. Venkat, Askew and Morrison (2020, p.398), in discussing the incorporation of Davydov’s focus on relationships between quantities into an intervention project in South Africa, apply the notion of ‘shape-shifting’ to consider how a contrasting approach might successfully be incorporated into another context and/or culture. They note the need to analyse original learning intentions of an idea and interpret how that might work within the cultural context. Thus, as noted by Coles and Sinclair (2022) and Askew, Venkat and Morrison (2020), ideas, such as those of Davydov’s (1992) approach to the multiplicative relationship, cannot be transposed simply into another context and there needs to be careful consideration of how novel and unfamiliar ideas might be developed in a specific cultural context.

This study has been developed to explore how an approach to introducing the multiplicative relationship involving measures might be incorporated into a curriculum that predominantly reflects a ‘counting world’ (Coles, 2017, p.206). Through developing tasks that might support learners in understanding the multiplicative relationship through measures, whilst recognising the ‘counting world’ from which they came, and through exploring learners’ and practitioners’ responses to the tasks developed, the aim is to explore the learning and teaching of the multiplicative relationship through measures contexts, whilst also developing tasks that might support practitioners and the learners using them.

Design research is the methodological framework applied within this study (Bakker, 2018); this methodological framework and the research approaches adopted within the study are discussed in Chapter 4. As noted by The Design-Based Research Collective (2003, p.5), design research ‘enables us to create learning conditions that learning theory suggests are productive, but that are not commonly practiced or are not well understood'. The design research involved the design and development of tasks according to specific design principles, reflecting a theoretical approach to learning multiplicative reasoning discussed in Chapter 2 and a view of learning and teaching discussed in Chapter 3. The design research involved two iterations: Cycle 1 and Cycle 2, discussed in Chapter 5 and 6 respectively. Points of learning are considered in both cycles, and themes from both cycles are discussed in Chapter 7.

1.2 RESEARCH QUESTIONS

Exploring the learning and teaching of multiplicative reasoning through measures: A design-based research project.

The key aim of this study is to explore the learning and teaching of multiplicative reasoning through measures tasks, in a context which predominantly reflects a 'counting world' (Coles, 2017, p.206). Using measures tasks, with a focus on relationships, it is envisaged that the tasks might act as a possible introduction, or bridge, to a 'measurement world' (Coles, 2017, p.206).

The sub-questions applied in the study are:

S1: What are teachers' and learners' prior experiences of teaching and learning multiplicative reasoning?

This question was a starting point for the study. Whilst my experience suggested teaching of multiplicative reasoning would typically evolve from counting experiences, seen as the 'predominant narrative' (Coles, 2017, p.206), it is important to consider the context and the way in which learners might be typically taught, as understanding developed from this would inform the design of tasks.

S2: What are learners' prior experiences of learning number and measures?

Like S1, this question was another starting point for the study. It is important to consider the way in which learners typically experienced number and measures, to inform the development of tasks.

S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?

This question focuses on the process of task development to support multiplicative reasoning. It allows for consideration of the efficacy of tasks in relation to learners' and teachers' prior experiences.

S4: What is the impact of learning multiplicative reasoning through measures on learners?

This question directs a focus on analysis of learning responses, to consider the possible learning.

S5: What are teachers' and learners' views on teaching/learning multiplicative reasoning through measures using the materials developed?

This question explores learner and teacher perceptions of their tasks and experiences.

Collectively, the research questions were designed to allow the efficacy of tasks and approaches to be considered, whilst recognising that it is the learning and teaching approach taken that is also being explored.

1.3 CONTEXT OF STUDY

The research was undertaken in one primary school in South Wales, over a period of two and a half years. In both cycles, the tasks were undertaken with groups of Year 2 learners (ages 6 to 7). This year group was chosen because, in Wales, Year 2 is a year group in which the multiplicative relationship is typically introduced through the explicit introduction of multiplication and division. Further information about the school and participants is provided in Chapters 4, 5 and 6.

It is recognised that many contextual factors are important to the study, and some of these have been considered as part of the research questions. However, there are also two national contextual factors that are discussed in relation to this study: curriculum development and the impact of the COVID-19 pandemic.

This study has been undertaken at a time of significant curriculum change in Wales. Although part of the UK, in Wales, since devolution in 1999, education has been controlled by Welsh Government (WG). Following a review of assessment and curriculum arrangements in Wales (Donaldson, 2015), between 2016 and 2020, a new national curriculum framework was developed, through a process of co-construction involving teachers from schools deemed to be successful by Welsh Government and named 'pioneer' schools during this process. Co-construction also involved support from national and international contributors and advisors. As part of this co-construction process, there were key review points in which stakeholders (schools, school staff, parents and learners as well as wider organisations) had opportunities to engage with the curriculum framework being developed, and to provide feedback. Thus, even though the final Curriculum for Wales framework, released in 2019, became statutory in 2022 for primary schools, engagement with it had, for the majority of schools, begun long before this. For example, the annual report of the Welsh education inspectorate, Estyn, for the academic year 2019-2020 (Estyn, 2021) reported that nearly all primary schools had begun engagement with the Curriculum for Wales.

The Curriculum for Wales Framework (WG, 2021) is noted by the Organisation for Economic Co-operation and Development (OECD, 2020, p. 20) as a 'cornerstone of the country's efforts to turn its education system from a performance-driven education with a narrow focus, to an education led by commonly defined, learner-centred purposes'. The recognition, within Donaldson (2015) and OECD (2020) that previous practice was performance-driven is important within the context of this study; national tests for literacy and numeracy were in operation in Year 2 and were used to consider learner and school performance prior to when the study began. This national performance-based practice changed during the study, and though national tests for literacy and numeracy still run, the results are not now used to consider school performance.

The focus on the move to learner-centred purposes and progression within the new curriculum in Wales is also of contextual importance. At the start of the study, schools were using national curriculum documentation that comprised year-on-year outcomes, performance driven statements for mathematics and numeracy. In contrast, the Mathematics and Numeracy Area of Learning and Experience (AoLE) within the Curriculum for Wales Framework (WG, 2021), is comprised of principles of progression and a progression framework, which aims to outline progression from the ages of 3 to 16 through broad descriptions of learning. As WG (2021) notes, the descriptions of learning are designed to support learning over a series of years and should not be used narrowly to design standalone lessons or assessments. In the new Mathematics and Numeracy AoLE, the term 'multiplicative relationship' is used as part of the descriptions of learning, and there had been no reference to this term (or terms such as multiplicative reasoning) in previous curriculum documentation.

For example, Table 1 illustrates differences between some prior curriculum statements and a description of learning relating to the multiplicative relationship and relevant to the Year 2 age group.

Foundation Phase Framework (WG, 2015a) Mathematical Development Year 2: Example statements relating to multiplication and division. Children are able to:	Curriculum for Wales Framework Mathematics and Numeracy (WG, 2020): Example of a description of learning progression step 2.
-counts sets of objects by grouping in 2s, 5s or 10s	I have explored and can use my understanding of multiplicative relationships to multiply and divide whole numbers, using a range of representations, including sharing, grouping and arrays.
-recall and use 2, 5 and 10 multiplication tables	
-begin to link multiplication with simple division, *e.g. grouping and sharing in 2s, 5s and 10s*	

Table 1: Comparison of some multiplicative relationship statements as outlined in the curriculum in Wales

Hence, the change in the way Mathematics and Numeracy, as an Area and Learning and Experience, is presented and envisaged is important to be noted, as the research was taking place within this period of change. The term multiplicative relationship was used for the first time in the curriculum documentation, giving prominence to a need for focus on relationships, and the broad descriptions of learning encompassing several years of learning generate a need for, and interest in, teacher professional development.

Another important contextual factor to note for this study is the COVID-19 pandemic, which began during Cycle 1 and therefore impacted on the way research could be completed. It had been originally intended to undertake post-implementation teacher and learner interviews following the initial implementation of the tasks, but due to national lockdowns and resulting school closures, these could not take place. Furthermore, due to national and local guidelines to mitigate the spread of the COVID-19 pandemic, when schools re-opened for all pupils, there were restrictions on visitors and the mixing of learners. Although it had been originally intended to undertake the cycles in two consecutive years, this was not possible. Cycle 2 took place as restrictions on visitors and mixing were easing, but some restrictions were still in place. Cycle 2 was also affected by short notice local authority directed school closures due to inclement weather; thus Cycle 2 took place in two time periods in an academic year (Cycle 2a and a follow up shorter Cycle 2b, with different learners involved in the two phases). A timeline of the research cycles is presented in Appendix A.

The next chapter focuses on the nature of mathematics and mathematics learning, and the learning and teaching of multiplicative reasoning and measures, introducing key ideas that inform the study.

CHAPTER 2: MATHEMATICS, MATHEMATICAL CONCEPTS AND THE LEARNING AND TEACHING OF MULTIPLICATIVE REASONING AND MEASURES

2.1 THE NATURE OF MATHEMATICS

It seems appropriate to begin a piece of writing about learning and teaching primary mathematics by considering the nature of mathematics itself. This is because beliefs about the nature of mathematics and the purpose of primary school mathematics will determine beliefs about how it can be learned and how it may be taught (e.g., Ernest, 1989; Askew *et al.,* 1997). Furthermore, White-Fredette (2010) argues that philosophical beliefs about the nature of mathematics itself are overlooked when considering mathematics education and mathematics education reform, suggesting that this can lead to a possible tension between how teachers view mathematics as a subject and beliefs about how it should be taught.

Within this work, mathematics is considered a construction; it has developed over time and through the contributions of many people into a subject that is studied and developed in learning environments across the world. Mathematics is a result of human and social activity which has developed into a set of shared understandings. As Freudenthal (1991) discusses, the word 'mathematics' looks like a plural and at one time, the word was a plural; encompassing four elements: arithmetic, geometry, astronomy and music. Indeed, it is through the analysis of areas such as music and astronomy and through attempts to work within and analyse the world around us that mathematics as we have come to know it today has been developed.

The Platonist view would be that mathematics exists naturally and is there to be discovered (e.g., as discussed by Ernest, 1991 and Greer, 2004). For example, a Platonist would argue that the special ratio between the circumference and the diameter of a circle which is the same for any circle is a natural phenomenon that has been discovered and labelled pi. However, pi is a construction; the notions of ratio, circumference and diameter are inventions that allow the analysis of objects such as circles. Furthermore, the notion of a circle can also be considered a construction. It could even be argued that a perfect circle rarely (if ever) exists in the natural world. Hence pi is a construction that allows analysis of the world, but it is a product of the human mind and a result of constructions that enable us to define it. Furthermore, these notions are social constructions; they exist as shared understandings in which the meaning has been socially and culturally constructed. As Hersh (1998, p.14) states 'Locating mathematics in the social-cultural realm means that it is human. For example, there is no sense to talking about mathematics existing before the human race existed or after it has vanished'. This is the view of mathematics taken in this work; mathematics is a social construction that has developed, and is still developing, over and through time, through shared and cultural understandings.

In some respects, the way children become aware of mathematics could be analogous with the way in which mathematics has developed as a set of constructed shared ideas. The earliest mathematics activity is commonly related to quantifying (e.g., through counting and/or measuring) and considering form (e.g., in relation to shape) and this is widely regarded as how mathematics itself began (e.g., Tall, 2013). Freudenthal (1991, p.18) argues that 'mathematics, unlike any other science, arises at an early stage of development in the then 'common sense reality' and its language in the common language of everyday life'. Indeed, it is through this 'common sense reality' that mathematics has evolved. Freudenthal (1991, p.31) discusses the term 'Mathematising' and considers it a process of generating and developing mathematics and is insistent that this term should include the 'entire organising activity of the mathematician, whether it affects mathematical content and expression, or more naïve, intuitive, say lived experience, expressed in everyday language'. Young children, before they even begin formal education, engage in what can be considered mathematical activity and mathematical reasoning; a young child sorting 3D shapes into a 'shape sorter' could be 'Mathematising' when he or she is able to identify shapes that will fit into the holes; a young child picking the larger quantity of treats could also be an example of 'Mathematising'. These examples may be more recognisable as mathematics if the child is able to articulate the reasons for choice using mathematical language but, even without articulation, the child is engaged in mathematical reasoning; the mind is at work and 'Mathematisation' is occurring. From a Vygotskian perspective (as discussed in Karpov, 2003, p.65) in the examples discussed above, the young child would be forming 'spontaneous' mathematical concepts; these are concepts formed through generalisation of everyday mathematical experience. For Vygotsky, those spontaneous concepts should then, through instruction, develop into 'scientific concepts' (Karpov, 2003, p.66) which are concepts which have developed, through human activity, into more formal, definable notions. The development of concepts in mathematics, discussed later in this chapter, is an important consideration in this work.

Mathematics may seem a very abstract subject to some. Progress through school mathematics undoubtedly involves increasingly more complex and abstract mathematics. Tall (2013) suggests that there are three stages to mathematics as it may be experienced in education: practical, theoretical and formal. A simplistic longitudinal overview would be that mathematics learning starts with practical experience, becomes more theoretical, developing into the more formal axiomatic mathematics commonly encountered at higher levels of education. This formal axiomatic mathematics has developed into shared understandings that seem to transcend cultures and language. However, shared mathematical understandings have not necessarily had a smooth development. Greer (2004) argues that mathematics, as a discipline, has a history of requiring 'conceptual restructuring' because its concepts have developed and evolved over time. For example, Greer (2004) cites the case of negative integers, once considered impossible by some eminent mathematicians, whilst Vamvakoussi and Vosniadou (2004) offer further examples of how the definition of number has needed to evolve. A famous example of this would be the Pythagoreans keeping $\sqrt{2}$ a secret, because the notion of an irrational number that could not be expressed as a ratio (fraction) between two integers seemed impossible. Greer (2009) notes that Piaget recognised that children are expected to learn the mathematics that has taken millenia to develop, and therefore argues the notion of conceptual restructuring should be recognised and accounted for within mathematics education.

Mathematics education and particularly 'school' mathematics may, and indeed does, differ in content and style across cultures; how and what mathematics is taught and experienced may depend on factors such as economy, cultural beliefs, and necessity for particular skills and knowledge. Hence the notion of ‘primary mathematics’ is itself a construction; a set of those mathematical ideas and skills deemed appropriate (often by policy makers) for learners of primary school age and this construct varies in different countries, regions and educational settings. As noted in Chapter 1, the recently co-constructed Mathematics and Numeracy curriculum in Wales (WG, 2020) reflects a change in approach from a focus on performance-based statements to broader descriptions of learning, indicating some key learning and experiences. Though comparison of mathematics curricula and policy documents from different countries and regions may at first glance show broadly similar content (e.g. concepts of number, operations and shape) and skills (such as reasoning, explanation, generalisation, classification), the organisation and expected application of the curriculum and seemingly subtle differences in expectation of depth and breadth or application would reflect a particular view of primary mathematics and, at the very least, will reflect a particular context in which that curriculum was developed.

2.2 MATHEMATICAL CONCEPTS

The view that mathematics is a socio-cultural construction has already been outlined and this perspective will naturally impact on the consideration of young children's learning of mathematics in this work. I believe learning is a highly complex process, which relates to many factors including individual disposition and interests, physiology and age, interactions, culture and interpretations. I also believe that there is no 'ultimate truth' in the way children come to learn mathematics; as Simon (2007) argues, learning theories are not proven but can be seen as perspectives, lenses, or philosophies. This work is underpinned by the philosophy of social constructivism as defined by Ernest (1991), adopting an adapted theoretical framework synthesising theories of cognitive development to enable consideration of learning within the mainstream classroom, discussed in Chapter 3. This section explores the notion of conceptual understanding in mathematics and considers, in particular, understanding of number and arithmetical concepts, with a particular focus on multiplication and division.

The term 'conceptual understanding' is used quite frequently in mathematics education; for example, it is noted as a key 'principle of progression' for Mathematics and Numeracy within the Curriculum for Wales (WG, 2022), informed by the strands of 'mathematical proficiency' in the work of Kilpatrick, Swafford and Findell (2001, p.115). However, concepts are recognised as difficult to define. Mason and Johnston-Wilder (2004, p. 198) in their work discussing fundamental constructs in mathematics, conclude that 'it is very difficult to be precise about what a concept is!'

Within this work, a mathematical concept is seen as an abstraction; it is an understanding formed about something mathematical, based on an experience or a collection of experiences. Freudenthal (1991) asks, what then is the difference between X (an object) and the concept of X? He concludes that the 'concept of X' is how that object is perceived in a certain perspective; that is when it may be analysed/reflected on/scrutinised. The notion that mathematical concepts develop through experiences and mental activity is also suggested by Skemp (1976, p.76) who states that 'A concept can be described as a mental awareness of something in common among a certain class of experiences'. The process of identifying commonalities and being able to group them into a set is, Davydov (1990) explains, often discussed in educational psychology, and is called generalisation. Hence a concept is a result of generalisation. For example, a child might experience 'five' in different ways, seeing five fingers on a hand, playing with five objects, singing and enacting a song about five little ducks, seeing five candles on a birthday cake. Generalisation would be being able to recognise that all these experiences involve a set of five objects. These experiences could then be cognitively filed to form a concept of 'five' being a set of five objects (whatever those objects may be). Tall (2013, p.81) discusses the notion of a 'concept image' which encompasses all the mental pictures or associations with a concept. Thus, a concept image for 'five' could include images and associations with 'five' things but would change over time and experience.

Clark (2011, p.32) defines a concept as a ‘big idea’ that allows the connection or making sense of lots of little ideas, and sees them as ‘cognitive file folders’, providing a structure within which information or ideas can be stored. This view of a concept as the connection or filing of ideas and experiences is similar to Skemp’s (1976) notion of awareness of commonality and suggests generalisation as discussed by Davydov (1990). Of note in Clark’s (2011, p.35) work is the view of learning as ‘the act of interpretation that emerges from the interaction between the learner and the object of learning’. This leads to the conclusion that a concept can never be complete because it is an internal interpretation informed by making connections between ideas and experiences; a concept is formed and owned by the learner and therefore may be continually developing. Each individual learner will hold images or ideas that may be cognitively filed to form the concept itself.

However, Shayer (2003) comments that:

> *‘a concept is more than the sum of certain associative bonds formed by memory, more than a mere mental habit; it is a complex and genuine act of thought that cannot be taught by drilling, but can be accomplished only when the child’s mental development has itself reached the requisite level’*
>
> Shayer (2003, p.465)

In Shayer’s definition, the use of the phrase ‘accomplished’ suggests concept formation reaches a limit. However, taking the example of a child's concept of a particular number, this will develop and change over time and through connection of different experiences. Different learners may, at different times, have their own interpretations of what that number means to them. This does not mean that the concept is not ‘accomplished’, rather it may be at a different stage of development. For example, a child's concept of a particular number such as 'five' might start with images of five and is then likely to develop to include the sum of other numbers such as 2 and 3, 4 and 1 and later may become an example of a prime number, a square root, a rational number, the quotient of numbers such as 20 and 4. As a learner's experience with particular numbers expand, a concept may become what Tall (2013, p.50) defines a 'thinkable concept', i.e. a concept which can be used, applied and acted on without having to think about what it means (a 'usable concept' would be another apt description). This developmental view of concepts recognises that concepts evolve with the learner.

In many respects, a mathematical concept (or indeed any concept) is a notion deemed important enough to be given a label or name. As von Glasersfeld (2001) discusses, interestingly in relation to translations of Piaget's work, in different languages, different words exist and there may not always be direct translations of some words. This can suggest that different concepts have developed. Indeed, as von Glasersfeld (2001) argues, even in the same language, people may hold different interpretations of words (labels). However, if we are to communicate effectively then conventions, which can be told, can be accepted and this can support mutual understanding. Mathematics is a discipline which has developed over time, and in which there exist shared conventions and definitions. Indeed, formal axiomatic mathematics (e.g., as discussed by Tall, 2013), usually more associated with 'higher' level mathematics, is founded on axioms (accepted notions) and definitions, and this axiomatic property, from which formal proofs are derived, is considered the power of mathematics. Thus, some concepts can also be defined and agreed.

As mentioned previously, Vygotsky (in Karpov, 2003) believed that, in young children's learning, concepts could be spontaneous or scientific. Hedegaard (2007) explains that Vygotsky saw spontaneous concepts as concepts that arose in everyday settings, mediated through interaction within family and community and appropriated through experience with everyday objects. In contrast, a scientific concept, though not limited to science, will involve a form of abstraction (such as a formal definition) and will need, as Schmittau (2003, p. 226) discusses ‘pedagogical mediation’. Noteworthy, as Hedegaard (2007) discusses, is that spontaneous and scientific concepts should not be considered discrete, rather intertwined. For Vygotsky, scientific concepts could be formed from spontaneous concepts and, indeed, Vygotsky's famous 'Zone of Proximal Development' can be seen as a way of linking spontaneous and scientific concept development (Daniels, 2007). Scientific concepts can also enrich and qualify spontaneous concepts. A mathematical example of this could be the concept of division. Children typically have everyday experiences of sharing items equally (e.g., between friends or siblings) or finding out how many groups of one number are in a number (e.g., sweets grouped into twos or threes) and could therefore develop spontaneous 'everyday' concepts around sharing and grouping even though they may not yet be familiar with division as a mathematical idea. Supporting children in moving from these spontaneous everyday concepts of sharing and grouping to the scientific concept of division would need to be mediated using tools such as language, manipulatives and symbols. Once developed, the scientific concept of division can enrich the everyday spontaneous concepts of grouping and sharing. The complexities of division as a mathematical concept are explored later in this chapter, with the example provided as a way of illustrating the relationship between spontaneous concepts and scientific concepts, which were proposed by Vygotsky. Shayer (2003), however, suggests such a relationship highlights a paradox in Vygotsky's thinking; on the one hand spontaneous concepts are seen to precede scientific concepts yet the development of scientific concepts is also seen to cause spontaneous concepts to evolve. A more conciliatory view would be that the relationship is interdependent.

Russian psychologist Davydov (1990), working in the 1960s, applied Vygotsky’s definitions of spontaneous and scientific concepts to his own work, but he also believed (1990, p.40) that ‘scientific knowledge is not a simple extension, intensification, and expansion of people’s everyday experience’. As Davydov (1990) explains, spontaneous concepts involve concrete experiences and a process of generalisation leading to an abstract notion, but they are distinguishable from scientific concepts because there will not be awareness of the concept itself. Indeed, for Davydov, a key feature of Vygotsky’s scientific concept was that the learner 'was more aware of the concept itself' (Davydov, 1990, p.86) and that the concept should arise 'not through a direct encounter with things' but through mediation resulting in 'movement from the concept to the thing – from abstract to concrete'. Thus, for Davydov, understanding the scientific concept is seen as the starting point for instruction. Thus Davydov (1990) offers a new, arguably radical, perspective on concept development.

Furthermore, Davydov (1990) considers the way disciplines have evolved, arguing that scientific disciplines, including mathematics, have developed theoretical concepts about objects, and these are different from the objects themselves. A mathematical example of this would be number. As discussed earlier in this chapter, children can develop spontaneous concepts of particular numbers (e.g., of the number 'five' being a word and symbol that represents any five things, and they might develop this through connecting images of five and experiences of making five in different ways). However, this is quite different to having a theoretical concept of number, which would recognise number itself as an abstract notion with 'five' as an example of this. As Schmittau (2003) clarifies, for Davydov, *all* mathematics concepts were scientific and not spontaneous. Davydov's (1990) view that all mathematics concepts are scientific and therefore should arise through awareness of the concept itself and from abstract to concrete experience seems contrary to deep rooted and common practice in the teaching of early number concepts.

2.3 THE DEVELOPMENT OF NUMBER CONCEPTS: CONCRETE TO ABSTRACT OR ABSTRACT TO CONCRETE?

Number concepts and number operations are traditionally and typically introduced through counting activity, involving counting of positive integers (counting numbers are also known as natural numbers). As discussed in Chapter 1, Coles (2017, p.206) refers to this as the 'counting world'. Tall (2013, p.7) summarises typical mathematical development as 'young children are introduced to counting physical objects to develop the concept of number and to learn to calculate with numbers'. Hence, common practice is that children will count objects and learn to use abstractions (number names and symbols) to communicate the number of objects in a set; they may then learn that a set could be comprised of sub-sets (the part-whole relationship) and that this additive relationship can be expressed symbolically. Such activity moves learners from concrete experiences, which may be supported with pictorial representations, to abstractions in the form of definitions, ideas and symbolic notation that could be applied to any concrete or pictorial representation. This practice seems, at first, perfectly acceptable. Such experiences also exemplify a current 'popular' model for teaching mathematics called Concrete – Pictorial – Abstract (CPA). The CPA heuristic, as Merttens (2012) explains, developed in countries such as Singapore and China and its principles, often likened to Bruner's enactive – iconic – symbolic theory (Hoong *et al*., 2015), are now a common feature of UK textbooks and teaching materials. However, starting with counting as a basis for understanding number concepts is not without its critics.

Schmittau (2003, p.227) points out that a flaw in starting number concepts with counting discrete objects is that it will 'ground children in their spontaneous notions of number'. For Davydov (1991, discussed in Schmittau, 2003) this will result in a concept of number heavily influenced by counting numbers, one consequence being that this will make fractions and irrational numbers more difficult to learn. The difficulties children experience learning fractions are frequently documented (e.g., Nunes and Bryant, 2009a). As Bobos and Sierpinska (2017, p.208) discuss, fractions are commonly introduced as 'special numbers' and they are also typically generated by counting (e.g., counting how many parts are shaded out of how many parts altogether). Fractions are examples of rational numbers (any number which can be expressed as a ratio between two integers) and yet pupils typically learn definitions for concepts such as rational numbers, irrational numbers and real numbers long after they have experienced particular examples of them. Hence it is typically only at later stages of education that the theoretical scientific concepts of rational number will be met. The implication of teaching number in this way, as Vamvakoussi and

Vosniadou (2004) suggest, is that understanding the rational number system then requires conceptual restructuring.

For Davydov (1990), the idea that theoretical concepts are met later in education and not from the outset was a flaw in curriculum design. Although Davydov (1990) critiqued the Russian curriculum in the 1960s (his work was translated into English in the 1990s), the typical mathematics curriculum he describes relates closely to current international practice, including that of the UK. For Davydov (1990), understanding the very essence and history of a theoretical scientific concept and finding a practical way of enabling learners to understand that scientific concept was vital from the outset. Enabling learners to understand the scientific concept, and then learn concrete examples, was what Davydov (1990, p. 128) advocates; he called this 'ascent from the abstract to the concrete'.

Davydov (1990) establishes that a central notion in the scientific concept of number is that of a unit. Quantification is achieved through identifying a unit and calculating how many of that unit represent the quantity being considered. This reflects the process of measurement; as Nunes and Bryant (2009a) note, measurement involves the identification of a unit and finding out how many times that unit fits into what is being measured. Thus, developing the theoretical concept of number, for Davydov (1990), would fundamentally involve the notion of a unit, as number is the result of finding a relationship between a quantity and a unit.

Davydov (1990) reports research undertaken by himself and colleagues in 1961 with first grade children (ages 6-7, 53 learners). This research involved five 'assignments', which, Davydov argues, involved counting and measure that the children had already mastered. The assignments are summarised below:

1) Pupils were given a wood panel measuring 50cm and asked to bring wood of the same length from another room (not being allowed to take the original panel). The only thing the child could take was a 10cm stick. This assignment was designed to assess whether the child could use the 10cm as a mediating unit.

2) 12 blocks were placed on a table in 4 groups of 3. Children were asked 'How many here'?'. There was a deliberate absence of indication of what was to be counted to assess whether the child might ask clarification as to what was being counted or could demonstrate, through their actions, what exactly was being counted.

3) A row, made of 20 blocks, was placed on a table. A row of 4 within the row of 20 was broken off and the child was asked 'How many of these here?'. If the child correctly identified that there were five of those rows of 4, then the child was asked to identify one of the five. This assignment was designed to explore whether the child could establish a relationship between an object and what was being counted, identifying a particular unit.

4) Two panels of 20cm were combined to make a panel of 40cm. A panel of 10cm was shown to the child and the child was asked to identify how many of the 10cm panel would make the 40cm panel. The child was then asked to show where two of the 10cm panels would go. This assignment was designed to assess whether the child could relate the object being used to measure with a number to measure the panels.

5) Two big jars and two little jars were placed on a table. The child was shown that two little jars would fill a big jar (this was demonstrated by pouring water). The child was then asked 'How many of these (little jars) will fill these (the two big jars and the two little jars)? The child was then asked how many of the big jars would fill the row of jars. This assignment was designed to explore how the child used a unit that did not directly relate to what was being considered.

Davydov (1990) categorised the results of these tasks with 53 children into those that were managed independently (without mistakes), those that involved mistakes but were then managed with support and those that were not managed at all. The support given is not detailed and it is not clear whether any support could have been given in the cases managed independently. Davydov (1990) reports that, considering all tasks for all 53 children, 31% were managed independently, 42% involved mistakes with some support and 27% were not managed at all. Only two children managed all five independently and only one managed four. Davydov (1990, p.69) used this data and further analysis of individual tasks to argue that 'many first graders experienced significant difficulties'.

Clearly these assignments were designed to explore the children's understanding of what they were counting. A sceptic might argue that the tasks were deliberately misleading, not assessing what the children may have experienced previously. Personal experience of working with children with groups of interlinking cubes would suggest that they would indeed need clarification of what was being counted. However, the key point is that the notion of a 'unit' was being assessed and many of the children appeared, across all the tasks, to lack understanding of this. This is perhaps unsurprising, because developing the notion of a unit had not been focus of teaching. However, Davydov (1990, p. 75) uses these results, together with results from research in language and history, to argue that there was, in the curriculum, 'a detachment of school instruction in concepts from their origin'. Put simply, Davydov (1990) argues for the teaching of concepts to consider the theoretical nature of the concept itself and its origin. Termed 'genetic analysis' by Schmittau (2003, p.232), the notion that the genesis of a concept should be reflected when teaching was central to the curriculum that was developed by Davydov and his colleague Elkonin.

Davydov (1990, p.76) argues that when introduced to young children 'numbers are taken as given and ready-made having representation in number configurations.' For Davydov, the concept of, and need for, number should be developed not through counting but through activity involving quantities. As Nunes and Bryant (2009b) clarify, numbers and quantities are not the same. Quantities are physical and may not always need number for comparison. Continuous quantities (such as length, area, mass, volume and capacity) can be related to number, or quantified, through measuring activity. Indeed, as Vergnaud (1979, p.264) asserts, 'the concept of number would not exist if man had not met problems of measurement'. In the curriculum devised by Davydov and Elknonin, early experiences for young children involve trying to measure continuous quantities such as length, area, mass, volume and capacity. As Schmittau (2003, p229) summarises, such activities 'reflect the essence of mathematics as the science of quantity and relation'. Coles (2017) emphasises the focus on relationships in such a curriculum. In Davydov's (1990) curriculum, at first, quantities may be perceptually comparable, and activities are then structured such that this perception becomes more difficult. Examples discussed by Schmittau (2003) include children being asked to compare the height of a bookcase and the length of a desk or the capacity of liquid in two containers of different shape. In such cases, the use of an intermediary becomes necessary, and hence the notion of a unit develops. Children will learn to count, but the counting activity is in the context of measure, with a need quantification and a focus on relationships. From the outset, the notion of a unit is central. Also noteworthy is that children may use algebraic notation to represent general relationships (e.g., a=b, a>b, a<b etc.). This is, again, an example of abstract before concrete. As Coles and Sinclair (2022) assert, starting with complex ideas might, in the long term, make learning simpler.

A central pedagogic theme within Davydov and Elknonin's learning and teaching activities is *necessity* (e.g., see Schmittau, 2003 and 2010; Davydov 1990 and 1992; Venenciano 2017); problems are set up which are too difficult or inefficient and so this necessitates a new way of working. Schmittau (2010) details a sequence of early activities, suggested by Davydov and trialled in the US, progressing from communicating the height of a mammoth through unit 'tokens' to then recognising that representing the height in actual tokens may be insufficient (e.g., one might be dropped or lost), leading into recording using tally marks. At each stage, the problem is extended in a way that the previous mode of working becomes insufficient and so a new way of working needs to develop. Thus, after children have learnt to use tally marks to record the number of units used, problems are set up where just using the tally is ineffective because what the tally represents needs clarification. This leads to a need to record relationships between units and then a need for communicating number names using words. In this way learners should develop, not only an understanding of the number system and its communication, but also its purpose, meaning and history.

In such a curriculum, the concept of a fraction as a relationship between quantities evolves and progresses from the early experiences with measure (e.g., comparing two lengths using an intermediary unit). Through these experiences the notion of any rational number (integer or fraction) and indeed irrational numbers can develop and, as Schmittau (2003, p. 229) argues, 'significantly, do not require a reconceptualization of number when they do occur'. Such sequences of progressive activity are designed to develop understanding of the *scientific* concepts of rational and irrational numbers from the very outset. It should be emphasised that learners are engaged in concrete activity, but the concrete activity and related problems are designed to develop scientific theoretical concepts of number rather than the traditional way in which scientific concepts are introduced after spontaneous concepts are deeply rooted. Situations are not set up for learners to apply ready-made mathematics, rather situations are set up to allow the construction of the concept itself.

Research into the effect of a Davydov and Elkonin style curriculum in mathematics seems sparse and elusive, at least within the English language. Davydov (1990, p. 163) reports that the mathematical programmes designed are 'experimental' and references studies on them, but these, as with much of the work of Davydov and Elkonin, appear unavailable in English. It is also difficult to ascertain the extent to which the work influenced current practice in Russia. However, interest in Davydov and Elkonin's work developed in the US in the 1990s, perhaps a consequence of the translation in 1990 of some of Davydov's work for the National Council of Teachers of Mathematics (NCTM). Schmittau (2010) claims that a three-year implementation of a Davydov and Elknonin programme was, to her knowledge, the first in a US school setting (New York). Details of implementation (e.g., number of participants, how the programme was introduced and evaluated) are unclear. Schmittau (2004, p. 20) reports that children ‘found the continual necessity to problem solve a considerable – even daunting challenge, which required virtually a year to meet as they gradually developed the ability to sustain the concentration and intense focus necessary for success’. Despite acknowledging this difficulty, Schmittau (2010) certainly makes bold claims about the Davydov and Elknonin curriculum, suggesting that it promotes the connection of ideas and allows for the resolution of several typical and commonly discussed mathematics learning and teaching dichotomies such as: procedural/conceptual learning, problem solving/routine practice, discrete/continuous quantity, action on objects/action on symbols, numerical/algebraic focus. Seemingly independently of Schmittau and colleagues, Dougherty (2003) also reports on a US (Hawaii) project, called ‘Measure Up’, which involved following a Davydov and Elknonin style programme in grades 1 to 5. Venenciano (2017) discusses some details of the project, summarising that it took place in a laboratory school with approximately ten students in each grade. Furthermore, this Venenciano (2017) paper discusses a study with 27 grade 12 students, thirteen of whom had followed a Measure Up curriculum in at least some of their early years of schooling and 14 of whom had not followed this curriculum. She cautiously concludes that data collected suggest that following a Measure Up curriculum in the early years appears to support students in generalising relationships and structures in non-numeric situations and, like Schmittau and Morris (2004), suggests that a Davydov and Elkonin curriculum could support learners in making a transition from arithmetic to algebra. Moxhay (2008), notably a translator of Davydov’s (2008) ‘Problems of Developmental Instruction’ book, researched a Davydov and Elkonin style mathematics curriculum in Maine, US, over a six year period, and reported that it took the first four years of the project to train the teachers to develop their understanding of such an approach. Furthermore, Moxhay (2008, p.21) implied success only occurred when learners ‘accepted Davydov’s form of instruction’. In Moxhay’s (2008) paper, details about what the difficulties in acceptance might have been are elusive, with the suggestion this related to behaviour of some learners. Nevertheless, Moxhay’s (2008) comments reinforce the argument made by Coles and Sinclair (2022, p.19), discussed in

Chapter 1, that there would need to be 'significant and sustained effort' to make changes to a whole curriculum paradigm.

Interest in the work of Davydov and Elkonin appears to have reignited recently, with a 2017 special issue of the International Journal for Mathematics Teaching and Learning devoted to their work, followed by a special edition of Educational Studies in Mathematics in 2021. Both journals show that there is international interest in research into the potential of Davydov's ideas. Indeed, more recently, there appears to have been a focus on the research of some specific aspects and adaptations of Davydov's work within particular cultural contexts, rather than an attempt to explore the transposition of the original curriculum. Mellone, Ramploud and Carotenuto (2021, p.382) argue the need for 'deconstruction'; they see deconstruction as an analysis of practice and beliefs from one setting whilst considering its compatibility within another setting. Similarly, in South Africa, Venkat, Askew and Morrison (2021, p.399) report on their 'shape-shifting' of Davydov's ideas, that is, the re-interpretation and adaptation of ideas to 'align with classroom cultures and conditions'. They re-interpreted Davydov's (1990) work as a 'straight-for-structure' approach (p.400), designing tasks to draw attention to relationships and structure in supporting number calculations, rather than focusing on calculating through counting. They acknowledge, however, that their work does not reflect all aspects of Davydov's work, particularly as they do not use continuous quantities.

In Sweden, Eriksson and Jansson (2017), Erisksson and Erisksson (2020) and Erisksson and Sumpter (2021) report on the development of algebraic tasks for learners (age ranges 7 to 13 over the three studies), inspired by the work of Davydov and Elkonin. Their work focuses on algebraic reasoning with Cuisenaire rods. Cuisenaire rods are a mathematics resource consisting of sets of ten differently coloured unmarked rods to support reasoning about relationships. Collectively, the studies show that both algebraic and quantitative reasoning can be developed successfully through tasks inspired by the approaches introduced by Davydov and Elkonin. Another example of a specific aspect of the Davydov and Elkonin curriculum being researched is the exploration of adaptations of the original curriculum tasks within an elementary year group (ages 6-7) in a school in The Netherlands (Jaffer, 2021). In Jaffer's (2021) work, reasons such as availability of materials, recognition of prior learning and a need to ensure problems were fulfilling were given for changes made to tasks in the curriculum documentation. Nevertheless, Jaffer (2021) concludes that the tasks offered rich opportunities for abstraction and that learners demonstrated skills beyond those typically expected.

To conclude, this section has considered the work of Davydov, and his theory of mathematical concept development, with a focus on the teaching of theoretical concepts from the start, with the use of continuous quantities, through situations that necessitate a new way of working and that reflect the way in which the concept itself may have developed. Furthermore, there is a focus on relationships within the mathematics being experienced. Although precise details about implementation and impact of Davydov and Elknonin's curriculum are difficult to find, the work of Davydov offers a socio-cultural approach to the construction of mathematical concepts that not only reflects the social constructivist approach to learning taken in this work, but also reflects the nature of mathematics as a social construction.

2.4 THE LEARNING OF MULTIPLICATION AND DIVISION

The learning of multiplication and division both as *arithmetical operations* and as *concepts* has been widely considered in analysis of mathematics learning, which is discussed within this section. Mathematics curricula and the related pedagogy reflect the context and culture within which they were developed; as Brown (2001) noted over 20 years ago, as technology has advanced, the emphasis on the need for pupils to learn algorithmic procedures to calculate can be reconsidered. An OECD position paper in 2018 reinforces that, with advances in technology, numeracy and data literacy are increasingly important skills for the future (OECDa, 2018). Numeracy is widely considered the application of mathematics in everyday life, for example, the OECD notes that numeracy involves the ability to 'access, use, interpret and communicate mathematical information and ideas' (OECDb, 2018 p.6). The OECDa (2018, p.5) also argues the need for learners to 'think like a mathematician' and 'apply their knowledge in unknown and evolving circumstances'. Hence being able to operate with arithmetical procedures proficiently is no longer considered sufficient for learning multiplication and division; learners need to be able to apply and use their understanding of the mathematics they learn to a range of contexts, and this will require understanding of concepts and reasoning about relationships (e.g., Nunes *et al.*, 2012).

The argument about the balance between arithmetical competence, conceptual understanding and reasoning is not new. Brown (2001) gives examples of British educationalists in 1850s and 1860s, who argued for attention to reasoning skills and understanding rather than over emphasis on rote learning of arithmetical procedures. In 1979, Vergnaud (p. 263) commented that arithmetic had a history of being associated with 'boring and out of date calculation' and argued that understanding the *concepts* involved in elementary arithmetic was fundamental to mathematics learning. Vergnaud (1979) suggested at that time that one of the most challenging questions in mathematics education was to establish a link between arithmetical concepts and arithmetical situations. Often, there is a perceived dichotomy between being able to use arithmetical procedures and understanding the concepts of the associated operations, sometimes termed the procedural-conceptual divide (e.g., Schmittau, 2004). Star (2005, p.1) calls such arguments in mathematics education a 'war'. In the same decade as Vergnaud, Skemp (1976, p.20), in his seminal writing, distinguished between 'instrumental' and 'relational' understanding. Instrumental understanding is knowing how to do something, whereas relational understanding involves knowing why and being able to make connections between ideas. Work such as Vergnaud's (1979) and Skemp's (1976) could be considered influential in subsequent mathematics education research which considered the relationship between children's understanding of number concepts and their use of arithmetical operations and/or procedures, some of which is discussed within this work. Star (2005) argues that, in past decades, there may have been an over emphasis on research into conceptual understanding, to the detriment of understanding of how procedural knowledge might develop. Rittle-Johnson, Siegler and Alibali (2001, p.346) suggest an 'iterative' relationship between conceptual and procedural knowledge, arguing that they are mutually dependent; gains in one area affect gains in the other. I adopt this position for considering how children learn multiplication and division; both conceptual understanding and procedural competence are important, and they are not mutually exclusive. Research into conceptual understanding in multiplication and division, what procedural knowledge might suggest about understanding in multiplication and division, and the relationship between conceptual understanding and procedural knowledge is considered within discussion of the literature.

In an extensive meta-analysis of research into children's learning of mathematics, Nunes and Bryant (2009a) reinforce that children can solve problems associated with multiplication and division before they learn about them as arithmetical operations, and attribute this to the notion of one-to-many correspondence. Piaget (1952) researched children's understanding of one-to-many correspondence, which is seen as an important factor in understanding of multiplication and division (e.g., Sophian and Madrid, 2003; Nunes and Bryant, 2009a). Furthermore, Correa, Bryant and Nunes (1998) comment that Piaget saw one-to-many correspondence as the origin of multiplication and division. One-to-many correspondence requires the understanding that one thing can represent many things, i.e., one thing can represent a set of objects. This contrasts with the notion of one-to-one correspondence where one thing corresponds with another one thing directly. Piaget (1952) investigated one-to-many correspondence by setting up tasks with flowers and vases. One task involved showing children that a vase held two flowers and, for a set number of vases, asking children to pick tubes to represent each flower needed. Although the notion of one-to-many correspondence is recognised as important within multiplication and division, situations involving one-to-many correspondence can be solved with repeated one-to-one counting. For example, a child asked how many flowers would be needed if two flowers are placed in each of six vases, could count tubes out in ones. This might not mean that one-to-many correspondence has been misunderstood, but it suggests that the procedure used to calculate is limited to counting in ones. Sophian and Madrid (2003) point out that Piaget (1952) noted that five-year olds were more successful in using one-to-many correspondence when allocating two flowers to each vase within a set of vases when there were two colours of flowers, suggesting that the distinction in colours allowed for two separate one-to-one correspondences, as opposed to one-to-many correspondence. It can be considered unsurprising that children might use one-to-one counting in one-to-many correspondence situations, since many of the activities they will have been involved in from an early age typically uses counting in ones.

Bryant (1997) argues that Piaget's notion of one-to-many correspondence can be overlooked when considering multiplication and division, but suggests that Piaget's work on this contributed to understanding of how children might learn multiplication and division. It is certainly true that the notion of one-to-many correspondence allows the modelling of a situation that, in arithmetical terms, could be described by a multiplication or division calculation. In the example above, with vases and flowers, a division situation could involve asking how many vases would be needed for placing a given total number of flowers (e.g., 12) so that there are three in each vase. Correa, Bryant and Nunes (1998) point out that Piaget saw division as the inverse of multiplication and therefore did not extensively research division. However, Bryant (1997) argues that Piaget's biggest contribution to understanding children's mathematics, with his work on one-to-one and one-to-many correspondence, was the distinction between additive and multiplicative relationships, although these terms were not used by Piaget himself.

2.5 ADDITIVE AND MULTIPLICATIVE RELATIONSHIPS

Vergnaud (1982 and 1983, in Nunes *et al.*, 2012) asserted that many problems involve distinguishing between an additive or multiplicative relationship, and he seems to be most associated with the first use of these terms. As Nunes (2009c, p.8) helpfully summarise, an additive relationship involves *difference* between quantities and a multiplicative relationship involves a *ratio* between quantities. As arithmetical operations, addition and subtraction are inverse of each other and a scenario involving an additive relationship could involve either addition or subtraction. For example, asking a child how many more apples, when one person has a certain number, and another has a different number could involve either addition or subtraction. Indeed, a simple example (e.g., Rhiannon has 5 apples and Elinor has 8, how many more does Elinor have?) could involve neither addition or subtraction as arithmetic procedures; if modelled by using pictures or objects (Figure 1 below), one-to-one correspondence could be used and then one-to-one counting to find the difference, Nevertheless, however the difference is calculated, the relationship and semantic structure within the example itself is considered additive.

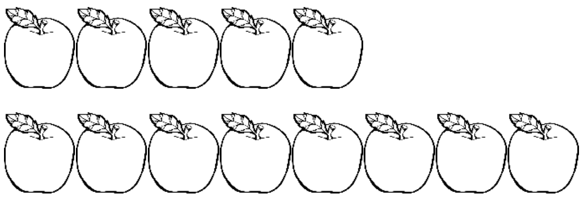
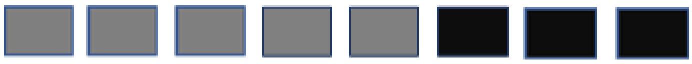


FIGURE 1: AN ADDITIVE RELATIONSHIP

The additive relationship shown in Figure 1 would make use of one-to-one correspondence and this is why Piaget believed one-to-one correspondence was important within additive relationships.

Another way of analysing the additive relationship is that it involves a relationship in which a whole is split into parts, which may or may not be equal. Nunes and Bryant (2009b) reinforce that the part-whole relationship was recognised by Piaget as important in children's understanding of number. Understanding the part-whole relationship involves recognising that when a whole is split into parts (e.g., a set of 8 can be split into a set of 5 and a set of 3), then those parts can be combined to make the whole, or if one part is taken away then only the other part will remain. This is illustrated in Figure 2:



5 + 3 = 8 or 3 + 5 = 8
8 – 3 = 5 or 8 – 5 = 3

FIGURE 2: A PART-WHOLE ADDITIVE RELATIONSHIP

An additive relationship involves the same class of objects, or quantities, being combined, separated, or compared; Nunes and Bryant (2009c, p.12) note that additive reasoning is used in 'one-variable' problems.

In contrast, a multiplicative relationship involves two variables which will be in a fixed ratio (Nunes and Bryant, 2009c). For example, Piaget's earlier example of flowers and vases (which were used to discuss one-to-many correspondence) demonstrates a multiplicative relationship with the variables being the vases and flowers and the ratio being how many flowers in each vase, illustrated in Figure 3.

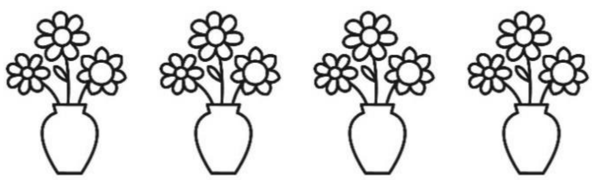


FIGURE 3: AN EXAMPLE OF A MULTIPLICATIVE RELATIONSHIP

Steffe (1994) argues that a key aspect in the multiplicative relationship involves being able to work with units that may be composed of more than one thing, called composite units. The notion of a composite unit reflects Piaget's construct of one-to-many correspondence; in the example of vases and flowers, the vase of three flowers would be the composite unit. As Clark and Kamii (1996) discuss, Piaget recognised that addition and multiplication require different levels of abstraction. Clark and Kamii (1996) illustrate (Figure 4) how 3 + 3 + 3 + 3 (repeated addition of 3) involves additive thinking in that each unit of three is seen as three 'ones' such that the resulting 12 is obtained by counting in an additive way (3, 3 more 'ones' to 6, 3 more 'ones' to 9 and 3 more 'ones' to 12). They compare this to the model of multiplication: 4 x 3, in which 'three' is itself seen as a unit. This involves one-to-many correspondence recognised by Piaget, but they also argue this requires simultaneous understanding of 'inclusion' relationships between the units of three containing three 'ones' and the four units of three containing three units of three, two units of three and one unit of three. Applying the distinctions of additive and multiplicative relationships discussed earlier and relating this to the example of flowers and vases above, model (a) would involve one variable (the flower repeatedly added), whereas model (b) would involve two variables (three flowers, four vases).

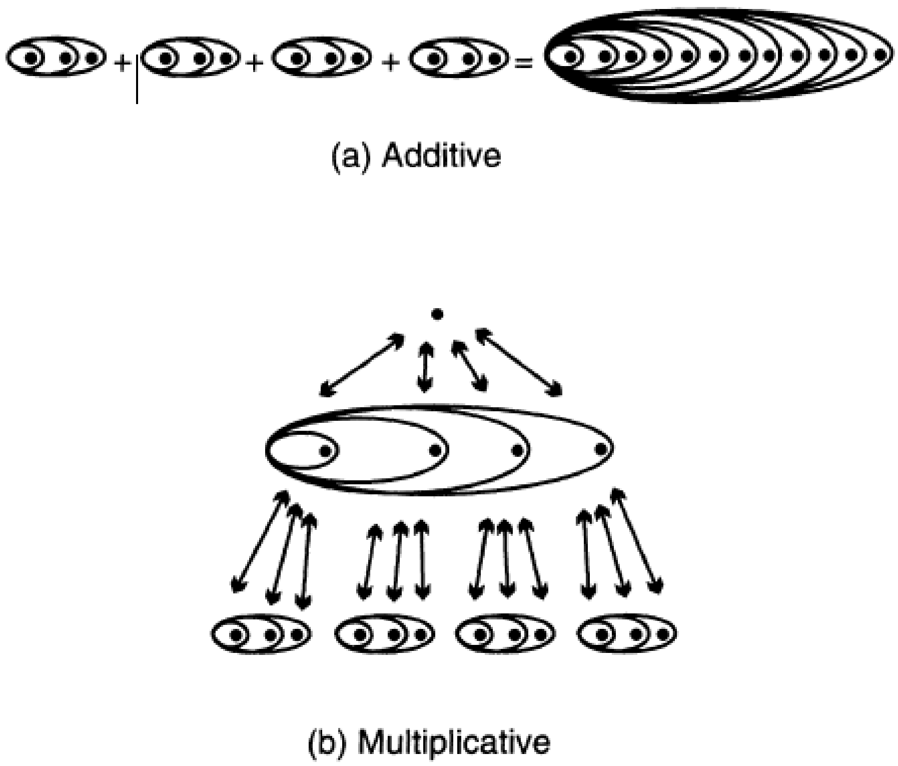


FIGURE 4: A COMPARISON OF ADDITIVE AND MULTIPLICATIVE APPROACHES, CLARK AND KAMII (1996, P.42)

In the example provided in Figure 3, the relationship between flowers and vases could also be described using fractions and ratio. For example, each vase contains one quarter of the flowers and the ratio of vases to flowers is one to three, or there are three flowers per one vase. These are all ways of describing situations involving proportion.

Vergnaud (1994) argues that concepts such as multiplication, division, fractions, ratio and rational numbers, whilst different, are interconnected and therefore proposes that a 'multiplicative conceptual field' should be considered, particularly in relation to research. This suggestion explicitly acknowledges that concepts such as multiplication, division, ratio and rational numbers cannot be considered in isolation; success in one area relies on understanding of others. Vergnaud (1994, p.46) considers the multiplicative conceptual field ‘a bulk of situations and a bulk of concepts’ which develops over time and through multiple experiences, and he emphasises the importance of recognising implicit knowledge that learners may bring within such a field. Vergnaud's argument for recognition of a 'multiplicative conceptual field' seems to have been realised, to some extent, in the development of research in the field of 'multiplicative reasoning'. It should be noted that the terms 'multiplicative reasoning', 'multiplicative thinking' and the 'multiplicative relationship' are often used interchangeably. In this work, the term 'multiplicative relationship' will be used to reflect a relationship as discussed above and the term 'multiplicative reasoning' will be used to reflect reasoning about a multiplicative relationship, although it is recognised the terms are not mutually exclusive. Furthermore, the research discussed in this review considers any research involving situations which involve multiplicative relationships, regardless of whether the terms multiplicative relationships or multiplicative reasoning are used or not.

As noted previously, many calculations involving a multiplicative relationship (such as those involving only whole numbers) could be approached using one-to-one counting. These could also be calculated using repeated addition (e.g., in the example above 3 + 3 + 3 + 3). It is therefore important to recognise the distinction between the *semantic structure* of a situation (e.g., whether additive or multiplicative) and the *calculation* approach to the situation. The way pupils have approached calculation and the structure of multiplicative problems has been a focus of much research into children's learning within multiplicative relationships and is discussed in the next section.

2.6 RESEARCH INTO STRUCTURE, MODELS AND CALCULATION WITHIN MULTIPLICATION AND DIVISION

Fischbein *et al*.'s (1985) seminal work analysed the responses of 628 Italian pupils (Grades 5, 7 and 9, ages 10/11, 12/13 and 14/15) to worded problems involving arithmetical operations, hypothesising that (p.4) 'Each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model'. It should be noted that Fischbein *et al*. (1985) seem to have used the term 'model' to consider the way pupils approached, calculated, or viewed a problem, but also to reflect the underlying structure of a problem. Their interest was specifically in the models used for multiplication and division and they conjectured that the model of multiplication as repeated addition would be prevalent. Repeated addition, as seen previously, is a model in which multiplication is seen as being the result of repeatedly adding a number (e.g., 3 x 4 could be 4 + 4 + 4, or 3 + 3 + 3 + 3).

For division, Fischbein *et al*. (1985), conjectured that two models would be prevalent. As Squire and Bryant (2003) note, there are two types of division problems. The first model for division is partitive (sharing) which involves identifying the dividend (number to be divided by) as being shared between groups (the number of groups represented by the divisor), so 12 ÷ 3 would be modelled as '12 shared into 3 equal groups'. The second model is quotitive, in which the dividend may be modelled as being made up of groups of the divisor, so 12 ÷3 would be interpreted as 'how many groups of 3 in 12'. Nunes and Bryant (2009a, p.27) note that quotitive division can also be called 'measurement division' because, like in measurement, the desire is to find out how many times one quantity fits into another.

The suggestion by Fischbein *et al.* (1985) that such models are 'primitive' may, at first, seem harsh as these models are the models that are commonly, almost universally, used to introduce young learners to the concepts of multiplication and division, particularly when modelling with concrete apparatus. For example, as seen in the curriculum statements in Chapter 1, (Table 1, p.8), refer to sharing and grouping. Fischbein *et al.* (1985, pp.5-6) hypothesise that 'the enactive prototype of an arithmetical operation may remain rigidly attached to the concept long after the concept has acquired a formal status'. For Fischbein *et al.*, the 'formal status' in this context appears to be the use of arithmetical algorithms without the need for concrete apparatus. Indeed, they acknowledge Piaget's suggestion that every mental operation, including arithmetic, is rooted in practical situations. However, as Fischbein *et al.* (1985) discuss, the prevalent intuitive models for multiplication and division have their limitations, particularly when rational numbers or decimal fractions are involved. For division, examples of this would be the notion of 5 ÷ 15 and 5 ÷ 12 (p.12), which could not be modelled in a quotitive way as 'how many groups of 15 (or 12) in 5? Although these examples can be modelled as '5 shared between 15' and '5 shared between 12', they rely on an understanding of fractional quantities. Other examples (p.12) would be 3.25 ÷ 5 and 0.75 ÷ 5, which cannot be modelled in a quotitive way as 'how many groups of 5 in 3.25 (or 0.75)'? although could be modelled in a partitive way as '3.25 (or 0.75) shared into 5 groups)'. Again, use of the model in this way would require an understanding of decimal fractions.

Fischbein *et al.* (1985, p.15) conclude that the intuitive models are 'incomplete'. Taken as a set of two models for division, it is possible to argue that they are collectively complete in that one of the models could be invoked to try to explain (model using concrete apparatus or visual images) any division situation involving real numbers. However, it is also clear that further understanding, particularly in relation to fractions and decimal fractions and relationships between numbers would be necessary, and furthermore, with that understanding, use of the models becomes unnecessary or over-complicated. For example, 5 ÷ 12 modelled as '5 shared into 12' results in the fraction 5/12 and thus an understanding that fractions are rational numbers (numbers which can be expressed as a relationship between two integers, in this case 5 and 12) would render the model unnecessary. Furthermore, recognising the relationship *between* the numbers involved in a division situation supports understanding. For example, an understanding that 5 is one third of 15 and that 5 ÷ 15 can be considered 5/15 which is 1/3 would seem less complicated than modelling this calculation in a partitive way.

Hence Fischbein *et al*.'s (1985) work suggests that, as learners mature and experience a wider range of numbers and situations, there can be a conflict between taught algorithms and early intuitive models, which, as they conclude, not only endure but can limit the arithmetical problem solving in multiplication and division. However, as the questions were administered as tests and not interviews, it could be argued that the effects of intuitive models were assumed rather than observed in practice. It could also be argued that such work assumed a high status on use of arithmetical algorithms. The role of arithmetical algorithms in UK mathematics curricula has varied since Fischbein *et al.*'s (1985) study. For example, with the introduction of the National Numeracy Strategy from 1997 onwards, pupils were encouraged to use a variety of approaches to solve problems rather than being expected to routinely use traditional algorithms. Indeed, the National Numeracy Strategy aimed to encourage the connection between calculation approaches and intuitive models; an example of this would be the introduction of a written method of division commonly known as 'chunking', which was introduced following studies comparing algorithms for division in England and the Netherlands (e.g., Anghileri, 2001). This written method builds on the quotitive model of division in that in a calculation such as 325 ÷ 17 could be considered as 'how many groups of 17 in 325'? which can then be calculated by working with multiples of 17, for example through repeatedly subtracting multiples of 17. It should be noted, however, that the expectation of the use of 'formal' algorithms for 'long' multiplication and division returned to the curriculum in England in 2013 (Department for Education, 2021) and, although explicit multiplication and division algorithms are not specified in the curriculum in Wales, it is expected that learners are able to use be able to calculate 'confidently, efficiently and accurately' with all four operations (WG, 2020).

Although Fischbein *et al*.'s (1985, p.15) conclusion was that the intuitive models 'come to conflict with the formal concepts of multiplication and division', it is not entirely clear what is meant by the formal concepts in this context, as this was not defined. Indeed, the word 'formal' can take many meanings in mathematics education, as noted by Tall (2013). One interpretation might be that, in Fischbein *et al*. (1985), the formal concepts of multiplication and division were considered akin to scientific concepts as discussed by Vygotsky (e.g., in Hedegaard, 2007) or theoretical concepts discussed by Davydov (1990). Another interpretation could be that the formal concepts of multiplication and division were considered the interpretation of a situation without the need for concrete or visual images for modelling; this, indeed, would align with what Tall (2013, p.150) describes as 'Piagetian formalism', in which thought 'no longer needs to involve physical referents'. Nevertheless, whatever the interpretation of the word formal, Fischbein *et al.*'s (1985) study, demonstrated that an understanding of the relations between numbers, including fractions and decimal fractions, would be necessary and thus it can be seen why Vergnaud (1994) suggested a multiplicative conceptual field.

Fischbein *et al*. 's (1985) seminal work raised important and highly relevant considerations for those teaching arithmetic, particularly for those involved in the introduction and modelling of arithmetic concepts and for those working with pupils expected to apply their arithmetical understanding to the solving of problems. Following their work, several studies considered children's use of models or solution strategies in multiplication and/or division situations. For example, Anghileri (1989) interviewed children (152) between the ages of 4 and 12 to research their understanding of multiplication. She analysed their strategies for solving tasks designed to reflect the following aspects within multiplication:

1. Equal grouping: making a pattern stick using coloured cubes, such that there were 5 colours with 3 of each colour.
2. Rate: each time a cotton reel was placed on a table, the interviewer took 3 counters and placed them in her hand.
3. Array: coins were stuck on a card making a 6 x 3 array, the child was shown the array and it was then hidden.
4. Number line: the interviewer showed how a model man could hop on a number line of 'stepping stones' in twos and threes and the child was then asked where the man would land after five jumps if he could jump in fours.
5. Scale factor: a lorry with four small square boxes was shown to the children and another lorry was shown which was three times the size. Children were asked how many squares would be on the big lorry.

6. Cartesian product: children were given cardboard cut outs of shorts (3 different colours) and shirts (4 different colours) and were asked to work out the combinations.

Anghileri (1989) categorised the way the children approached the calculations in the tasks in the following ways: unitary counting (counting in ones), rhythmic counting (counting the complete number sequence but using rhythm or acknowledgement of every group), pattern counting (counting in steps of a number, e.g., 3, 6, 9) and use of multiplication facts. She found that that the majority (78%) of children (41) successful in all tasks used at least three different strategies and that 81% of test items solved successfully involved either a counting strategy or direct modelling using apparatus. It is not clear from the results whether those using direct modelling were considered (or observed) to be counting in a unitary way with the apparatus, although it seems this might be the case. It is also, perhaps, unsurprising that children used a range of strategies in the scenarios within the tasks, as these arguably invoked a range of strategies (e.g., the rate task could invoke rhythmic counting in threes and the number line task might invoke pattern counting). Indeed, the results do not include an analysis of actual strategy in relation to the task, only whether counting, direct modelling or multiplication facts were used. There also appears to be little analysis in relation to the age of children and their strategies, which is surprising given the age range considered. However, a conclusion was that to understand multiplication, children need to recognise that a multiplication calculation can represent a range of situations and that research needs to be undertaken to consider how and when children need to be encouraged to use facts rather than less efficient counting strategies.

At a similar time to Anghileri (1989), Kouba (1989) analysed the strategies used by 128 children in the US (grades 1, 2 and 3, ages 6-9) in relation to the semantic structure of multiplication and division problems. Synthesising the work of previous research (Vergnaud, 1983; Schwartz, 1976; Usiskin and Bell, 1983, in Kouba 1989), she suggested that there are two semantic factors in multiplication and division problems. These factors, Kouba (1989) suggested are firstly the quantities and the nature of the relationship between them and secondly the quantity which is the unknown (i.e., whether a direct multiplication, a partitive or quotitive relationship). Essentially it is the relationship between the quantities that are important factors rather than quantities themselves. Kouba suggested there are three types of multiplicative relationships: scalar problems (e.g., three times as many of something), cross-product problems (e.g., combinations of shirts and shorts, note these were termed 'Cartesian product' by Anghileri, 1989) and equivalent set problems (e.g., four sweets in each bag). In contrast to Anghileri (1989), who investigated each problem type with all children, Kouba (1989) chose only to use problems involving equivalent sets, stating that scalar and cross-product (Cartesian product) problems were too difficult. It is noteworthy that Anghileri's (1989) results supported the assertion that such problems would be difficult.

Kouba (1989), although making no reference to Piaget, drew on Piaget's (1952) notion of one-to-many correspondence in her analysis of multiplicative problems. She asserted that the clarity of the one-to-many correspondence would be a factor in the complexity. Take the following examples (adapted from Kouba's original, for purposes of clarity and relevance):

1. 15 marbles are placed in 3 bags so that there is an equal number in each. How many marbles in each bag?
2. In a game, stars earned can be converted to medals. Lewis earns 15 stars and swaps them for 3 medals. How many stars can be converted into one medal?

Kouba (1989) argued that, although the one-to-many correspondence is the same in both problems (1 thing representing 5 things), the correspondence is clearer in the first example than in the second. Kouba's suggestion was that the notion of a bag being a container meant children could relate to the idea of the correspondence more easily. It is interesting to note the second example given could be classified by Anghileri (1989) as a rate problem

(1 medal per 5 stars). Kouba's (1989) point about the clarity of the relationships is an important one as it demonstrates that the social and cultural context of a problem needs to be considered (and it is noteworthy that the examples used above were changed for the purposes of this discussion to reflect the cultural context in which they are being discussed). Hence the semantic structure alone may not be sufficient explanation for difficulties learners may experience.

Kouba (1989) conducted one-to-one interviews with the children, and the children had access to physical apparatus. Questions (Figure 5) are shown below, and number triples were changed randomly (all whole numbers below 30).

Table 1
Multiplication and Division Word Problems

Multiplication—Grouping
You are having soup for lunch. There are ____ bowls. If you put ____ crackers in each bowl, how many crackers do you need altogether?

Multiplication—Matching
Pretend you are a squirrel. There are ____ trees. If you find ____ nuts under each tree, how many nuts do you find altogether?

Measurement Division—Grouping
You are making hot chocolate. You have ____ marshmallows to use up. If you put ____ marshmallows in each cup, how many cups do you need?

Measurement Division—Matching
You are making lunch. You have ____ carrots to use up. If you put ____ carrots with each apple, how many apples do you need?

Partitive Division—Grouping
You are having a party. You have ____ cookies and ____ plates. You put all the cookies on the plates so that there is the same number of cookies on each plate. How many cookies are on one plate?

Partitive Division—Matching
You are shopping. You paid ____ pennies altogether for ____ toys. You paid the same number of pennies for each toy. How many pennies did you pay for one toy?

FIGURE 5: MULTIPLICATION AND DIVISION PROBLEMS, KOUBA (1989, P. 150)

Kouba (1989) analysed the children's responses using some similar features to Anghileri (1989). The categories used are summarised below:

1. Direct representation of problem using physical objects resulting in one-to-one counting. Kouba (1989) noted that the way the one-to-one counting occurred could reflect differences of sophistication in the counting procedure (e.g., tallying a group with emphasis on a word, which Anghileri (1989) called rhythmic counting or recognising one group and then counting on from that in a one-to-one manner).

2. Double counting. Kouba (1989) used the term double counting to acknowledge that in a division situation children may be co-ordinating two counts simultaneously. For example, 21 ÷3 as a measurement (quotitive) situation as in the third question in Figure 5 could involve the forming of groups of 3 and counting up to 21 whilst also counting how many groups of 3 were formed. This is different to starting with 21 and grouping into 3s. In a partitive situation such as the fifth question, double counting could occur by sharing out one-by-one whilst also co-ordinating the total to be shared out such that it is not counted out to start with.

3. Transitional counting. This involved what Anghileri (1989) called pattern counting and is essentially counting in steps or multiples (e.g., counting in twos or threes) although Kouba (1989) used the term transitional to reflect that there may have been an indication of this rather than its full use.

4. Additive or subtractive. Kouba (1989) used this category when there was an indication that addition or subtraction was being used. This reflects the idea of repeated addition and repeated subtraction discussed previously, although Kouba (1989) included examples where repeated addition and/or subtraction may have been used partially.

5. Use of known facts. This term was used to encompass use of the fact or use of derived facts that did not directly relate to the repeated addition or subtraction.

Kouba's (1989) categories and resulting analysis provide insight into the sort of strategies that might be used within multiplication and division problems. Of note, is the way partitive problems were approached. Kouba (1989) discussed that there were two main approaches to question five. Giving the example of 24 cookies to be shared equally onto 3 plates, she discussed that one approach was the sharing out (one-to-one) on each plate. The other approach was to use trail-and-error to estimate then adjust so that the groups were equal. As Kouba (1989) noted, this highlights an important distinction between quotitive and partitive division structures because in a partitive situation (modelled using apparatus) the number within a share (group) is fixed, yet unknown. In a quotitive situation the number within a group can be the starting point and so this can allow more flexibility in approach. Furthermore, as Correa, Bryant and Nunes (1998) later point out 'sharing' as an action is different to division as an operation because in sharing out equally the children have no concern other than the equality of the share.

Kouba's (1989) main conclusions were that Fischbein *et al.*'s (1985) intuitive models were not sufficient to explain the strategies used by learners to solve multiplication and division problems, particularly in the case of division. For quotitive division, Kouba (1989) concluded that repeated taking away and repeated building up were the predominant models and for partitive division sharing by dealing, sharing by repeated taking away (guessing the number to be taken) and sharing by repeated building up (guessing the number to build). Furthermore, Kouba (1989) noted that repeated subtraction and repeated addition were used as strategies in *both* partitive and quotitive situations so the distinction between the structure of the question as partitive and quotitive may not be as important for the children as it might be expected. Kouba's (1989) work showed that the structure of a question and the strategies used to calculate it are not necessarily related and her thorough analysis of the strategies used provided insight into the relationship between direct modelling of a problem and its calculation.

Through applying and extending Anghileri's (1989) and Kouba's (1989) solution categories and extending this research approach, Mulligan and Mitchelmore (1992) investigated children's solution strategies for both multiplication and division. They also made a distinction between the semantic structure of a problem and the solution strategy, which they termed intuitive model, being used within their analysis. They researched 70 girls'

(Australia, ages 6-7) approaches to six multiplication and six division questions chosen to represent specific structures as shown in Figure 6 and used two different selections of numbers to consider whether the numbers used impacted the solution strategy in any way. Furthermore, through conducting interviews over three periods they were able to analyse the effect of maturation.

Table 2
Multiplication and Division Word Problems

Multiplication	Division
Equivalent groups	*Partition*
1. There are 2 tables in the classroom and 4 children are seated at each table. How many children are there altogether? (4, 7)	7. There are 8 children and 2 tables in the classroom. How many children are seated at each table? (28, 4)
2. Peter had 2 drinks at lunchtime every day for 3 days. How many drinks did he have altogether? (3, 7)	8. Six drinks were shared equally among 3 children. How many drinks did they each have? (14, 7)
Rate	*Rate*
3. If you need 5 cents to buy 1 sticker, how much money do you need to buy 2 stickers? (5, 7)	9. Peter bought 4 lollipops with 20 cents. If each lollipop cost the same, how much did 1 lolly cost? (8, 40)
Comparison	*Comparison*
4. John has 3 books, and Sue has 4 times as many. How many books does Sue have? (6, 5)	10. Simone has 9 books. This is 3 times as many as Lisa. How many books does Lisa have? (40, 8)
Array	*Quotition*
5. There are 4 lines of children with 3 children in each line. How many children are there altogether? (3, 8)	11. There are 16 children, and 2 children are seated at each table. How many tables are there? (36, 4)
Cartesian product	
6. You can buy chicken chips or plain chips in small, medium, or large packets. How many different choices can you make? (8, 2)	12. Twelve toys are shared equally among the children. If they each had 3 toys, how many children were there? (24, 6)

Note. The text gives the small-number problems. The large-number problems were worded identically except for the substitution of the numbers given in parentheses.

FIGURE 6: MULTIPLICATION AND DIVISION WORD PROBLEMS, MULLIGAN AND MITCHELMORE (1992 P.314)

Children's responses to multiplication were categorised as either: direct 'unitary' counting, repeated addition (which included rhythmic and pattern/skip counting and additive calculation), and the use of multiplicative operations. Responses to the division scenarios were categorised as either: direct 'unitary' counting, repeated subtraction (including rhythmic counting backwards and pattern (skip) counting backwards), repeated addition and multiplicative operations.

Mulligan and Mitchelmore (1992) suggest their results, like Kouba's (1989) findings, show that the semantic structure of a problem may not relate directly to the solution strategy used. In particular, their results suggest that pupils tended to use three strategies for multiplication: direct counting, repeated addition and multiplicative operations, and four for division: direct counting, repeated addition, repeated subtraction and multiplicative operations. They argue, like Kouba (1989), that their findings contrast with Fischbein *et al.*'s (1985) findings, because the structure of the question did not necessarily relate to how it was solved, for example partitive and quotitive structures within division questions were not necessary solved using those approaches. However, the numbers involved in both Kouba's (1989) and Mulligan and Mitchelmore's (1992) studies were limited to whole numbers and were relatively small. Furthermore, in both studies the children (notably much younger than those within Fischbein *et al*.'s study) were able to draw or use cubes to model. For example, question 7 in Mulligan and Mitchelmore's (1992) study (Figure 6) could invoke a partitive approach (8 split between two groups) even when repeated addition is used. Although Mulligan and Mitchelmore (1992) state, again like Kouba (1989), that non quotitive problems were harder for pupils to repeatedly subtract or add (due to having to guess a number to group), the use of cubes and paper could have allowed the problem to be *physically modelled/pictorially represented* using a partitive model and *calculated* using repeated addition and subtraction. This, again, highlights a need for consideration of the distinction between the way a problem is physically modelled or pictorially represented and the way it may subsequently be calculated. Modelling and calculating are not the same process.

Mulligan and Mitchelmore's (1992) results suggest progression in the use of the strategies over time, with greater use of the multiplicative operation as children mature. This is perhaps unsurprising as children were learning multiplication in grade 3. Their results also suggested that some semantic structures (in particular comparison/scale and Cartesian product) caused particular difficulties, similar to Anghileri's (1989) findings and supporting Kouba's (1989) claim.

As noted previously, a multiplicative relationship involves terms in a fixed ratio. Correa, Bryant and Nunes (1998) focused their research on children's understanding of the relationship between quantities within multiplicative (division) situations. They set up tasks, with 20 children in each case, which did not require any computation, but that were designed to consider children's understanding of the relationship between quantities. The tasks are summarised below:

Task 1 Partitive division: Four pink rabbits and four black rabbits, each with a box on their backs were used. Plastic blocks (to represent sweets): 24 red blocks (for the pink rabbits) and 24 blue blocks (for the blue rabbits) were available overall. The children were shown how the blocks could be shared between the rabbits (an initial control task was set up to identify those that understood sharing). Children were then, in situations involving the same number of blocks to be shared but varying numbers of pink and blue rabbits, asked to predict whether the pink or blue rabbits would get more blocks.

Task 2 Quotitive division: Six pink and blue rabbits with boxes on their backs were used. 24 red and 24 blue blocks were used to represent sweets. Pictures of the blocks on plates (separate red and blue) to represent the quota (amount given to the rabbits) such that either 2, 3 or 4 blocks was given. Children were presented with pictures of the cubes on a plate and a total number of blocks. They were told that the investigator wanted to invite rabbits to a picnic but did not know how many to invite. In some cases, the number of blocks on each plate was the same for both sets of rabbits and in other cases the quota was different. Children were asked, in each case, to say whether more pink or blue rabbits could go to the picnic.

These tasks were designed to consider the situations of 'shared between' (partitive division) and 'shared into' (quotitive division).

Kouba's (1989) point about the clarity of the one-to-many relationship could be applied to the above tasks particularly as, arguably, the tasks seem rather convoluted. Sharing sweets to rabbits and the different colours involved could be considered confusing. Nevertheless, Correa, Bryant and Nunes (1998) used their results to conclude that children can make inferences about the inverse relationship between divisor and quotient (the bigger the divisor the smaller the quotient) and that this is easier in partitive than in quotitive tasks. They also reinforced that the notion of sharing, whilst an important starting point for division, and the understanding of division, in both partitive and quotitive situations, are not the same.

Sharing as a process has not been widely considered in discussion up to this point. Correa Bryant and Nunes (1998) consider sharing as an 'action schema' (action schemas are repeatable actions often with concrete materials, suggested by Piaget). As indicated by Kouba (1989) and Mulligan and Michelmore (1992) sharing can be actioned in a one-to-one distributive (dealing out) manner or through 'guessing' and then adjusting. Without being able to physically distribute objects, it can be argued that Correa, Bryant and Nunes' (1998) experiments relied on the anticipation of a sharing process rather it being used as an action scheme. Furthermore, the 'guessing' and adjusting process of sharing shows understanding of the one-to-many correspondence indicated as so important by Piaget whereas in the one-to-one sharing (dealing) process the one-to-many correspondence is a result of the actual process. Indeed Bryant (1997) suggests that sharing out in a one-to-one manner is only indication of one-to-one correspondence. Hence, sharing as a process can support understanding of division but, as Correa, Bryant and Nunes (1998) correctly point out, cannot be considered sufficient to indicate understanding of division, or the multiplicative relationship, as this requires understanding of the relationship between quantities involved. To apply Vygotsky's distinctions of spontaneous and scientific concepts, the spontaneous concept of sharing is not a sufficient indicator of understanding of the scientific concept of division.

To explore the relationship between reasoning in division and the procedure of sharing, Kornilaki and Nunes (2005), building on the work of Correa *et al*. (1998), considered the ability of children (5-7 years) to generalise, in non-computational tasks, the results of sharing in both partitive and quotitive situations. They compared sharing of discrete and continuous quantities. A discrete quantity is a quantity in which the value may be limited, usually to whole numbers. In contrast, continuous quantities can have an unlimited range of values because they can be split in different ways. Experiments were set up involving cats (brown and white cats) and food (discrete quantity being fish and continuous quantity being fish cakes) as follows:

Partitive situations: In the case of a discrete quantities, children were shown two groups of cats (brown and white cats – the number in the groups differing on occasions) and a number of fish. They were asked to predict whether the cats would get the same or different shares when the total amount of fish to be shared amongst cats would be the same each time. In the case of continuous quantities, children were asked about the share of fishcakes.

Quotitive situations: Two cats (one brown and one white) were shown to the children. For discrete quantities, pictures of plates with either 2, 3, 4 or 6 fish on each were shown and a total number of fish was given. For continuous quantities fractional portions of fish cake (either one half, one third, one quarter or one sixth) were shown and a total number of fishcakes given. Children were asked about the number of recipients.

Kornaliki and Nunes' (2005) tasks are arguably quite convoluted. As with Correa, Bryant and Nunes (1998), the quotitive situation, in particular, does not seem intuitive or realistic. Furthermore, the notion of fish being a discrete quantity can depend on whether a fraction of fish can be accepted. Nevertheless, Kornaliki and Nunes (2005) suggest their results replicate Correa, Bryant and Nunes' (1998) findings and also show that children appear to be equally competent understanding the relationship between terms when sharing discrete quantities and continuous quantities. It is particularly noteworthy that they comment on children's competence in recognising the inverse relationship between divisor and quotient in continuous quantities where fractions of fishcakes were involved. They point out that it is well documented that older children have difficulty in ordering fractions and that more attention needs to be paid to the relationship between sharing and fractions. This is further argument for the consideration of the multiplicative conceptual field, and for the involvement of continuous quantities in teaching the multiplicative relationship.

It is noteworthy that the research in relation to division considered so far (Kouba, 1989; Mulligan and Mitchelmore, 1992; Correa, Bryant and Nunes 1998; Kornilaki and Nunes, 2005) has not explicitly considered a relationship between the children's understanding of multiplication and their understanding of division. Indeed, the notion of multiplicative reasoning was not raised in any of the above the research, although clearly it relates to the field of research into multiplicative reasoning. Furthermore, the work discussed so far, although adding much insight into the semantic structure and children's modelling and calculation strategies used for multiplication and division, does not offer significant pedagogic advice. For example, Steffe (1994, p.11) advocates the importance of 'mathematical interactions' (social interactions in mathematics) and the importance of recognising intuitive models children may use in multiplicative reasoning, but there is little advice on how pedagogy can support an understanding of concepts within the field of multiplicative reasoning. The following section includes research that explicitly considers the multiplicative relationship and implications for the related pedagogy and progress.

2.7 THE MULTIPLICATIVE RELATIONSHIP AND SUPPORTING PEDAGOGY

Clark and Kamii (1996) report research involving 336 participants from grades 1-5 in the US (5-10 years old). They modified a task designed by Piaget and colleagues by creating plywood fish of length 5cm, 10cm and 15cm. The relationship between the fish lengths was highlighted verbally and through placing the fish on top of one another. As part of this explanation the children were also told that the 10cm fish ate twice the amount of food as the 5cm fish and the 15cm fish ate three times the amount of food as the 5cm fish. Children were then asked to identify how much food each fish would get in particular situations, e.g., when fish A (5cm) was given one pellet, or when fish B (10cm) received 4 pellets. Children were allowed to demonstrate their thinking through actions (e.g., by making groups of pellets) and were given a counter suggestion to reinforce the multiplicative relationship if an incorrect response was given. Children's responses were categorised as additive if they (incorrectly) applied an additive relationship (e.g., fish B +2 more and fish C +3 more) and multiplicative (with categorised levels of success within this) if the x2 and x3 relationship was recognised. Clark and Kamii (1996) report that multiplicative reasoning starts early (e.g., 45% of grade 2 demonstrated some level of multiplicative reasoning) but develops slowly, with only 49% of grade 5 pupils demonstrating successful multiplicative reasoning. Their results seem to support Piaget's claim (discussed in Nunes and Bryant 2009c) that understanding of multiplicative relationships starts early. As part of their discussion of results, Clark and Kamii (1996) suggest that memorising multiplication facts, without understanding the multiplicative relationships involved, is a factor in children's inability to demonstrate multiplicative reasoning. They suggest that when multiplication problems are given to children, they should be allowed to work them out in their own ways and should not be forced to learn facts and procedures. This seems rather a weak conclusion to the otherwise informative research because it does not support planning for the transition from additive to multiplicative reasoning with specific pedagogical suggestion.

In contrast, Hurst and Hurrell (2016) report the results of research into multiplicative thinking in Australia and offer some quite specific pedagogical suggestions to support multiplicative reasoning. They report the results of semi-structured interviews with 38 pupils Australian in Years 5 and 6 (10-12 years) in two schools, and a questionnaire administered to 180 pupils Years 4, 6 and 6 in another school (9-12 years). Questions involved asking pupils to explain a multiplicative relationship in words and using counters, understanding of factors and multiples, and understanding of the inverse relationship between multiplication and division. Hurst and Hurrell (2016) liken their results of the semi-structured interviews to those of Clark and Kamii (1996) in terms of proportions of pupils and levels of multiplicative thinking demonstrated. They also suggest the questionnaire implied a link between the responses and the learners' pedagogic experiences, potentially indicating a relationship between the way the children had been taught multiplicative relationships and their understanding of it.

Hurst and Hurrell (2016) argue that a powerful visual model in supporting multiplicative thinking is the array. For example, they discuss how an array can be made of small tiles (counters could be used as an alternative) to show multiplicative relationships for a particular number. Examples of arrays are shown in Figure 7.

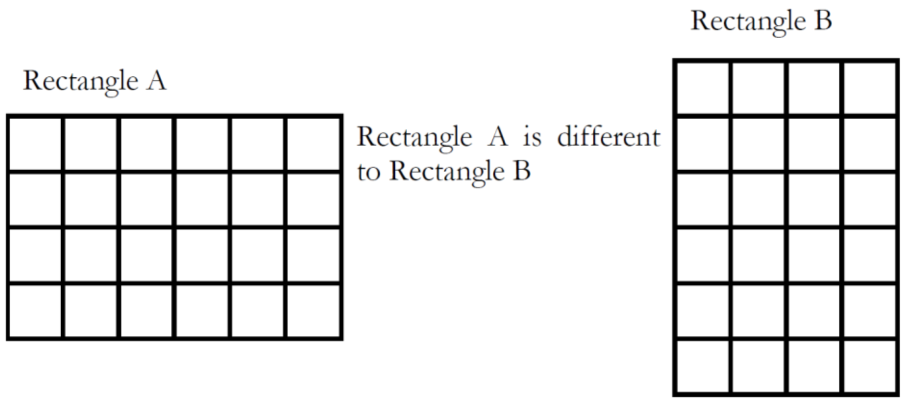


FIGURE 7: ARRAYS TO SHOW 24 AS 6 X 4 AND 4 X 6, HURST AND HURRELL (2016, P.8)

Arrays, as discussed by Hurst and Hurrell (2016), allow for reinforcing the commutative relationship within multiplication (i.e., that 6 x 4 and 4 x 6 are equivalent). They also allow for discussion of factors (different arrays can be found for 24: 1 x 24, 2 x 12, 3 x 8, 4 x 6, and, by finding these, the factors can be identified). Arrays also allow the relationship between multiplication and division to be reinforced. The notion of an array is not new, for example, it was referred to by Anghileri (1989) when discussing the structure of multiplication, and it is difficult to trace its history as a visual, or concrete, model for multiplication and division.

However, Hurst and Hurrell (2016) explicitly link the array to the development of multiplicative reasoning by emphasising its use as a way of reinforcing properties (the commutative property) and relationships (multiplication, division, fractions, and factors). Furthermore, they suggest that learners who are exposed to the array as a model for the multiplicative relationship are likely to use it to support their multiplicative reasoning. The array as a model for supporting learning of the multiplicative relationship has been explicitly mentioned in a description of learning in Mathematics and Numeracy within the Curriculum for Wales (see Table 1, p.8).

Downton (2008) makes use of the notion of the multiplicative relationship and reports the results of research into children's understanding of division. She explored the relationship between the multiplicative structure 'equivalent groups' and 'times as many' within division contexts, considering both partitive and quotitive situations. Examples of questions used are shown in Figure 8.

Table 1

Division Word Problems Used in the Study

Semantic structure	Aspect of division	Level of difficulty	Problem
Equivalent groups	Partition	Easy	I have 12 cherries to share equally onto 3 plates. How many cherries will I put on each plate?
		Medium	I have 18 cherries to share equally onto 3 plates. How many cherries will I put on each plate?
		Challenge	I have 48 cherries to share equally onto 3 plates. How many cherries will I put on each plate?
	Quotition	Easy	There are 12 children in the class. Three children sit at each table. How many tables are there?
		Medium	There are 24 children in the class. Four children sit at each table. How many tables are there?
		Challenge	72 children compete in a sports carnival. Four children are in each event. How many events are there?
Times as many	Partition	Easy	Sam read 20 books during the read-athon, which was 4 times as many as Jack. How many books did Jack read?
		Medium	Sam read 36 books during the read-athon, which was 4 times as many as Jack. How many books did Jack read?
		Challenge	Sam read 72 books during the read-athon, which was 4 times as many as Jack. How many books did Jack read?
	Quotition	Easy	The Phoenix scored 18 goals in a netball match. The Kestrels scored 6 goals. How many times as many goals did the Phoenix score?
		Medium	The Phoenix scored 28 goals in a netball match. The Kestrels scored 7 goals. How many times as many goals did the Phoenix score?
		Challenge	The Phoenix scored 48 goals in a netball match. The Kestrels scored 16 goals. How many times many goals did the Phoenix score?

FIGURE 8: DIVISION WORD PROBLEMS REFLECTING MULTIPLICATIVE STRUCTURE, DOWNTON (2008, P.172)

Downton (2008) compares the responses of children involved in an intervention designed to develop multiplicative thinking to a control group. The reported intervention involved a total of 24 days teaching on multiplication and division. It is not clear what the intervention teaching activity involved, although it is suggested that it involved a problem-solving environment. It is also unclear whether the control group were receiving, as part of their standard curriculum, teaching on multiplication and division.

Applying and extending categories used by Kouba (1989), Anghileri (1989) and Mulligan and Mitchelmore (1992), Downton (2008) categorised the solution strategies as shown in Figure 9.

Table 2

Solution Strategies for Whole Number Division Problems

Strategy	Definition
Unclear	Strategy reflects lack of understanding of task, or is unrelated to task.
Direct modelling	Uses sharing or one to many grouping with materials, fingers or drawings and calculates total by skip or additive counting.
Partial modelling	Partially models situation with concrete materials, or drawings using sharing or one to many grouping. Consistently uses skip or double counting to find the total.
Building up	Skip counts using the divisor up to the dividend. May use fingers to keep track of number counted. Records a number sentence in symbolic form.
Repeated Subtraction	Repeatedly taking away a specific number from the dividend until reaches zero, or skip counts back in multiples of the divisor from the dividend. Partial modelling in some instances. Records a number sentence in symbolic form.
Doubling and Halving	Derives solution using doubling or halving and estimation, attending to the divisor and dividend. Recognises multiplication and division as inverse operations. Records a number sentence in symbolic form.
Multiplicative Calculation	Automatically recalls known multiplication or division facts, or derives easily known multiplication and division facts, recognises multiplication and division as inverse operations. Records a number sentence in symbolic form.
Wholistic Thinking	Treats the numbers as wholes—partitions numbers using distributive property, chunking, and or use of estimation.

FIGURE 9: SOLUTION STRATEGIES USED FOR DIVISION, DOWNTON (2008, P.173)

It is noteworthy that none of the categories refer to one-to-one counting as a strategy and it is not clear from the discussion of results whether this was observed in practice. Downton's (2008) findings certainly suggest that the intervention group were able to use multiplicative calculation or greater levels of multiplicative thinking such as doubling and halving relationships more frequently than the control group. Downton (2008, p.177) advises 'Placing emphasis on the relationship between multiplication and division and the language associated with both operations before any use of symbols or formal recording needs to be a priority'. This supports the suggestion made by Nunes *et al*. (2012) and Coles (2017) for a need to focus on relationships rather than procedures or calculations. Extending her work further, Downton and Sullivan (2017) explore multiplicative reasoning further by analysing the response of an intervention project involving 13 Australian Grade 3 (aged 8 and 9) pupils to problems involving multiplicative relationships, involving challenges considered 'outside the factor structure stipulated by the curriculum' (p. 311). They provided a choice of challenges for the pupils, (challenge or extra challenge) and posed questions in line with Anghileri's (1989) structures of multiplication including those previously mentioned as difficult (Cartesian product and scale/times as many). Of note, is that they not only used structures outside of those the pupils may be familiar with, but the extra challenge questions involved sets of numbers in a multiplicative relationship, i.e., number triples that they would not typically be familiar with. They conclude that posing more complex problems, involving more complex number triples to learners can support their development of multiplicative thinking. They also noted that when less complex number triples were used, on occasions, the same pupils used less sophisticated strategies than when they used more complex triples. This supports the point made earlier, that some questions may invoke the use of particular strategies. It seems using more complex number triples might 'force' multiplicative reasoning.

Downton and Sullivan (2017, pp.323-324) suggest four key factors that support the shift from additive to multiplicative thinking: moving beyond physical representation to mental representation, understanding the relationship between the numbers involved in multiplicative situations, having experiences of a variety of semantic structures with multiplication and division and finally engagement in number triples beyond the fact range. It should be noted that results discussed by Downton (2008) and Downton and Sullivan (2008) both involve analysis of intervention studies in Australia specifically focusing on multiplicative reasoning. Multiplicative reasoning became an important consideration in Australia, following a middle years numeracy project discussed by Siemon, Brood and Virgona (2008) which generated data (7000 pupils), focusing on the middle years of schooling (Years 5-9) in Victoria, Australia. Siemon, Brood and Virgona (2008, p.1) found '22.2% of students overall (31% at Year 5, 18% at Year6, 25% at Year 7, 19% at Year 8, and 18% at Year 9) were relying on simple 'make-all, count-all' models, skip counting by twos or doubling to solve problems that could be solved more efficiently using multiplication. A conclusion of their research was 'the transition from additive to multiplicative thinking is one of the major barriers to learning mathematics in the middle years' (Siemon, Brood and Virgona, 2008, p.1).

Using a longitudinal study over five years, Nunes *at al*. (2012) report data with 4, 259 participants, and involving analysis of Key Stage 2 (8-11 years) and Key Stage 3 (11-14 years) achievement. Overall achievement was measured by using national test data. Quantitative reasoning (i.e., reasoning about relationships between quantities) was measured through administration of mathematical reasoning tasks (which involved additive and multiplicative reasoning) and arithmetical competence was measured through administration of an arithmetic test. Aspects such as working memory and general achievement (indicated by English language and Science national test data) were also considered. The key finding of this study was that quantitative reasoning was a particularly good predictor of progress in mathematics, more so than arithmetical competence. The data also supports the notion that reasoning about relationships between quantities and recognising additive and multiplicative relations is a key part of supporting pupils in making progress in mathematics.

Siegler *et al.* (2012) analysed longitudinal test data from the UK (data from a cohort of 3, 677 children tested at age 10 in 1980 and then at age 16 in 1986) and the US (data from a cohort of 599 children tested at age 10-12 in 1997 and then at age 15 to 17 in 2002), specifically focusing on pupils' understanding of fractions and proportional relations, hypothesising there would be a relationship between understanding of fractions and later algebra performance. They found that there was a strong relationship between understanding of fractions and algebra, but also to general mathematical performance and, acknowledged as more of a surprise finding, there was also a strong relationship between early understanding of whole number division and later general mathematical performance. It is noteworthy that different tests were used in the different countries, and that they were administered in different decades. Algebra is also a broad term, and it is not clear what questions were classed as involving algebra. Though Siegler *et al*. (2012) acknowledge the differences in samples, tests and times for data collection, they argue that the predictive relationships in both sets of data were very similar. Siegler *et al*. (2012) acknowledge that fractions and division can be difficult to master (for learners and teachers), and they argue that the predictive relationship was seen across all achievement groups and therefore suggest that 'the unique predictive value of early fractions and division knowledge seems to be due to many students not mastering fractions and division and to those operations being essential for more advanced mathematics' (Siegler *et al.*, 2012, p.696), suggesting a causal relationship. Such work, although not specifically mentioning multiplicative reasoning, contributes to the body of research evidence linking multiplicative reasoning to progress.

Hence, from the literature, it is becoming increasingly clear that multiplicative reasoning is a key aspect in mathematics progress and the ability to recognise and apply the multiplicative relationship, incorporating a range of numbers, including fractions and decimal fractions, appears vital. Although it is clear that certain pedagogical strategies can support multiplicative reasoning (e.g., the use of a range of semantic structures, the use of visual supports such as arrays, encouraging a move from physical modelling to use of the multiplicative relationship between quantities to support thinking), there seems to be a lack of literature supporting specific pedagogical approaches to teaching multiplicative reasoning, particularly research which might support teachers in moving children from physical modelling to symbolic representation in a way that supports understanding of the multiplicative relationship.

2.8 A POSSIBLE PEDAGOGICAL APPROACH

Davydov (1992) offers a very different interpretation to the common view of multiplication as repeated addition and suggests a model for the introduction of multiplication and division concepts that reinforces the relationship between them. In an analysis of textbooks (from Russia in 1965 but bearing striking similarity to the way multiplication is introduced in some textbooks in the UK), Davydov (1992) argues that the introduction of multiplication as repeated addition will consistently imply that the first action is the taking and counting out of objects, one by one. Though Clark and Kamii (1996) conclude that for multiplicative reasoning to be present, one-to-many correspondence is necessary (in their 3 x 4 example shown in Figure 4, p.34, the 3 must be seen as a unit), it seems Davydov's (1992) conclusion is that this interpretation would be insufficient, because there would be an implication that, in a concrete example, the unit of 3 would need to be counted out and thus the first action is still counting, one by one. In fact, if concrete materials are not used, the only way a solution could be obtained without counting in ones would be to have the product of 3 x 4 available, or to rote count in threes. Davydov's (1992) conclusion is that repeated addition is an ineffective model for introducing the idea of multiplication because, in a repeated addition model, the action of counting one by one seems necessary if number facts are not already known. With the array, a model discussed in Section 2.7 (see Figure 7, p.52), counting one by one is possible.

In Section 2.3 of this chapter, it was noted that Davydov (1990) believed that all concepts in mathematics instruction should be treated as scientific and they should be introduced in a way which reflects the way in which they evolved, reflecting 'genetic analysis' (Schmittau, 2003, p.232). For Davydov (1992, who cites Lebesgue, 1960), multiplication should be seen as a 'change in the system of units' (Schmittau, 2003, p. 233). Though Steffe (1994), discussed earlier, emphasises the notion of a composite unit in the multiplicative relationship, for Davydov (1992) the notion of the *change* in units is an important one. Davydov (1992) gives an example, from everyday life, of a large number of identical coins accumulating and needing counting. It would be inefficient and time consuming to count them all, one by one, and so finding out how many coins would be equivalent to a chosen unit of weight (e.g., 1kg) would be more efficient. Thus, whatever the weight of all the coins in kilogrammes, the number of coins could be found by multiplying the known weight of coins per kilogramme by the total weight of the coins. There is a change in unit from the weight of one coin to the weight of 1kg of coins. One might argue that it may be possible to calculate the weight of one coin and divide by the total weight of coins thus avoiding the need for an intermediary unit. Yet in this approach, a new unit would become the weight of a coin rather than the number. The example highlights a situation in which both multiplication and division concepts can be introduced. Indeed, Davydov (1992) gives this example as a way of demonstrating, not only the importance of the notion of a change in units as central to the concept of multiplication, but also to reinforce that the *concept* of multiplication has arisen out of necessity and not as another way of considering addition. Although repeated addition is a commonly used model to introduce multiplication, it does not reflect the scientific concept of multiplication sufficiently to support Davydov's (1990) assertion that the abstract concept should be the starting point of teaching, albeit in a practical way. As Nunes and Bryant (2009c) note, the link between multiplication and repeated addition is a *procedural* link (and similarly the link between division and repeated subtraction), but these need not be the models used to introduce the ideas. Indeed. Davydov's (1992, p.12) analysis of multiplication suggests that the 'premise of multiplication – in this discussion of it – is the refusal of directly counting out one by one all the elements of a calculated set'.

Davydov (1992) sees the following features as characteristic of situations (problems) in which the necessity for multiplication arises:

i. It is impossible or inefficient to determine the quantity of a unit by counting.
ii. A larger scaled intermediary (with a known/identifiable relationship to the smaller unit) is introduced.
iii. The original unit's quantity is measured by comparison with the intermediary unit resulting in a relationship between the original unit, the intermediary unit, and the original unit.

For point iii, Davydov (1992) is expressing a multiplicative relationship. Although he never seems to have used the terms multiplicative reasoning, it is evident that his interpretation of multiplication and his view of how it should be learned and taught reflects a view that multiplication, division, fractions and ratio are not just connected concepts, but they are part of what Vergnaud (1994) terms a 'multiplicative conceptual field'. Indeed, the use of measures as a context, means that ideas such as fractions can also develop through such activity.

As discussed in Section 2.3, in a Davydov and Elkonin curriculum, learners will have much experience of quantification through measuring length, area, mass, volume and capacity. The notion of the unit was also highlighted earlier in this chapter, in discussion of how the concept of number would evolve. This is central, not only to the development of the concept of number and measure but also as a precursor to multiplicative reasoning. Davydov (1990; 1992) is not unique in recognising the importance of the concept of a unit in mathematical development; for example, Lamon (1994, p.92) notes that 'the ability to construct a reference unit or a unit whole, and then reinterpret a situation in terms of that unit, appears critical to the development of increasingly sophisticated mathematical ideas.' Davydov is also not unique in recognising that measure contexts can allow introduction and exploration of relationships. For example, as Coles and Sinclair (2022) note, Gattegno, in the 1960s, recognised the importance of teaching relationships and popularised the use of Cuisenaire rods (sets of rods measuring 1cm to 10cm) in supporting this. However, Davydov's (1990; 1992) important contribution is the use of a range of measures contexts, and continuous quantities, in which the unit can be used to introduce concepts of number, additive and multiplicative reasoning.

Schimttau (2010, p.269) explains that in developing understanding of a unit being used to measure a quantity, a schematic can be used to show the relationship between an object A and the unit used to measure it U, shown in Figure 10.

$$U \xrightarrow{?} A.$$

FIGURE 10: THE SCHEMATIC USED TO REPRESENT A RELATIONSHIP BETWEEN A QUANTITY (A) AND UNIT (U), FROM SCHMITTAU (2010, P.269)

In this form of notation, the arrow represents a relationship between the unit (U) and the quantity (A). A number placed on the arrow, at ?, would represent how many times the unit fits into the quantity A.

In the context of multiplicative reasoning, an example (discussed by Schmittau, 2004) would be children being asked to measure the volume of water in a large jug (J) using a small unit such as a very small cup (C). Recognising this could be inefficient, some children might suggest the use of another container (e.g., a glass G). To establish how many small cups are in the jug, a relationship between the small cup and the glass and between the glass and the jug would need to be identified. Thus, change in units would be the use of a glass. Schmittau (2004) explains that such a relationship could be represented using a schematic shown below. For example, in the case of a relationship where the volume of water in a jug may be measured by a glass which fills the jug 8 times, where the cup fills the glass 4 times would be represented by the schematic in Figure 11.

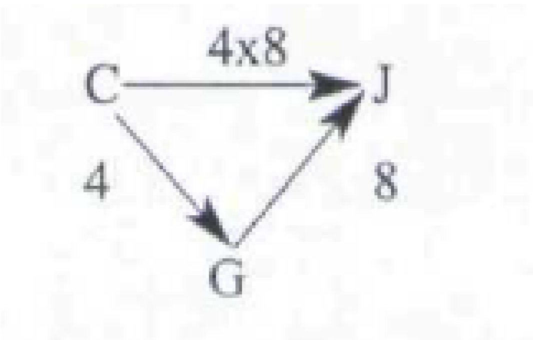


FIGURE 11: A SCHEMATIC TO SHOW A MULTIPLICATIVE RELATIONSHIP, SCHMITTAU (2004, P.29)

Such a schematic, as Schmittau (2004) argues, unites the concepts of multiplication and division, with the ideas being taught together rather than separately. Indeed, Schmittau (2004, p.39) sees this schematic as a ‘psychological tool’ or ‘semiotic mediator’ which ‘orients students toward the theoretical essence rather than the empirical features of problem situations’. In Davydov’s (1992) examples of introduction to the multiplicative relationship, situations are set up to highlight the inefficiency of counting in ones. However, Davydov (1992) does note that in establishing solutions to problems such as the one described above involving the jug, cup and glass, additive reasoning might be used to support finding a solution. For example, recognising the glass represents four cups, and that the glass is needed eight times to fill the jug, the result might be calculated by repeated addition of 8, or counting in 8s. In such discussion, Davydov (1992) seems to acknowledge Nunes and Bryant's (2009c) point that the link between multiplication (or division) and addition (or subtraction) is procedural more than conceptual.

The way in which Davydov argues for the teaching of multiplication and division seems not only to reinforce and emphasise multiplicative reasoning, but also seems to demonstrate Freudenthal's (1991, p.31) 'mathematisation', and the view that mathematics concepts can be socially constructed through meaningful problems. A key difference, however, is Davydov's (1990; 1992) belief that the *theoretical concept* should be introduced from the outset, with concrete examples of it explored; the 'ascent from abstract to concrete' (Davydov, 1990, p.128) as discussed earlier in this chapter.

The examples discussed by Davydov (1990; 1992) were based in a 'measurement world' (Coles, 2017) in which children use the measurement of continuous quantities (length, area, mass, capacity) from an early age and where measurement and quantification of that measure were the predominant way of working (see discussion of number development in Section 2.3).

This study seeks to explore the learning and teaching of multiplicative reasoning through measures tasks, focusing on relationships between quantities, to introduce the scientific concept of the multiplicative relationship, in particular multiplication and division. The study, taking place in Wales, involves learners who are typically familiar with a 'counting world' (Coles, 2017, p.206) and applies a theoretical and pedagogical perspective of social constructivism, to the development, implementation, and analysis of the research, outlined in the next chapter.

CHAPTER 3: THEORETICAL PERSPECTIVE

3.1 INTRODUCTION TO THE THEORETICAL PERSPECTIVE

As Simon (2007) argues, learning theories are not proven but can be seen as perspectives, lenses, or philosophies. The theoretical perspective in this work is informed by the philosophy of social constructivism as defined by Ernest (1991 and 1998). Social constructivism in mathematics is considered by Ernest (1991, p.42) to be a 'new philosophy of mathematics', but he acknowledges that it draws on the work of others. Notably, in Ernest's (1991) initial description of this philosophy, he draws particularly on the work of those within the field of the philosophy of mathematics (e.g., Lakatos, 1978, in Ernest, 1992). As Ernest (1991, p.16) notes, Lakatos challenged the 'absolutist' view of mathematics, arguing its 'fallibility'. Ernest (1991) argues that, though rejection of the view that mathematics can be considered an absolute truth could be seen as negative, a more positive perspective on this is the acceptance that mathematics knowledge can be continually developed and revised.

Ernest's philosophy also acknowledges, reflects and synthesises theories of cognitive development, and significant and seminal cognitive development theories from those, amongst others, such as Piaget (1972, in Ernest, 1997), Vygotsky (1978; 1979; 1986, in Ernest, 1997) and von Glasersfeld (e.g., von Glasersfeld, 2001). Thus, although Ernest's (1991) social constructivism is a philosophy of the nature of mathematics and the development of mathematics as a body of socially constructed knowledge, it can also be used to consider the way in which mathematics knowledge is constructed in the mind of the learner. Indeed, this is its strength as a philosophy, because it links the nature of mathematics to how it may be learned.

It is important to note, however, that I do not see mathematics as a discrete body of knowledge in this work, but as connected with other forms of knowledge. This position can be further reinforced by recognising that pioneers in the field of cognitive development draw on examples of mathematics learning to support their arguments into more general cognitive development, e.g., Piaget (1952), Vygotsky (1982, in Meschcheryakov, 2007), Bruner, 2006). Ernest (1998, p.49) argues that 'all knowledge is rooted in basic human knowledge and is thus connected by a shared foundation'. Therefore, whilst the nature of mathematics learning is the central focus of this work, it is not considered entirely distinct to other learning. Although the theoretical perspective is informed by a philosophy of mathematics and mathematics learning, it also reflects a perspective of learning any knowledge.

Ernest (1991, pp.43-44) outlines seven assumptions which underpin his social constructivism philosophy of mathematics. These are summarised below:

1. An individual possesses subjective knowledge of mathematics.

2. Publication is necessary (but not sufficient) for subjective knowledge to become objective mathematical knowledge.

3. Published knowledge becomes subject to scrutiny, which may result in its reformulation and acceptance as objective (i.e., socially accepted) knowledge of mathematics.

4. The scrutiny depends on objective criteria.

5. The objective criteria for scrutiny of published knowledge are based on objective knowledge of language, as well as mathematics.

6. Subjective knowledge of mathematics is largely internalised, reconstructed objective knowledge.

7. Individual contributions can add to, restructure or reproduce mathematical knowledge.

Ernest (1991; 1998) sees subjective knowledge as that which exists in the mind of the individual and objective knowledge as ‘shared’ and ‘socially accepted’ knowledge, which are the definitions applied in this work. Objective knowledge is not seen as truth, rather as an interpretation that may be agreed or accepted. Cobb, Yackel and Wood (1992, p.104) in their study of interaction and learning in mathematics classrooms, refer to ‘taken-as-shared’ interpretations. Indeed, as Cobb, Yackel and Wood (1992) note, discrepancies can exist in individual interpretations (subjective knowledge) and, through interaction, that which may be ‘taken-as-shared’ becomes subject to discussion, thus providing further opportunity for learning.

In a process akin to that which Ernest (1991) describes, I re-formulate his assumptions to reflect the theoretical perspective of learning mathematics that will be applied within this work. These are summarised in Table 2 and discussed further below.

Ernest's (1991) assumption about how mathematics develops as a body of knowledge	Re-formulation into a position about mathematics learning
An individual possesses subjective knowledge of mathematics.	An individual constructs subjective knowledge of mathematics, which is an interpretation of experiences.
Publication is necessary (but not sufficient) for subjective knowledge to become objective mathematical knowledge.	Through mediation with cultural tools (such as language, symbols or manipulatives) subjective knowledge can be shared with others. This process is necessary (though not sufficient) for subjective mathematical knowledge to become objective 'socially accepted' mathematical knowledge.
Published knowledge becomes subject to scrutiny, which may result in its reformulation and acceptance as objective (i.e. socially accepted) knowledge of mathematics.	When shared with others, mathematical knowledge may be scrutinised, which may result in its reformulation and acceptance as 'taken-as-shared' knowledge.
The scrutiny depends on objective criteria.	The scrutiny depends on criteria that have been socially accepted
The objective criteria for scrutiny of published knowledge are based on objective knowledge of language, as well as mathematics.	Criteria for scrutiny depend on shared understandings of mediating tools, as well as mathematics.
Subjective knowledge of mathematics is largely internalised, reconstructed objective knowledge.	Subjective knowledge of mathematics can be largely internalised, reconstructed objective knowledge, but it may also be 'spontaneous' (derived from everyday experiences).
Individual contributions can add to, restructure or reproduce mathematical knowledge.	Individuals can add to, restructure or reproduce mathematical knowledge. This relies on mediation and social interaction.

TABLE 2: THE RELATIONSHIP BETWEEN ERNEST'S (1991) ASSUMPTIONS FOR THE PHILOSOPHY OF MATHEMATICS AND THE POSITION FOR HOW MATHEMATICS IS LEARNED

In relation to individual knowledge construction, the theoretical perspective of this thesis adopts a cyclical view, albeit with certain necessary conditions, and this is summarised in a simplistic way in Figure 12.

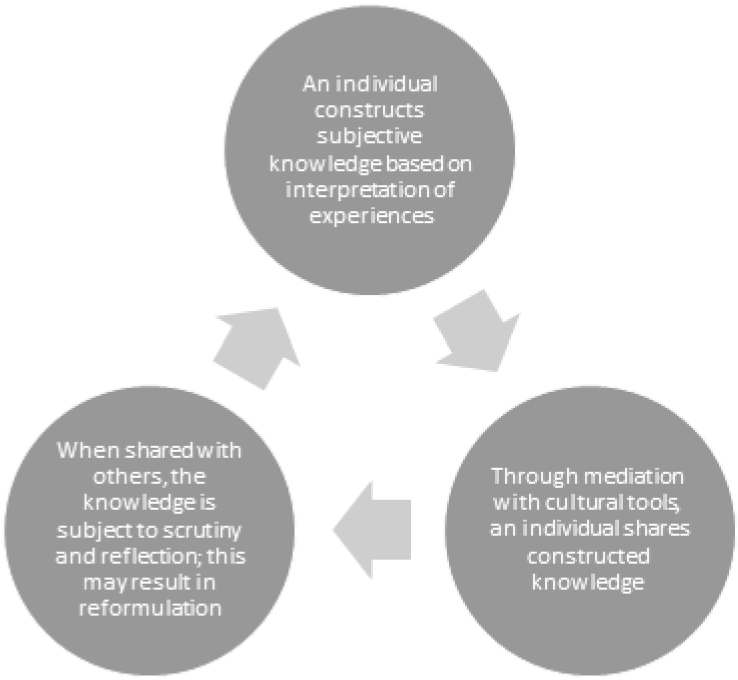


FIGURE 12: CYCLICAL (SIMPLISTIC) OVERVIEW OF INDIVIDUAL KNOWLEDGE CONSTRUCTION

An important note of caution here is that Figure 12 offers only a simple overview of the main points and does not imply a fixed procedure with a distinct start or end. Furthermore, whilst philosophical debates about genesis, ontology (the nature of being) and epistemology (the nature of knowledge) can be thought provoking and insightful, they are alluded to but not dwelt on within the proceeding discussion. Ultimately, the purpose of this writing is to establish a coherent framework which can be used as a perspective to consider mathematics learning rather than to establish a truth about how that takes place. A socio-constructivist theoretical framework accepts mathematics as a social construct and sees it as a product of human beings, and objective knowledge is developed through sharing knowledge, subsequent scrutiny and social acceptance of that shared knowledge (Ernest, 1991).

3.2 INFLUENTIAL THEORIES OF LEARNING

Discussion of how knowledge develops within a learner must consider epistemology because any theoretical framework for knowledge development must also reflect beliefs about knowledge itself. Piaget's seminal contribution to the field of cognitive development is the notion of constructivist learning (e.g., Campbell, 2009; Piaget, 1952). In constructivism the learner is seen as an active constructor, rather than a passive recipient, of knowledge. As Bruner (1997, p.66) summarised 'For Piaget, knowledge of the world is made, not found'. Yet an implication of Piaget's theory of constructivism would be that *made* knowledge must also be *individual* knowledge. Consequently, the question of how shared or socially accepted knowledge develops, and whether that accepted knowledge is true, becomes a key issue. However, Smith (2009) argues that Piaget's theory of developmental epistemology did not seek to explain the objectivity or coherence of human knowledge, rather it sought to consider how knowledge (of any type) can develop in the learner, and Piaget provided detailed frameworks for explaining this.

Von Glasersfeld (1984) implies that Piaget demonstrated ambiguity in his view of the relationship between the construction of knowledge and the nature of knowledge itself. He proposed the theory of radical constructivism to reflect the belief that, if individuals construct knowledge that is subjective and relative to their own experiences and interpretations, then knowledge itself, whilst this can be 'shared', can only ever be an interpretation of reality. It should be noted that von Glasersfeld (1984) founded his theory on Piaget's work and, rather than refuting Piaget's work, it can be considered an extension of it. Von Glasersfeld (1984, p.5) argues that this theory 'is radical because it breaks with convention and develops a theory of knowledge in which knowledge does not reflect an 'objective' ontological reality, but exclusively an ordering and organisation of a world constituted by our experience'.

Ernest's (1991) suggestion that an individual possesses subjective knowledge of mathematics therefore reflects von Glasersfeld's (1984; 1996; 2001) belief that any knowledge that is constructed is an individual interpretation and will be relative to the experiences of the learner. Furthermore, this recognition of the possession of subjective knowledge, when applied as a theory of learning, implies the importance of prior knowledge and experiences. The very youngest of learners will construct mathematical knowledge before 'formal' mathematical education, derived from of experiences involving, for example, quantity, space, measures and/or communication. No learner is a 'blank slate' and he or she will bring, to any situation, individual interpretations of language and experiences that will shape his/her learning of mathematical knowledge.

The belief that every individual possesses subjective knowledge leads to a conclusion that there may be no universal knowledge or 'ultimate truth'. However, as highlighted by von Glasersfeld (1996), the act of teaching requires that accepted knowledge is conveyed and communicated. Von Glasersfeld's (2001) solution to this philosophical quandary is that it is necessary for the teacher to distinguish between 'conventional' knowledge (i.e., accepted convention) and 'rational operations' (something derived from reasoning). For von Glasersfeld (2001), the application of this distinction would inform teaching methods. For example, in mathematics, the use of numerical symbols or words could be seen as convention which might not be possible to discover through reasoning and might therefore need to be told. In contrast, the result of combining two units with another two identical units resulting in four units would need to derive from reflection of experiences leading to such a conclusion. Von Glasersfeld (1996) asserts:

> *Whatever one intends to teach must never be presented as the only possible knowledge – even if the discipline happens to be mathematics. Indeed it should be carefully explained that a fact such as "2 + 2 = 4"may be considered certain, not because it was so ordained by God or any other extra human authority, but because we come to construct units in a particular way and have agreed on how they are to be counted.*
>
> von Glasersfeld (1996, p. 5)

Hence, for von Glasersfeld (1996), it appears that is important to reflect, in teaching, the notion that mathematics is socially constructed, and involves shared and agreed understandings. Indeed, von Glasersfeld's (1996) example can be related to Davydov's (1990) claims regarding how learners can come to understand number, discussed in Chapter 2; through contexts such as measures, learners can be supported in understanding the notion of a unit and number. However, it seems that von Glasersfeld and Davydov demonstrate contrasting beliefs about how such understanding may be obtained; von Glasersfeld's (1996) statement above suggests that number being a results of counting units needs to be told, whereas Davydov (1992) argues that this should be reasoned through experiences, reflected the way in which numbers developed historically.

Ernest (1991, p.46) defines shared knowledge as objective knowledge, that is all knowledge that is 'intersubjective and social'. Biesta (2007, p.12) might argue that the theoretical perspective taken in this work reflects a premise of 'dualism between the immaterial mind and the material world' because it considers 'the impossible question' of the relationship between subjective and objective knowledge development. However, Biesta's (2007) view seems to depend on the definition of objective knowledge applied. The perspective taken in this work is that knowledge development is an active and dynamic process; knowledge ('subjective' and 'objective') is seen as evolving, something that is constructed and re-constructed through interaction. There is, of course, socially, and culturally, accepted knowledge but this evolves over and through time. Ernest's (1991; 1998) framework reflects this view. Mathematics, as a body of shared knowledge, has evolved and developed over time, and continues to do so. Individual knowledge of mathematics also evolves and develops, as learners construct and reconstruct concepts. Furthermore, the theoretical perspective within this work, whilst recognising radical constructivism, also emphasises the social and cultural dimensions in knowledge construction, drawing on the cultural-historical-framework of learning founded by Vygotsky, developed further by his colleagues and followers, including Davydov. As Schmittau (2004, p.20) notes, Davydov's work was 'grounded in Vygotskian theory'.

For Vygotsky a key emphasis was on the role of the social, cultural and historical dimension in learning (Daniels, Cole and Wertsch, 2007). As Ernest (2006, p.5) summarises, social constructivists see the individual and the social realm as 'indissolubly interconnected' and this relationship is clearly reflected in Ernest's (1991; 1996) philosophy of mathematics. Indeed, Ernest (1991, p.106) notes that 'Vygotsky's social theory of mind offers a strong parallel with social constructivism'. Furthermore, Ernest (2016, p.106) argues that Vygotsky's theory has direct relevance to social constructivism because the belief that thought and language develop together implies that 'conceptual evolution depends on language experience, and, of particular relevance to social constructivism, that higher mental processes have their origin in interactive social processes'. A key idea in Vygotsky's work is internalisation, the move from the social to the individual (Lerman, 1996). As Bakhurst (2007) explains, this is not seen as the transplanting of a social activity into an inner plane, rather it is viewed an active process involving inner thought. Furthermore, Lerman (1996) draws on the work of a contemporary of Vygotsky's, Leont'ev (or Leontiev), to argue that internalisation is not the result of something but is the process of the formation of an inner plane of consciousness or awareness, and a vital aspect of the internalisation process is mediation.

Wertsch (2007) notes that, for Vygotsky, a pervading theme in his work was how interaction between the learner and the outside world is mediated, through cultural tools such as language, signs or symbols. Wertsch (2007) suggests that, for Vygotsky, teaching involves encouraging learners to master the use of cultural tools; becoming more expert means that the learner becomes more accomplished within a social order and will be able to use the cultural tools flexibly and fluently. Wertsch's (2007) suggestion is not a full summary of Vygotsky's position regarding the relationship between teaching and learning, rather it is a recognition of the importance Vygotsky placed in the role of mediation. Cultural tools such as language, signs and symbols not only allow us to communicate about mathematics, but they also influence the way concepts in mathematics may be formed. For example, the symbolic notation for a particular fraction informs us about a relationship between a part and a whole and allows us to communicate this. The ancient Egyptians expressed nearly all fractions in terms of unit fractions (i.e., fractions with a numerator of one) and this not only suggests a different way of communicating about fractions but also a particular way of conceptualising fractions. The schematic for the multiplicative relationship introduced in Section 2.8, Chapter 2 (Figure 11, p.62) can also be seen as a cultural tool that facilitates the communication of an idea around a change in unit and the representation of a multiplicative relationship. Furthermore, mathematics itself can be seen as a cultural tool because it allows us to communicate about the world around us.

Daniels (2015, p.1) describes mediation as ‘the process through which the social and the individual mutually shape each other’. It is not a one-way process. This interdependent relationship between the individual and the social is reflected in Ernest’s (1991; 1996) philosophy of mathematics. Indeed As Daniels (2015) discusses, experts in the work of Vygotsky seem to agree that, although there appeared to be differing emphases in the way mediation was discussed in his work, Vygotsky was clear that the individual was an active agent in this process. However, Agrievitch (2009) argues that, although mediation was a central message in Vygotsky’s work, the actual processes involved in mediation were unexplored. This could be, as Agrievitch (2009) acknowledges, because of Vygotsky’s untimely death.

Wertsch (2007) distinguishes between two types of mediation discussed in Vygotsky’s work; explicit mediation and implicit mediation. Explicit mediation is where an external stimulus is intentionally used to mediate learning; a simple mathematical example could be a teacher using a manipulative to represent a mathematical concept. Implicit mediation, as Wertsch (2007) explains, and as its name suggests, is less obvious and more difficult to detect; it involves internal meaning making. A mathematical example of this could be an individual reflecting on a mathematical calculation, trying to make meaning of the symbols. These mathematical examples may be somewhat crude in nature, but the distinction between the two different types of mediation reinforce the Vygotskian belief that meaning is mediated through cultural tools. In mathematics, those cultural tools could be language, symbols, or representations.

Ernest (1998, p.105) reinforces that a vital aspect of social constructivism is 'negotiation as a shaper of thought':

> *A central thesis of social constructivism is that the unique subjective meanings and theories constructed by individuals are developed to 'fit' the social and physical worlds.*
>
> (Ernest, 1998, p.105).

When an individual shares knowledge (through mediation), this knowledge may then be subject to scrutiny and hence a negotiation of meaning might occur. This can be seen as a two-way process; negotiation of meaning might occur when an individual's meaning does not 'fit' that of the social, but also 'accepted' social knowledge might develop when an individual offers new insight. Thus, through social interaction, mediated with cultural tools such as language and symbols, subjective knowledge can become objective knowledge or objective knowledge develops further. As Ernest states, interaction is 'necessary but not sufficient' (Ernest, 1991, p.43). Hence if learning is not experienced on a social plane, as indicated as necessary by Vygotsky, then it cannot become internalised. An important point to note here is that social interaction need not involve direct contact between people; engaging with a text would be an example of learning that is not direct interaction, but still social.

In early years of learning spoken language will naturally be a key mediating tool for most children, and thus social interaction in the form of mathematical discourse is seen as an important aspect of learning mathematics. Cobb, Yackel and Wood (1992) report how the research into learning of children in the US second grade (7-8 years old) informed their development of a theoretical framework for mathematics learning. They discuss how they initially began with a radical constructivist framework, focussing on individual cognition and development of 'taken-as-shared' knowledge. However, as they developed interest in the teacher's learning and the social interactions between learners, they adopted a more social constructivist view of teaching and learning. Applying a socio-cultural stance enabled them to consider the enculturation of learners and social norms, as well as individual, group and teacher learning. Their research also caused them to reflect on the 'circularity' of learning within mathematical discourse in which there was 'continual mutual adjustment as the children each influenced each other's activity while themselves being influenced by their interpretations of that activity' (Cobb, Yackel and Wood, 1992, p.117). Their position shows striking similarity with that of Ernest's (1991) philosophy, yet it was reached through analysis of classroom discourse and seemingly independently of Ernest's perspective.

It should be noted, however, that there are critics to the merging ideas of radical constructivism and social constructivism as seen in Ernest's (1991; 1996) and Cobb, Yackel and Wood's (1992) work. Lerman (1996, p.135) raises Goldin's (1990) argument that 'radical constructivism does not in principle ever permit us to conclude that two individuals have the same knowledge'. This is certainly a valid point; we can never be entirely sure that two individuals have identical knowledge, but those two individuals can, through negotiation and mediation, reach positions where they seem to share understanding, and the negotiation process can further develop knowledge. Applying a mathematical notion, perspectives may sometimes tend, or converge, towards seemingly similar conclusions. This seems to be the case with Ernest's (1991) development of a philosophy of social constructivism and Cobb, Yackel and Wood's (1992) move towards a social-cultural framework to consider mathematical discourse and learning. Studying the history of mathematics, or any other discipline, reveals examples where similar ideas have been developed seemingly independently, for example Leibnitz and Newton who have both been attributed with developing the idea of calculus (e.g., Hall, 2002). Of course, a sceptic may suggest that these were discoveries, suggesting an ultimate truth towards which something may tend, but this is not the view adopted here; the ideas themselves are social constructions, based on other constructions. The view adopted within this work is in line with Berger (2004, p.4), who notes ‘I assume that order in the world does not exist independently of the human mind; rather we impose order on the universe through our various theoretical constructions’.

Davydov’s beliefs about learning (e.g., Davydov, 1990; 2008) are built on the foundations of Vygotsky’s cultural-historical theory and those who followed and developed Vygotsky’s work (Lektorsky and Robbins, 2008; Fellus and Biton, 2017). Although often now discussed in relation to ‘Activity Theory’ because of the focus on activity, whether individual or collective (Lektorsky and Robbins, 2008), Davydov’s (1990; 2008) theoretical perspective of learning reflect cultural-historical views of learning (Schmittau, 2003) and can be considered complementary to the social-constructivist view taken within this work. For example, in relation to cultural-historical theory of development, Davydov notes:

> *This theory really does not admit any immanent development of the separate individual detached from sociocultural values, from communication and cooperation with other individuals, from instruction and upbringing. On the contrary, many theories admit and maintain the presence of precisely this sort of immanence. But, at the same time, cultural-historical theory admits the immanence and the presence of an internal logic of development of each individual, who from the moment of birth and throughout his life constantly communicates and cooperates with other individuals (either directly or in ‘ideal form’). This immanence of development is inherent in the social individual, who is situated in interaction with other people.*
>
> (Davydov, 2008, pp.198-199)

Furthermore, though Davydov (1990) discusses the importance of developing scientific (theoretical) concepts from the start, as discussed in Chapter 2, he notes that concepts are ‘social’ and ‘collective’ (Davydov, 2008, p.202) and acknowledges:

> *Concepts that have developed historically in society exist objectively in the forms of man's activity and in its results – in propitiously created objects. Particular persons (and children, above all) receive and assimilate them before they learn to act with particular empirical manifestations of them. The individual must act and produce things according to the concepts which exist as norms in the society beforehand – and he does not create them but accepts or assimilates them.*
>
> (Davydov, 1990, p.118)

Thus, Davydov (1990) reflects a view that there are 'objective', socially accepted concepts, as noted by Ernest (1991) and that learners also develop spontaneous concepts through their experiences.

It should be reinforced that a social constructivist view emphasises the role of social interaction within learning but does not imply a fixed direction such as movement in learning from social to individual or individual to social. Rather, the social constructivist perspective considers how individual 'subjective' knowledge might become 'objective' or socially accepted knowledge, whilst recognising that they are mutually dependent and constantly developing (see Figure 12 p.68). Though Davydov (1990) argues that there should be a focus on the development of theoretical concepts from the start of education, this does not imply that he believed in a different process of conceptual development to that proposed by Vygotsky, rather that he believed that theoretical concepts should be a focus of teaching from the start of education.

To conclude, a social constructivist perspective is taken in relation to all forms of activity involved in this study; from how mathematics is seen as a body of knowledge, as outlined in Chapter 2, to how research and learning tasks will be designed, implemented, and analysed, discussed further in Chapters 4, 5 and 6.

CHAPTER 4: RESEARCH METHODOLOGY AND APPROACHES TO DATA COLLECTION AND ANALYSIS

4.1 INTRODUCTION

In this chapter, the approach taken to the research and the thinking (logic) around this are discussed. Research methods, data collection, ethical considerations and the process taken to the analysis of data are outlined.

Hammersley (2011) notes that the term 'methodology' has evolved over time and now tends to incorporate not only the discussion of, and thinking about, methods, but also the philosophical approach to research. Additionally, Cohen, Manion and Morrison (2018) note that an important part of research design is the exploration and recognition of assumptions about the world and the ways of looking it (paradigms); this involves considerations of ontology (what is considered real) and epistemology (how knowledge is developed). Put simply, methodology, at its core, involves reflection of research process and beliefs (Hammersley, 2011).

In previous chapters, I have argued that mathematics is a social construction, and I have outlined the social constructivist theoretical perspective within this work. These views inform the research approach, the collection, analysis and interpretation of data, set out in this and subsequent chapters.

4.2 METHODOLOGICAL FRAMEWORK: INTRODUCTION TO DESIGN RESEARCH

Design research (or design-based research) is the methodological framework used within this study. As Bakker (2018) notes, most experts in design research agree that it is not a methodology nor a research method, but rather it is a methodological framework that uses existing research methods to gain research-informed insights. In particular:

> *In design research, design and research are intertwined: The design is research based and the research is design based.*
>
> (Bakker, 2018, p. 4)

In this study, though the intention is the design of research informed tasks to support the learning of multiplicative reasoning through measures, the product should be far more than the tasks; theoretical insight can be gained into the learning and teaching, and this is developed through the research undertaken in developing the designs and their relationships with learning and teaching. Hence the ultimate ambition of this research is to understand more about learning and teaching of multiplicative relationships through measures, via the design of tasks that could support this.

Design research in education has developed out of a desire to understand learning in real educational settings, yet it is not naturalistic as it seeks to analyse the learning that occurs through designing, reflecting on and improving interventions. American psychologist Brown (1992), often credited as a first developer of design-based research (e.g., Anderson and Shattuck, 2012), discussed how her research evolved from laboratory learning experiments to 'design experiments' in the classroom, arguing that school-based research involving designed interventions should allow for greater learning for all stakeholders (students, teachers and researchers) and that the complex factors that exist in educational settings must be considered as part of research, so that any designed intervention should have positive impact in typical settings.

Cobb *et al*. (2003, pp.9-11) synthesise ideas and approaches from a range of US based design experiments, the majority of which involve research into mathematics learning and cognitive development, to identify key features of the approach, in particular:

- there is a study of function of both the design and of the 'ecology of learning' at the heart of the approach, with the purpose of developing theory about the process of learning and the way in which it can be developed;

- it is interventionalist, investigating possibilities for educational improvement by bringing about new forms of learning;

- it is both prospective and reflective. It is prospective because designs are implemented to account for hypothesised learning and explore a possible way to support this and it is reflective because there is an analysis of the learning taking place with consideration of how this can be developed further. It can possibly reveal new pathways in learning;

- it is iterative, incorporating hypothesis, trial, analysis and further conjecture, each time refining the production of an explanatory framework for learning;

- it makes the theory 'do real work' because the theory is applied both to design something specific and to analyse the learning that may occur.

Table 3 below summarises the design research focus in this study, noting how these key features are enacted. Further discussion in relation to the features is provided in Chapters 5 and 6, in which the research cycles are introduced.

Exploring the learning and teaching of multiplicative reasoning through measures: A design-based research project	
Feature of design research	How it is enacted in this study
Study of function of both design and ecology of learning	Cobb *et al*. (2003, p.9) use the term 'learning ecology' to recognise the complex and interrelated nature of learning situations, which involve learning tasks, discourse that may be encouraged, norms of participation, tools and materials provided and the ways in which teachers may practically support the learners in engaging and relating these elements. As Tabak (2004, p.226) notes, '*there is a theoretical commitment to the idea that learning is a complex enterprise derived from a synergy of factors and interactions*'. The study involves analysis of learner interactions, with each other, with the tasks and materials provided and with me, as research and teacher. Norms of participation, familiarity with approaches and materials and resources are considered in preparation for tasks and as part of analysis. Learner engagement and interaction with tasks are also considered as part of analysis. Teacher views on tasks are sought.
Interventionalist	The intervention is the use of tasks involving measures, to introduce the multiplicative relationship as involving a change in unit, when this is typically taught through other approaches, as discussed in Chapters 1 and 2.
Prospective and Reflective	The design focuses on development of sequences of tasks reflecting Davydov's (1992) view of the development of the multiplicative relationship, recognising that learners may not have previously learnt number through measures contexts. There is reflection and analysis of learning, including learner views and teacher views to further develop the tasks and task sequences.
Iterative	The hypothesis is that tasks involving measures can support the learning of the multiplicative relationship. There are two phases of research and development.
Theory is put to work	The design and analysis of tasks draws on Davydov's (1992) theory of the learning of the multiplicative relationship. In addition, the social constructivist approach and criteria for tasks such as those by Erikson and Jansson (2017) are used to develop, and reflect on, task designs. Theory about learning in the multiplicative relationship (e.g., Steffe, 1994) and understanding measures concepts (e.g. Nunes and Bryant, 2009a) is also considered in analysis of learning.

TABLE 3: THE DESIGN RESEARCH STUDY AND HOW IT REFLECTS COBB *ET AL*.'S (2003) FEATURES

4.3 THE DEVELOPMENT OF DESIGN RESEARCH

Research into mathematics learning and the evolution of design research are strongly connected. Furthermore, it can be argued that the nature of mathematics is a reason for this connection. Hjalmarson and Lesh (2008), in discussing the role of engineering in design research, note that design cycles begin with the identification of a problem; Burkhardt and Swan (2017) indicate that such a problem is the need for coherent lessons in a subject like mathematics. Whilst all lessons, whatever the subject, need to be coherent, Burkhardt and Swan (2017) suggest that the technical nature of original literature in Science, Technology, Engineering and Mathematics (STEM) subjects mean that teachers need more support in designing tasks and task sequences that allow learners to understand mathematics. It should be noted that both Burkhardt and Swan's extensive research profiles in task design focused predominantly on secondary and higher-level STEM subjects. However, a similar, and perhaps more compelling, argument could be used for primary mathematics, where teachers are typically non-specialists in mathematics and schools commonly seek planning, task design and sequencing support in mathematics, e.g., through published schemes (WG, 2015b).

Wittman (2021) asserts that mathematics education should be viewed as a design science because successful mathematics learning relies on the design of meaningful, coherent tasks for learner engagement. Wittman (2021, p.87) even controversially suggests that the design of substantial teaching units should not be 'left to the teachers', but rather to 'experts'. Although he does acknowledge the important contribution teachers can make to task development, Wittman suggests this is more valuable when they are familiar with underlying research. I have no intention to enter the complex debate of the relationship between teacher understanding in mathematics and the way it is taught, and certainly no desire to underestimate the understanding a teacher can bring to task design, but it can be reasonably asserted that effective mathematics teaching relies, to an extent, on the application and sequencing of appropriate meaningful tasks which reflect understanding of mathematics and how it can be learned. Hence design research can make an important contribution to supporting the learning of mathematics. Furthermore, the desire to both understand and improve mathematics learning at all stages of mathematics through the design, implementation and evaluation of learning sequences has initiated research which can be viewed as roots of the methodological framework now known as design research.

It is particularly noteworthy that Bakker (2018) traces one of the roots of design research to Russian 'transformative experiments', involving the creation of educational situations to transform learning. Davydov's work was founded on such 'transformative experiments'; he, along with colleague Elkonin, developed a curriculum for 'School 91' in Moscow (Dougherty and Simon, 2014) and, indeed, Davydov's (1992) discussions of multiplication learning draw on learning tasks taking place in this school setting. Davydov (1992) identified a sequence of 'instructional situations' for the introduction of multiplication, founded on the focus of a change in units. His writing outlines detailed implementation of these situations including tasks, teacher questions and typical learner statements. He also provides results of the teaching situations, indicating how they were tested (1961 - 1967), providing examples of learner responses. He gives detailed results and analysis of assignments (administered with 92 learners), including percentages of success on questions. Davydov (1992) concludes his discussion of results by emphasising that the learners must master the concept of transference of one unit of count or measure to another and that further instructional situations must be developed and analysed to enable all learners to master multiplication. Thus, in implementing a theoretically informed intervention to develop understanding of multiplication, considering the instructional situations, reflecting on outcomes, and recognising how the work could be further developed, Davydov was reflecting aspects of Cobb *et al*.'s (2003) design research criteria.

A further mathematical and international root of design research, which began in the early 1970s, is developmental research from the Netherlands. The Freudenthal Institute, founded by Freudenthal, based its work on 'guided reinvention' (e.g., Freudenthal, 1991) which involved a belief that learners should experience mathematics as 'human activity', reinventing it through support from teachers and tasks. This reflects a view of 'mathematising', discussed in Chapter 2. Through a process of designing and analysis of tasks used in real classroom settings through a developmental research approach, a significant body of work around Realistic Mathematics Education evolved in the Netherlands, influential in teacher education and curriculum development (e.g., Wittman, 2021). Furthermore, work of the Freudenthal Institute has influenced development and perspectives of mathematics education across the world (van den Heuvel-Panhuizen, 2020).

In the US, the development of design research began with design experiments by Brown (1992) and Collins (1992). Although, as Cobb *et al*. (2003) note, pedagogical design had informed theories around instruction for at least a century before, the development of design experiments signalled a move from research in education taking place in laboratories to research taking place in authentic educational contexts. As Collins, Joseph and Bielaczyc (2004, p.20) note, an implication of this is that design research takes place in 'messy situations' with 'multiple dependent variables' involving 'complex social situations'. Cobb *et al*. (2003, p.9) note that design experiments can vary in type and scope; examples in schools include 'small scale ecology' (e.g., small group of pupils) and examples with researchers and teachers working together. It is noteworthy here that two authors of the Cobb *et al*. (2003) paper, Paul Cobb and Jere Confrey, are predominantly researchers in mathematics education. Paul Cobb was, for example, researcher in a year-long teaching experiment involving a teacher and a second grade (7-8 years) class, developing teaching materials to support group work, drawing on cognitive models of children's arithmetical learning (Cobb, Yackel and Wood, 1992). In this teaching experiment, all mathematics lessons were video recorded in order to develop 'a conceptual framework…to cope with the complexity of classroom life in a more general way' (Cobb, Yackel and Wood, 1992, p.100). As discussed in Chapter 3, their analysis also caused them to adopt a social constructivist view of learning. Additionally, Jere Confrey has an extensive research background in design research and, through such work, has developed researched informed learning trajectories in key areas of mathematics, including the development of a learning trajectory in rational number which considers the multiplicative relationship (e.g., Confrey and Maloney, 2015; Penuel *et al*., 2014). Hence not only is the history of design research strongly connected to mathematics education, but the outcomes of design research have also contributed to specialised understanding of mathematics learning.

Design research involving mathematics, and other STEM areas, has also taken place, at scale, in the UK. The most notable example of this is the 'engineering research' approach by the Shell Centre at the University of Nottingham (e.g., Burkhardt and Swan, 2017). The centre was founded in 1968 with the aim of improving teachers' understanding of mathematics and its applications. Burkhardt and Swan (2017, p. 176) note that after eight years its focus changed from getting teachers to 'know more maths' to more ambitious aims, reflecting a view that 'large-scale impact can only be achieved through reproducible materials' which should be developed through 'engineering-style' research with 'a focus on design – strategic, structural and technical'. It is noteworthy here that the emphases in those aims, improving classroom practice (and presumably learning) and engineering tasks at scale, differ from other international examples discussed previously, most notably because the focus was on producing materials that would support meaningful learning rather than developing insight into the learning process. Nevertheless, outcomes of the activity have provided insight into mathematics teaching and learning; the most notable example in school mathematics is the work of Malcolm Swan who focused on the development of materials to support collaborative learning in mathematics in the secondary school (16-18 years). Swan's (2006) book gives detailed discussion and analysis of the design research process, and the outcomes that emerged from the process were not only materials which were shared at scale, but principles for the design of teaching which encompassed social constructivist views about learner understanding and teacher roles in facilitating interaction and providing support.

Burkhardt and Swan (2017, p.181) identify features that can guide considerations around task difficulty in design research. These are:

- complexity (aspects such as number of variables, modes of presentation of information);
- (un)familiarity (similarity to a task that might have been practised previously);
- technical demand (the level of mathematics required);
- student autonomy (the level of guidance from teacher or from structuring or scaffolding of task);

These features are used in task development and analysis in this study.

Hence, as outlined in this section, the outcomes of the design research processes involving mathematics are foundations of the theoretical approach taken within this study and are used to support the development and analysis of both the learning and the task design.

4.4 RECOGNISING LIMITATIONS OF DESIGN RESEARCH

Many critiques of design research focus on the potential for researcher bias. For example, Barab and Squire (2004) argue that researcher involvement in the conceptualisation, design, development, implementation and research of a pedagogical approach might affect the credibility of findings or assertions. Tabak (2004) also suggests a key risk in design research may be researcher bias; she argues that the focus on context (or ecology as discussed by

Cobb *et al*. 2003) can mean other factors that may play an important role in the learning may be missed. Tabak (2004, p.227) applies two constructs 'exogenous design' and 'endogenous design' to consider context, where 'exogenous design' refers to the materials, strategies and activity structures that have been developed for the research and the term 'endogenous design' refers to materials and practices that are in place in a local setting, including the way in which teachers and students may engage in enactment of materials. Tabak (2004) argues that both constructs must be explained and applied in any explanation of learning, and failure to do this could limit the credibility of any assertions. Indeed, Tabak (2004) notes that design research projects have the potential to focus attention and analysis on exogenous design, with insufficient attention to endogenous design.

In this study, I act as designer and enactor, researcher and teacher. To support endogenous and exogenous design, the first phase of Cycle 1 research involves observation of mathematics practice and the learning environment, a semi-structured interview with practitioners to explore mathematics practice, and a pre-assessment with learners. Cycle 2 involves an interview with practitioners prior to enacting the re-developed tasks and a post enactment interview. By acting as designer and researcher, and by developing understanding of the mathematical environment for learners and accounting for this within the design, the possibly of tensions between exogenous and endogenous design, as suggested by Tabak (2004), are reduced. Nevertheless, Barab and Squire's (2004) suggestion that researcher involvement in all aspects of design research could be applied to conclude this furthers the risk of bias. Yet, as Anderson and Shattuck (2012) note, this criticism of researcher involvement risking bias is not unique to design research and is often used as a critique of any research involving qualitative data. A discussion on the process of data collection and analysis follows, and this includes considerations undertaken to try to minimise any susceptibility to researcher bias.

Shavelson *et al*. (2003) note that design research tends to involve narrative approaches, whether this is recognised by the researcher or not . They note this might involve considering perceptions and experiences over time as well as taking a narrative approach to the communication of findings. However, the authors demonstrate a positivist view of knowledge, for example suggesting the use of randomised control trials to establish cause and effect. Though not seeking narrative accounts, this research involves seeking learner and practitioner reflections and perceptions on the tasks and experiences, through recorded semi-structured interviews, and includes my own reflections, based on reflective notes written at the time. I also apply a chronological approach to reporting the development of the design principles, tasks, enactment, and the points of learning. Shavelson *et al*. (2003) suggest narrative elements in design research could raise questions about knowledge claims and generalisability to other situations. These aspects are addressed further in the proceeding discussion on research quality, data collection and analysis. However, it is important to reinforce that a social constructivist lens is applied throughout this work, acknowledging that individuals will interpret experiences to form their own concepts or views. Through sharing these interpretations, they become subject to scrutiny. This is the case for the participants in the research and for me as a researcher sharing my interpretations. I will not claim that results can be generalised to other situations, rather, I will share my interpretations for scrutiny.

To conclude, it seems that many of the benefits of design research such as integrating and researching theory and practice in real settings, involving the participants through seeking perceptions and reflections, and accounting for the ecology of learning, can also be used as part of the critique. However, Bakker (2018) and, indeed, Shavelson *et al*. (2003), suggest that clear argumentation is a key factor in ensuring findings from design research can be considered credible. Furthermore, as Bakker (2018) notes, consideration of the internal validity and reliability of the research methods, data collection and data analysis will support quality design research, discussed further in this chapter.

4.5 RESEARCH METHODS, DATA COLLECTION AND PARTICIPANTS

This design research involves mixed methods, using a variety of approaches to qualitative data collection. As noted by The Design-Based Research Collective (2003), design research typically triangulates multiple sources and data, which can allow for connections to be made between any outcomes (intended or unintended) and implementation of the design. Cohen, Manion and Morrison (2018) reinforce that triangulation allows for an aspect of enquiry to be considered from different standpoints. In this research, the approach of developing multiplicative reasoning through measures tasks is considered through learner, practitioner, and researcher perspectives.

All research activity took place in a primary school in South Wales. The school was selected because it had an established research relationship with the university. Initially the head teacher was contacted, asking if the school might be interested in taking part in the research and initial interest was indicated. As outlined in Section 4.12, consent from practitioners and parents, and assent from learners, were also sought as part of each cycle.

No school can be considered representative, each having its own unique circumstances and culture, with wide-ranging contributing factors such as school environment and socio-economic backgrounds of learners. Nevertheless, in my experience as a teacher educator in Wales, I felt the school could be considered typical, as far as this is possible, of a primary school in Wales. The school typically had two classes for each year group, and its percentage of learners entitled to free school meals was slightly above the national average.

Practitioners involved in the research were those involved in teaching the age ranges considered. In Cycle 1, Phase 1 a practitioner interview was conducted with teachers in the Foundation Phase (ages 3 – 7); these were teachers who were available at the time and who had consented to take part. In Cycle 2a, an interview took place with the two Year 2 teachers; this was arranged with their consent and at a time convenient to them. The Year 2 teachers were interviewed because I would be working with this year group. At the end of Cycle 2b, a Year 2 teacher who had been involved, through interview and through having learners involved in both cycles, was interviewed. This teacher also happened to be the co-ordinator for Mathematics and Numeracy within the school.

All participants involved in task implementation were Year 2 learners. This year group was chosen because, in Wales, Year 2 is a year group in which the multiplicative relationship is typically introduced through the explicit introduction of multiplication and division, as discussed in Chapter 1, Section 1.3. Learners involved in the research were those for whom parental consent was obtained, through the support of their teachers. Learner assent to take part in the research was also then sought. In Cycle 1, the learners involved in the research were described by their teacher as ‘average to high attaining’ in mathematics. In Cycle 2, the learners varied more widely from those described as ‘low attaining’ to those described as ‘high attaining’ in mathematics, according to their teachers. As I sought to explore the response of a range of learners to the tasks, the variety of learners involved across both cycles can be seen as an advantage. No learner is considered representative.

Table 4 below outlines the research questions, and the related research methods employed to collect qualitative data.

Research Question	Cycle 1	Data	Cycle 2	Data
S1: What are teachers' and learners' prior experiences of teaching and learning multiplicative reasoning?	Phase 1: Observation of learning environment 'learning walk' using a semi-structured observation schedule Semi-structured focus group interview with practitioners (4 practitioners)	Observation notes Reflective notes Audio recording of interview	Phase 1: Semi-structured focus group interview with practitioners (2 practitioners)	Audio recording of interview
S2: What are learners' prior experiences of learning number and measures?	Phase 1: Observation of learning environment 'learning walk' using a semi-structured observation schedule Semi-structured focus group interview with practitioners (4 practitioners)	Observation notes Reflective notes Audio recording of interview	Phase 1: Semi-structured focus group interview with practitioners (2 practitioners)	Audio-recording of interview
S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?	Phase 2: Trial and implementation phase in one school involving teaching, learner feedback through semi-structured interviews (8 learners) Researcher unstructured reflective notes.	Audio recording of tasks Reflective notes Audio recording of semi-structured interviews with learners	Phases2a and Phases 2b: Trial and implementation of iterated tasks, involving learner feedback through semi-structured interviews (Phase 2a = 8 leaners, Phase 2b = 5 learners) Researcher unstructured reflective notes.	Audio recording of tasks Reflective notes Audio recording of learner perceptions of tasks
S4: What is the impact of learning multiplicative reasoning through measures on learners?	Phase 2: Pre- assessment, implementation of tasks and learner feedback through semi-structured interview (8 learners) *	Reflective and observation notes of pre-assessment Audio recording of tasks Audio recording of semi-structured interviews with learners	Phases 2a and 2b: Pre-assessment, implementation of tasks and learner feedback through semi-structured interview (Phase 2a = 8 leaners, Phase 2b = 5 learners) ***	Reflective and observation notes of pre-assessment Audio recording of tasks Audio recording of semi-structured interviews with learners
S5: What are teachers' and learners' views on teaching/learning multiplicative reasoning through measures using the materials developed?	Phase 2: Learner feedback through semi-structured interview (8 leaners) **	Audio recording of semi-structured interviews with learners	Phases 2a and 2b: Learner feedback through semi-structured interview (Phase 2a = 8 leaners, Phase 2b = 5 learners) *** Practitioner feedback through semi-structured interview (1 practitioner)****	Audio recording of semi-structured interviews with learners Audio-recording of semi-structured interview with practitioner

TABLE 4: AN OVERVIEW OF RESEARCH QUESTIONS AND DATA FOR CYCLES 1 AND 2

As indicated in Section 1.3 and summarised in Appendix A, COVID-19, and resulting national and local lockdowns, along with operational restrictions once learners did return to school, did impact on time between cycles and data collection. Furthermore, a locally imposed school closure due to inclement weather, and practitioner illness, resulted in two phases of data collection in Cycle 2 (Phase 2a and Phase 2b). In reference to the table above:

*A post-assessment was planned but could not take place due to school closure (COVID-19 pandemic)

**A semi-structured interview with practitioners was planned but could not take place due to school closure (COVID-19 pandemic)

***Due to school closures for inclement weather and COVID-19 restrictions in operations, time in school was affected resulting in focus group interviews with learners rather than individual interviews

****Due to staff illness the practitioner interview took place with one practitioner

To conclude, the methods for data collection in this study can be summarised into these main categories: observation, interview, audio-recording, and the process of taking reflective field notes.

4.6 ENSURING QUALITY DESIGN RESEARCH

Discussion of methodology typically includes consideration of the validity and reliability of research; as Bakker (2018) notes, the term validity refers to the extent to which the focus of study is being investigated and the term reliability refers to whether findings can be considered independent of the researcher and whether similar findings could be developed by other researchers under similar circumstances.

As Cohen, Manion and Morrison (2018) and Bakker (2018) discuss, use of terms such as validity and reliability in qualitative data are contested by many researchers. Yet Cohen, Manion and Morrison (2018) apply both these terms in the context of qualitative research and, indeed, Bakker (2018) uses these terms when discussing design research, whilst recognising design research typically involves qualitative data.

Whatever terminology is used, an important aspect of ensuring research quality is transparency of approach and, helpfully, Bakker (2018, p.90) outlines aspects of validity and reliability to be considered through a design research study; this has been applied to this study to ensure transparency (see Table 5).

Table 5 is not seen as exhaustive or complete in terms of ensuring design quality within this research study, as further discussion is undertaken in subsequent sections; rather, it is intended to give a broad overview of how aspects such as validity and reliability are addressed.

Aspect of study Questions by Bakker (2018, p.90) Response notes how applied within study	**Validity issue**	**Reliability issue**
Theoretical constructs	*Are they well defined in line with literature?* The theoretical constructs of multiplicative reasoning and multiplication as a change in the system of the unit of measure is defined by Davydov (1992) and applied within this study.	*Can sources be found?* Sources are discussed within Chapter 2 and within Chapters 5 and 6 in the context of cycles.
Research design and procedure	*Is the research design suitable for the question raised?* As discussed earlier in this chapter, Davydov's work (e.g., 1990; 1992) reflects design research approaches and design research offers the opportunity to explore an approach, guided by theory, and the efficacy of tasks developed.	*Can the procedure be (virtually) replicated?* As Bakker (2018, p.93) notes it is 'impossible' to replicate design studies fully, but 'virtual' replication would mean that the research process could be followed so that main ideas can be used in different situations. Though replication of procedure in a different context may generate different interpretations, the research procedures followed are outlined in this chapter and in Chapters 5 and 6, so that another researcher might be able to undertake a similar study.
Sampling procedure	*Representative sample?* In recognition that every school, practitioner and learner is different, it can be argued that no sample involving these is representative, though the school can be seen to be seen as 'typical' of schools in Wales, based on my experience. Practitioners were those based within the Foundation Phase for Cycle 1, Phase 1. Year 2 teachers (teachers of relevant year group) were interviewed in Cycle 2 Phase 2 and Cycle 2. Learners were those in Year 2 for whom parental consent and their own assent was attained.	*Can the sampling procedure be replicated?* The procedure could be replicated though it should be acknowledged that the resulting sample would be different.

Aspect of study Questions by Bakker (2018, p.90) Response notes how applied within study	**Validity issue**	**Reliability issue**
Instruments	*Are the instruments valid?* Interviews, observation and reflective notes are the research instruments. These are widely used instruments within education research (Cohen, Manion and Morrison, 2018). Discussion of their application is outlined in proceeding sections.	*Are the instruments reliable?* Proceeding sections notes steps taken to ensure the instruments are as reliable as possible given the context of the research.
Data collection	*Are the data of high quality?* Data collection approaches for each instrument are shared and examples given.	*Have audio/video recordings been used to avoid memory issues? Have transcripts been used?* Audio recording and transcripts are used, discussed in proceeding sections.
Data analysis	*Has triangulation been applied?* Learner, teacher and researcher perspectives and interpretations have been considered. In discussion of results, these multiple sources of data are considered.	*Has coding been used and, if so, is there interrater reliability?* In Cycle 1, two fellow researchers 'critical friends' (not involved in study) were asked to apply coding to data. This allowed for consideration of whether the codes could be interpreted and applied accurately.
Drawing conclusions	*Have conclusions been drawn in a valid way?* Data are used to support any conclusions. The process of drawing conclusions is also discussed within this chapter.	*Is the argumentation transparent? Could another researcher arrive at the same conclusions?* It may not be possible to claim that another researcher would arrive at the same or similar conclusions at this stage, but argumentation is addressed for transparency.

TABLE 5: VALIDITY AND RELIABILITY CONSIDERATIONS ACCORDING TO BAKKER (2018, P.90)

4.7 OBSERVATION

Observation offers the opportunity for 'live' and 'in situ' data to be collected (Cohen, Manion and Morrison, 2018, p.542). It can allow for information gathering, for events to be noted and for behaviours to be observed. However, Cohen, Manion and Morrison (2018) also note that an important aspect of observation is consideration of what will be noted as acceptable evidence. In this study, observation is used as part of Phase 1, Cycle 1 research, which involved a 'learning walk'. A 'learning walk' is used in inspection of schools in Wales (Estyn, 2021), and this approach of walking through the learning environment to gather information about the teaching and learning practices was considered a useful approach for Phase 1 of Cycle 1. The focus of this learning walk was to observe a range of mathematics experiences across the Nursery to Year 2 age ranges within the setting. This involved three aspects noted above:

-information gathering (such as what resources were used to support teaching and learning)

-events (mathematics activity taking place, whether whole class, small group, with teacher or of learners independently)

-behaviours (mathematical interactions observed)

It is possible, as noted as a potential limitation to observation by Cohen, Manion and Morrison (2018), that behaviours and events are altered because of the presence of an observer. The purpose of the learning walk observation is set out in Appendix B; this was shared with practitioners prior to visiting the school. As a school with an established research and teacher education relationship with the university, the school frequently hosted observers. The learning walk took place throughout a morning in which I moved between classrooms and settings. I did not observe *all* mathematical activity (nor had I intended to), rather I sought an overview of the sort of activity that was taking place and how it was organised.

As Cohen, Manion and Morrison. (2018) note, observations can be prone to bias because they are likely to be selective. Indeed, I was selective about what was being observed, focusing solely on finding out information about approaches to mathematics teaching and learning in the setting, observing any mathematical events that happened to occur and considering behaviours within those. However, as noted above, the purpose of the observation was to gain an overview of the approaches employed within the setting, which would then be followed up through interview. Data collected were in the form of semi-structured notes made at the time. Appendix C outlines the observation schedule.

The learning walk observation took place over one morning, for approximately an hour and a half. I began in the school hall and moved into all the classrooms within the Foundation Phase (Nursery, Reception, Years 1 and 2), spending around 10 to 15 minutes within each setting. I noted aspects such as mathematics learning resources and learner access to these, I observed some mathematics whole class starters and plenaries, work with a practitioner (teacher or teaching assistant) and groups of learners, and some independent activity in the form of enhanced provision.

Notes taken were used to inform questions asked at interview or to consider in relation to other data (e.g., comments made within interview, learner responses to tasks) and thus, in this study, observational data collected are not considered in isolation. No judgements were made about quality; for example, in observing mathematical interactions; the focus was on what they might involve (e.g., how learners and/or practitioners were interacting, where this took place and how). Thus, the observation in this study is low inference, noted by Cohen, Manion and Morrison (2018, p.562) as being possibly the ‘safest’ form of observation when considering reliability and validity. Furthermore, interpretations made through observation are triangulated with other data sources such as the practitioner interviews.

4.8 INTERVIEWS

As noted by Kvale (1996, in Cohen, Manion and Morrison, 2018) an interview can be considered as 'inter-view' in which different views are exchanged and discussed and, as Cohen, Manion and Morrison (2018, p.506) note, the use of an interview 'sees the centrality of human interaction for knowledge production and emphasises the social situatedness of research data'. Hence, the use of interviews as part of this study is in line with the social constructivist perspective taken within the work, as it allows for interpretations to be shared, explored and analysed, and what is said is not viewed as an ultimate truth.

Brinkmann and Kvale (2018) note that a qualitative interview can seek information as well as meaning, though it is usually harder to seek meaning. In the interviews within this study, both information (e.g., about approaches to teaching multiplicative reasoning and measures) and meaning (e.g., what effective approaches to teaching mathematics might involve) were sought. Table 6 (p.101) gives an overview of the interviews undertaken, including the purposes of each interview, the participants involved and the approximate time taken for each interview.

As King, Horrocks and Brooks (2019) reinforce, meanings are co-constructed within an interview; this is an important consideration for this study as meanings or views may alter, or be altered, through an interview. This is perhaps more likely when there are multiple interviewees in a group interview but can also occur during interviews with one participant.

All interviews conducted in this study were semi-structured because I used pre-planned open-ended questions, with further prompts available if needed. Interview questions and responses are provided as appendices and discussed further within subsequent chapters. In all practitioner interviews, I provided copies of the main questions and gave time for these to be read within the first part of the interview. Practitioner interviews took place in designated private workspaces for practitioners, at times chosen by them, when they were not teaching.

When interviewing learners, I did not provide copies of questions, because of their age, but I did provide images of the tasks that had been undertaken, to support learner recollection of the tasks, and to provide stimulus for the discussion. Learner interviews took place in the same spaces where tasks had been conducted; this was an open plan space outside of the classroom but within very close proximity to it. The space was familiar to the learners and was frequently used for small group/breakout work.

Commonly claimed limitations of interviews are that they are invalid because they are subjective (indeed inter-subjective), unreliable as they can be prone to leading questions and ungeneralisable because they have a small number of participants (Brinkmann and Kvale, 2018). Nevertheless, these criticisms can be considered strengths of the interview in the context of a qualitative study which seeks to explore an issue; as Brinkmann and Kvale (2018, p.99) point out, inter-subjectivity allows for ‘a distinctive and sensitive understanding of everyday life’ and ‘controlled’ use of questions can lead to ‘well-controlled knowledge’. As they note, ‘the plurality of interpretations enriches the meanings of the everyday world’.

Cohen, Manion and Morrison (2018, p.508) note that an interview can ‘be prone to subjectivity and bias on the part of the interviewer and interviewee’. It is possible that interviewees may say what they believe the interviewer may want to hear. As a teacher educator with an interest in mathematics, it is possible the teachers I interviewed expressed views they thought I may wish to hear. Similarly, it is possible that learners, who may consider adults within a school as authority figures, would wish to please. In all cases, I reinforced I was seeking genuine perspectives, to try to mitigate these possibilities. I also aimed to use different sources of data (observation, transcription of learning episodes and my own reflective notes) to try to ensure things that were said could be considered in reference to other data.

It is possible that as the interviewer and transcriber, I noted and interpreted what I wanted to hear, as a form of confirmation bias. However, whilst acknowledging that transcription is a construction, I audio-recorded and transcribed all that was said in the interviews; this is discussed further in proceeding sections.

Cycle 1			
Interview type	**Participants**	**Purpose**	**Notes**
Semi-structured group interview	Practitioners (4) Practitioners all worked from Nursery to Year 2	To explore approaches to teaching multiplicative reasoning and measures To inform design of tasks	Followed a 'learning walk' observation (one week later) Approximately 40 minutes
Semi-structured individual interview	Learners (8 Year 2 learners)	To explore perceptions and experiences of tasks undertaken	Used images of tasks as stimulus. Took place on Day 5 of 5 consecutive days. Approximately 8 minutes per learner.
Cycle 2a			
Interview type	**Participants**	**Purpose**	**Notes**
Semi-structured group interview	Practitioners (2 Year 2 teachers)	To explore whether any approaches to teaching multiplicative reasoning and measures had changed. To inform design of tasks in Cycle 2.	Took place after a 2 year period (due to national lockdown and Covid-19 restrictions in schools). Approximately 25 minutes.
Semi-structured group interview (x2)	Learners (4 Year 2 in each group)	To explore perceptions and experiences of tasks undertaken	These were intended to be individual interviews but time constraints due to inclement weather and school closure resulted in them being small group interviews. Used images of tasks as stimulus. Took place on Day 4 of 4 consecutive days. Approximately 8 minutes per group.
Cycle 2b			
Interview type	**Participants**	**Purpose**	**Notes**
Semi-structured group interview	Learners (6 Year 2)	To explore perceptions and experiences of tasks undertaken	Due to time constraints and to follow approaches taken in Cycle 2a, this was conducted as a focus group. Used images of tasks as stimulus. Took place on Day 2 of 2 consecutive days. Approximately 8 minutes.
Semi-structured individual	Practitioner (Year 2 teacher and mathematics coordinator)	To gain feedback and perceptions on tasks undertaken. To share some preliminary data and explore practitioner views.	This had been intended to take place with both class teachers with learners involved in Cycle 2, but one teacher was unavailable due to long-term sick leave. Approximately 30 minutes.

TABLE 6: INTERVIEWS WITHIN THE STUDY

4.9 AUDIO-RECORDING

Audio-recording is not usually considered a research method, requiring little from the researcher at the point of implementation, but it warrants discussion as part of the data collection approach because audio data, both from interviews and from task implementation are a key source of data within this study. An advantage of audio-recording is that it allows concentration on the task in hand rather than simultaneously having to take field notes (Bloor and Wood, 2006); this is particularly important during task implementation as I acted as both teacher and researcher in this study. Bloor and Wood (2006) and Bakker (2018) note that the use of audio-recording improves the reliability of data collection, and this is the case for both the interviews (recording exactly what was said) and the recording of task implementation (to record discussion and to compare against my own reflective notes).

In this study, video recording was not used; although this might provide further visual data, it is time-consuming to analyse (e.g., Cohen, Manion and Morrison, 2018) and may have been a distraction for the participants. Furthermore, the use and storage of video data could invoke more ethical concerns by participants or parents of learners than audio data alone, thus audio recording was selected as an inobtrusive data collection approach. Nevertheless, as Bloor and Wood (2006) note, awareness of audio-recording could risk credible data collection as it might distract or inhibit participants. In the case of interviews and task implementation, participants were made aware of the recording, were assured of the right to withdraw (see ethical consideration in Section 4.12 for further detail) and were assured of confidentiality. Furthermore, it was reinforced that I was seeking to explore views and perceptions rather than make judgements. Indeed, Fielding and Thomas (2001, in Bloor and Wood, 2006) note that the awareness of audio recording can indicate to participants that their views are valued. Hence this could support the credibility of what may be said.

For practitioner interviews, recording was undertaken on a voice recorder laptop application. Small digital dictaphones were used for recording learning tasks; these were introduced to participants. Learners were shown how the dictaphones worked through asking them to introduce themselves and playing back the recording; this allowed for learner familiarisation with the equipment, but also supported the later identification of individual learner voices. Although several dictaphones were used during task implementation (placed on the table near each group/pair), the dictaphones recorded all conversation in the immediate area and thus typically only one recording was needed for transcription. Whilst this had advantages because typically only one audio file was required, this did sometimes mean it could be difficult to distinguish between different small-group conversations taking place, which consequently made transcribing more time-consuming, as cross referencing between recordings was then applied.

To summarise, the use of audio-data allowed for reliable data collection, and its use to compare what was said with other data can add validity to any interpretations made.

4.10 TRANSCRIPTION

Hammersley (2010, p.556) reinforces a view that transcription is a process of researcher 'construction', rather than simply a case of noting everything that has been said. This is because of the many decisions necessary in making a transcription, such as how much to transcribe, how to represent recorded talk (e.g., noting intonation or dialect), indication of whom is being addressed in group talk, inclusion (and possibly timing) of non-talk elements, lay out and labelling. This decision making process involves the researcher's selection and cultural knowledge (Hammersley, 2010). Nevertheless, as Hammersley (2010) argues, for transcripts to be accepted as data for analysis, they need to be viewed as representing actions and events and, in the case of this research, perspectives. Transcription, as

Hammersley (2010, p.564) notes, could be seen as a 'slowing down and reflexive re-routing' of the process of interpretation that occurs in everyday social interaction. In offering practical advice, Hammersley (2010, p.565-566) sensibly suggests that decisions should be guided by research questions, ensuring that care is taken to try to include descriptions of everything that might be relevant to understanding what is occurring; I applied this advice in this study, as noted below.

In this study, for the interviews, *all* audio data were transcribed, because the interviews were defined events with clear start and end points. For the learning and teaching episodes, some data from some events were not transcribed; the decision not to transcribe these was taken after listening to the recordings of Cycle 1. Some events were seen as 'non-learning/non-teaching events' and involved aspects such as distributing and clearing equipment with no discernible discussion about the tasks or mathematical ideas, and these typically occurred at transition points (the start or end of a task).

For all transcribed talk, transcription involved noting exactly what was said, including utterances and noises such as laughter. Silences were not noted (e.g., through timing them); this is because time taken to respond or consider a question was not viewed as an important factor. As the transcription was from audio rather than audio-visual recording, instances where talk may have been directed at a particular person were not explicitly noted. However, in learning and teaching episodes, learners were often working in groups, and these group work episodes involve learner to learner discussion where it is understood that learners were addressing each other; as I was present at the time of all events, I was aware that this was the case.

The software package NVivo was used to support qualitative data analysis; teaching and learning episodes were transcribed directly into NVivo, and sections of talk were coded. Saldaña (2016, p.3) defines a code as 'most often a word or short phrase that symbolically assigns a summative, salient, essence-capturing, and/or evocative attribute for a portion of language-based or visual data'. It can be seen as an item or unit for analysis of qualitative data. Further detail of coding is provided in Section 4.11 and in Chapters 5 and 6.

4.11 REFLECTIVE NOTES

Reflective practice can enhance a research study through providing another perspective on what is being considered (Billups, 2021). It can involve two elements: reflection (stepping back and considering what happened) and reflexivity (considering one's own position, questioning assumptions and interpretations). The use of reflective notes is not consistently recognised as a research method. However, as Bold (2012, p. 81) argues, 'a research diary's capacity to support sustained self-reflection, critical reflection and reflexivity should justify its role as a research method' and therefore the use of reflective notes as data is discussed here.

Cohen, Manion and Morrison (2018, p.302) emphasise that taking a reflexive stance is an integral part of qualitative research because the researcher 'is in the world and of the world that they research' and 'what we focus on, what we see, how we understand, describe, interpret and explain are shaped by ourselves and what we bring to the situation.' McAteer (2013, p. 70) discusses a reflective diary as a place in which theorising can occur, and where 'contextualised understanding' around issues might develop. Although discussing the use of reflective diaries in action research, McAteer (2013) argues that reflective diaries are a useful support for triangulation, providing data that might be used to find points of similarity or difference, or to identify areas that need further exploration. Though design research differs from action research, with an explicit focus on design and instructional theory, the boundaries between them can be 'fuzzy' (Bakker, 2018, p.15), and it is often the case that they use similar data collection approaches.

Through making reflective notes, I monitored my own interactions with participants, noted my own observations and reactions to what occurred and what was said, and considered this in relation to theory. I word-processed notes and thoughts in an ongoing research diary as soon as possible after all research activity in both cycles. There was no set format to diary entries, I noted what had happened, my thoughts about some responses and any implications I felt were important for future tasks. An example of a diary entry from Cycle 1, Phase 1 is provided in Appendix D. I also made reflective notes before and during the transcription of the audio data from the task implementation; I did this using two functions in NVivo. Whilst transcribing data in NVivo, I used the 'annotations' function to add commentary to what I thought might be of interest, and I also created a 'memo', which was a reflective note on the episode/s after listening. Examples of annotations and a memo are provided in Appendix E.

As King, Horrocks and Brooks (2018, p182) note, reflexivity can facilitate accountability in research; indeed, they argue that its practice makes the researcher 'visible' in the construction of knowledge. Through the social constructivist lens applied in this study, my reflective notes are my own interpretations of what occurred and why things might have occurred, and they provide a source of data that can be used as part of the triangulation process.

4.12 APPROACH TO DATA ANALYSIS

In much that is written about design research, little attention seems to be given to the process of data analysis, other than discussion of its iterative nature and its role in contributing to understanding about learning and how that may be learning may be attained (e.g., The Design Based Research Collective, 2003). This is, perhaps, understandable because, as Bakker (2018) notes, design research is a methodological framework that uses existing research approaches and can involve a range of data collection methods. Indeed, Prediger, Gravemeijer and Confrey (2015, p. 880) acknowledge the wide variation in approaches that may be labelled design research in education; they identify two main types, the first being that with a curriculum focus, which typically takes place at scale and has 'a rather well articulated research method' and the second, smaller scale approach, with a focus on learning processes, which they acknowledge can involve a much wider variety of methods and data analysis procedures. Nevertheless, Prediger, Gravemeijer and Confrey (2017) and Bakker (2018) emphasise the importance of articulation of approach to data analysis within design research and argue that explicit attention to this supports credibility of any claims. In this section, I aim to provide an overview of the approach taken to data analysis; further detail, with examples, is provided in the chapters related to each research cycle.

This study involves qualitative data. The analysis of qualitative data requires moving from data in a search for understanding of what is being researched (Cohen, Manion and Morrison, 2018). As Cohen, Manion and Morrison (2018, p.643) note, this sense making can involve a range of activity in relation to data, such as organising, describing, understanding, accounting for, explaining, noting patterns, themes, categories and irregularities. Qualitative data analysis is often described as messy; as Cohen, Manion and Morrison (2018, p.644) note, the process of data analysis is 'recursive, non-linear, messy and reflexive, moving backwards and forwards between data, analysis and interpretation'. It is important to note that in this study, data analysis occurred in three iterative phases: Cycle 1 data analysis, Cycle 2 data analysis, followed by analysis of both Cycle 1 and Cycle 2 data, drawing together key interpretations. Within each data analysis phase, a similar process of moving back and forth between data, analysis and interpretation was applied.

Thomas (2006, p.237) introduces an approach to qualitative data analysis in evaluative research called 'a general inductive approach'. Thomas (2006) defined this approach through analysing approaches evident in other qualitative data analyses. As Thomas (2006) argues, such an approach is evident in much qualitative data analysis, though has often not been defined or labelled.

Thomas (2006, p.238) explains that the three main purposes of the general inductive approach are to:

- condense data so they can be summarised
- establish clear links between research objectives and the summary findings derived from raw data
- develop a model or theory about the underlying experiences or processes that are evident in the text data.

Thomas (2006, p.239) notes that a key strategy in this approach is that 'the analysis is guided by the evaluation objectives'; this is certainly one aspect involved in this study as I sought to explore and evaluate the approach taken to teaching multiplicative reasoning through measures. Thomas (2006, pp.239-240) notes analytical strategies that are features of the general inductive approach, key features of this approach are summarised below:

- *multiple readings and interpretations of the raw data*
- *findings arise directly from analysis of the data, not from a priori expectations of models.*
- *the development of categories from the raw data into a model or framework, where the model contains key themes and processes identified and constructed by the evaluator during the coding process.*

Thomas (2006, pp.239-240)

The approach to data analysis applied in this study reflects these aspects of Thomas' (2006) general inductive approach; a priori expectations were not sought, and points of learning and themes are induced from data. After listening to initial recordings of task implementation in Cycle 1, a Behaviour -Emotion - Awareness framework was developed to support coding of these data. This framework, developed from an existing task-design/learning approach framework introduced by Mason and Johnston-Wilder (2006), is discussed further in Chapter 5.

Thomas (2006) notes that trustworthiness of data can be supported through coding consistency checks; two work colleagues were introduced to the coding framework and were asked to independently apply the codes to an extract of transcription. Their feedback suggested that, for the majority of codes, the coding framework was clear, and could be applied consistently, although some adjustments were made to clarify some particular codes, discussed further in Chapter 5.

Thomas (2006, p.244) also suggests 'stakeholder or member checks' to support credibility. In both cycles, learners were interviewed about their thoughts on the tasks. At the end of Cycle 2, a teacher was shown tasks and an overview of learner responses. This teacher, also the mathematics and numeracy co-ordinator, had been involved in interviews in both cycles and was teacher of learners involved in both cycles. Seeking the teacher views on the learner responses and data collected can be seen as a form of stakeholder checking.

As Cohen, Manion and Morrison (2018) caution, with qualitative data, there are many possible interpretations. Furthermore, Thomas (2006) acknowledges that it is possible for there to be more than one credible interpretation as these are influenced by the researcher's own experiences, values and assumptions. For transparency of approach, further detail of data analysis is discussed in subsequent chapters.

4.13 ETHICAL CONSIDERATIONS

It seems inappropriate to discuss ethical considerations at the end of a chapter on methodology because ethical considerations should permeate a research study, being evident at planning, implementation, analysis and reporting (BERA, 2018; Cohen, Manion and Morrison, 2018). However, as understanding of the research approach is necessary so that ethical considerations are clear within the context, explicit discussion of the ethical considerations for this study takes place within this final section of the chapter.

As noted by BERA (2018), a key ethic for any educational researcher is respect. At all points during this research study, this has been a central consideration as I have attempted to treat the school, its practitioners and learners with respect, for example, by exploring and attempting to faithfully report experiences and viewpoints, by treating participants respectfully when implementing interviews (e.g., timing practitioner interviews at times chosen as convenient by them and timing learner interviews as part of the learning time, not undertaken at times designated for play or other activity) and by treating learners respectfully when teaching. I believe it is also respectful of participants to acknowledge, in line with the social constructivist approach undertaken in this work, that any finding is based on *my* interpretation of data; though I have attempted to report what has been said or what happened faithfully, I do not claim that what happened or what might be found is truth or that any conclusion is the only possible conclusion.

An outline of key ethical considerations was shared, and approved, by the university ethics panel prior to commencement of the research study (Appendix F), and some key decisions in relation to that approved ethical practice are summarised below.

Voluntary informed consent was sought at the start of the study; firstly, from the school and then from practitioners and parents involved in both cycles. The school was initially approached because it supported educational research in partnership with the university. Although the school supported research, it was not assumed all practitioners would want to take part, and therefore informed consent was sought from practitioners (all teachers) taking part. Whilst it may be argued there was an expectation from the school management for practitioners to participate, it was reinforced that it was the choice of each practitioner whether they wished to be a part of the research. An example of this informed consent for practitioners can be seen in Appendix G, with similar letters being shared prior to each practitioner interview.

In the case of consent with learners (Year 2, ages 6 and 7), it was clarified that all learners in the Year 2 cohort could be potential participants and learners were identified, with teacher support, through the process of attaining parental consent (see Appendix H, as an example). The right to withdraw was reinforced as part of the consent forms. To seek learner assent, learners were asked whether they wished to take part, and were also informed of their right to withdraw, through sharing a 'traffic light' fan with red, amber and green cards. Learners were encouraged to show an amber colour if they were unhappy or anxious at any point and a red colour should they wish to stop and go back to their classroom (see Appendix I for the text that was read out to learners). The fans were available throughout task implementation in both cycles but were not used by learners to show amber or red at any point. At the start of Cycle 2a, one learner, after being involved in the initial introduction, appeared anxious, and verbally indicated a wish to withdraw; this learner was encouraged to return to class. Throughout this work, the school, practitioners, and learners are discussed in a way that should preserve their confidentiality, and they should be anonymous to anyone other than me as the researcher.

Further ethical guidelines by BERA (2018) including data storage and transparency have been followed throughout the study.

To conclude, this chapter has included discussion of the methodological framework, the research methods and tools employed, the approach to analysis and the ethical considerations, providing an overview of the approaches used in both cycles. Further detail, relevant to each cycle, is discussed in subsequent chapters.

CHAPTER 5: RESEARCH CYCLE 1: FROM DESIGN PRINCIPLES TO POINTS OF LEARNING

5.1 INTRODUCTION TO RESEARCH CYCLE 1

The focus of this chapter is the first cycle of task design, from design principles to implementation, including the analysis and reflection of tasks to support the learning of the multiplicative relationship through measures.

Research Cycle 1 was essentially an exploratory cycle of research to consider the following research questions:

Exploring the learning and teaching of multiplicative reasoning through measures: A design-based research project, in particular:

S1: What are teachers' and learners' prior experiences of teaching and learning multiplicative reasoning?

S2: What are learners' prior experiences of learning number and measures?

S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?

S4: What is the impact of learning multiplicative reasoning through measures on learners?

S5: What are teachers' and learners' views on teaching/learning multiplicative reasoning through measures using the materials developed?

An outline of the research activity in Cycle 1 is provided in Table 7. This can also be considered in relation to a timeline of research cycles (Appendix A) and an overview of all research activity, provided in Table 4 (Chapter 4, Section 4.5, p.91).

Research Question	Method of exploration in Cycle 1	Data collected
S1: What are teachers' and learners' prior experiences of teaching and learning multiplicative reasoning?	Phase 1: Observation of learning environment Semi-structured focus group interview with practitioners (n=4)	Observation notes Reflective notes Audio recording of interview
S2: What are learners' prior experiences of learning number and measures?	Phase 1: Observation of learning environment Semi-structured focus group interview with practitioners (n=4)	Observation notes Reflective notes Audio recording of interview
S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?	Phase 2: Initial trial, implementation and iteration phase in one school involving teaching, learner feedback through semi-structured interviews. (n = 8)	Audio recording of tasks Reflective notes Audio recording of semi-structured interviews with learners
S4: What is the impact of learning multiplicative reasoning through measures on learners?	Phase 2: Pre- assessment, implementation of tasks and learner feedback through semi-structured interview (n=8) *	Reflective and observation notes of pre-assessment Audio recording of tasks Audio recording of semi-structured interviews with learners
S5: What are teachers' and learners' views on teaching/learning multiplicative reasoning through measures using the materials developed?	Phase 2: Learner feedback through semi-structured interview (n=8) **	Audio recording of semi-structured interviews with learners

TABLE 7: CYCLE 1 RESEARCH ACTIVITY

*A post-assessment was planned but could not take place due to school closure (COVID-19 pandemic)

**A semi-structured interview with practitioners was planned but could not take place due to school closure (COVID-19 pandemic)

<u>Participants in Cycle 1 were:</u>

Four Foundation Phase (Nursery – Year 2) teachers, who participated in a semi-structured interview. These are identified as Teachers 1 to 4 in any discussion.

Eight Year 2 (ages 6-7) learners. Learners were identified through parental consent (those learners with parental consent). The class teacher described the learners as being average to higher attaining in mathematics within the class. These learners took part in pre-assessment and task implementation activity and are identified as Learners 1-8 in all discussion.

5.2 DESIGN PRINCIPLES

Van den Akker (2013, p.67) provides a set of heuristic statements for considering design principles:

> *If you want to design intervention X [for purpose/function Y in context Z]*
> *then you are best advised to give that intervention the characteristics C1, C2,…, Cm [substantive emphasis]*
> *and to do that via procedures P1, P2, …, Pn [methodological emphasis]*

because of theoretical arguments T1, T2, …, Tp

and empirical arguments E1, E2, …, Eq

(Van den Akker, 2013, p.67).

Though Van den Akker's (2013) approach should not be seen as formulaic, and certainly cannot guarantee success, the format provides a reminder of the need to consider the substantive elements (what should happen) and the methodological elements (how that might happen), whilst also paying attention to the theoretical and empirical arguments. Whilst I do not apply the syntactic structure suggested by Van den Akker (2013) in articulating my design principles, I do consider theoretical and empirical arguments for the principles, based on literature, and also articulate important characteristics and procedures.

It is important to note that if, in design research, as Cobb *et al*. (2003, p.9) argue, theory is 'doing real work' then it seems right that design principles may develop and evolve during iterations. Indeed, Anderson and Shattock (2012) note the evolution of design principles as one of the features of design research.

Design principles in Research Cycle 1 were informed predominantly by Davydov (1990; 1992). I also draw on work by Eriksson and Lindberg (2016) and Eriksson and Jansson (2017), who analyse Davydov's work and discuss design principles for tasks inspired by his work on learning activity (Davydov, 2008).

As discussed in Section 2.3, Davydov (1990) believed that all concepts in school mathematics are scientific and that, from the outset, mathematics should be taught in a way that develops scientific 'theoretical' concepts and awareness of the concept itself. He believed that children should progress from abstract to concrete, becoming aware of the scientific concept first, and then specific concrete examples of it. Davydov (1990) also believed in what Schmittau (2003, p.232) calls 'genetic analysis'; that is the teaching of a concept should reflect the way the concept has evolved:

> *Consequently, instructional subjects must include, not ready-made definitions of concepts and illustrations of them, but problems requiring the ascertainment of the conditions by which these concepts originated.*
>
> (Davydov, 1990, p. 162).

Davydov's (1992) genetic analysis of the concept of multiplication asserts that it should be seen as a change in the use of units. As discussed in Section 2.8, the concept of multiplication has developed through situations where it is inconvenient to use a particular unit for counting or measure; this may be because the unit is too small and therefore its use would be inefficient. In these situations, the unit can be altered, for example taking a larger unit which has a numerical relationship to the smaller. Then, through using that unit to count or measure, a number can be obtained which expresses the relationship of the original object to the new (intermediate) unit. As there is a relationship between the intermediate unit and the original object, a multiplicative relationship can be established. Davydov (1992, p. 12) asserts that the premise of multiplication is 'the refusal of directly counting out one by one all the elements of a calculated set' and his critique of typical approaches to multiplication in schools, through a focus on discrete number situations rather than measure contexts, is that counting one by one is nearly always possible.

Davydov (1992, p.21) summarises these views into a 'system of instructional situations in introducing multiplication'. This involves:

- taking problems which require determining the relationship of some object to a given unit of count(measure) and revealing the unsuitability or impossibility of a direct application of this unit;
- replacing the unit and determining the relationship of the large and small units (finding the multiplicand);
- performing the count with the new unit (finding the multiplier);
- composing a formula for the product;
- determining the result through the use of a table or by means of addition (arriving at a solution to the problem by indirect means).

It is noteworthy that Davydov (1992) provides an overview of the multiplicative reasoning tasks carried out with children and that he comments that tasks were 'repeatedly tested under experimental conditions' (Davydov, 1992, p.22). As discussed in Chapter 4, it is reasonable to conclude that Davydov's (1992) work (taking place in the 1960s) was an early form of design research, with instructional principles, tasks which were reviewed and refined, and quite detailed notes for teachers explaining how the tasks can develop.

In developing the principles for Phase 1, Davydov's (1992) work was analysed. However, Davydov's tasks were developed in experimental schools, where learners had already been introduced to the notion of units. As discussed in Section 2.3, Davydov (1990) argued that children should be introduced to the theoretical, or abstract, concept of number before working with concrete examples of it, and thus number was introduced through measure activities that were set up to necessitate a unit that could be counted to allow quantification.

A central theme of the teaching and learning activities set up within the experimental schools run by Davydov and his team is *necessity*; problems are set up which are too difficult or inefficient and so this necessitates a new way of working (e.g., Schmittau, 2003 and 2010; Davydov 1990 and 1992; Venenciano 2017). Eriksson and Lindberg (2016), also note the notion of restriction as being a central idea in Davydov's approach to tasks; problems are shaped so that familiar tools or solutions cannot be used.

Davydov (2008, p.85) asserts that 'practical, object-orientated productive activity – labor [sic] – is the basis of all human cognition'. It should be noted here that Davydov (2008) strongly outlines his belief in cultural-historical activity theory, seeing activity as being more than being 'active', but as encompassing motives, goals, tasks and operations. Furthermore, he saw labour activity as being social, with communication being central to this. Davydov (2008) outlines that learning activity, seen as the leading activity of children aged 6-10 years, should involve the solution of problem-based cognitive tasks to aimed to support the development of concepts. Whilst this work does not adopt a full activity theory approach, tasks will be set up as problems for the learners and they will be invited to share and discuss ideas for the solution of the problems. The approach of encouraging social interaction, will reflect the social constructivist theoretical perspective taken in this study, discussed in Chapter 3.

In a pilot study exploring student engagement in Davydov's (2008) learning activity, Eriksson and Jansson (2017) applied the theoretical principles of Davydov's learning programmes in the design of tasks and classroom activities to support the development of algebraic understanding. They (p.259) reinforce the importance of learner agency within learning activity, noting that 'the teacher can plan for learning activity to occur, but its realisation is dependent on the development of students' joint agency in the process'. Eriksson and Jansson (2017, p.261) helpfully introduce the idea of a 'key task'; an open-ended task which can be a starting point for learning activity. They also discuss 'warm-up tasks' (*ibid*); these were not necessarily open-ended tasks, but tasks suggested by teachers in the project to support recall of previous learning. Through their trialling and subsequent analysis of the tasks, they developed the following criteria for tasks:

- *A task should be designed to enable the joint extension of the content via unfolding, rather than several small, disparate items*: here they reinforce the importance of student agency in terms of possible courses of action, with the possibility of a task unfolding in several different directions.

- *The design of the task and its development should be related to what the students do or do not do and know or do not know. How the task develops is not solely determined by the teacher but by the teacher in collaboration with the students*: they reinforce here that a task can develop as the teacher attends to learners' responses and interactions and cite an example of how a warm-up task was developed into a key task.

- *The task is designed to introduce a situation containing a problem that hinders the students from using familiar solutions but is still intriguing enough for them to try to solve using joint action*: discussed above

- *The tasks must contain problems that are content-rich and culturally and historically relevant*: here they reflect the attention to cultural-historical activity theory, suggesting that a task being culturally and historically relevant means it involves mediating tools, identifying the most powerful tools as language and symbols. They identify measurement as a source of content, as an idea developed in Davydov's work.

- *Students can discern the specific core principle of a concept and its conceptual relations, symbol, or model by identifying concrete instances of the theoretical knowledge*: here they suggest that, within a task, learners are able to transfer between a theoretical concept and specific instances of it.

(Eriksson and Jansson, pp.266-269), with my notes following each point.

Eriksson and Jansson's (2017) work is particularly informative for this work because it explores a learning activity approach and Davydov's approach to mathematics learning in an educational system (Sweden), where the whole programme has not been used. Like this study, their research explores how Davydov's approaches could support learning (albeit of a different concept) within such a system. Furthermore, the notion of warm-up tasks and key tasks, suggested by practitioners involved in Eriksson and Jansson's (2017) study, is also applied. The use of warm-up tasks could act as a bridge into a key task, to support the recall of prior learning that could be exploited within key tasks. Hence, for Research Cycle 1, the following design principles are applied:

1. Through the context of measure, the task should support the development of the theoretical concept of multiplication involving a change in the system of units.
2. The task should be set up as a problem, where counting in ones is restricted, inefficient or impossible.
3. The problem, with the facilitation of the teacher, should invite social interaction, discussion and possible debate in order to suggest possible approaches to finding a solution.
4. The task should facilitate transfer between the theoretical concept of multiplication as a change in units, and particular instances of this.
5. The task should be able to unfold in a range of possible directions, according to learner agency and teacher facilitation.
6. The tasks should involve a range of measures contexts.

FIGURE 13: DESIGN PRINCIPLES RESEARCH CYCLE 1

Warm up tasks were also used, which incorporated mainly principles 1 and 4.

5.3 TASKS: DIFFICULTY AND CONTEXT

As introduced in Section 4.3, Burkhardt and Swan (2017, p.181), through their work on task design, identify aspects of tasks that affect the difficulty, or accessibility of tasks. These include:

- complexity (aspects such as number of variables, modes of presentation of information)
- (un)familiarity (similarity to a task that might have been practised previously)
- technical demand (the level of mathematics required)
- student autonomy (the level of guidance from teacher or from structuring or scaffolding of task)

Burkhardt and Swan (2017) note that consideration of student performance in relation to tasks needs to take the balance of these factors into account.

In discussing the context of mathematics tasks, van den Heuvel-Panhuizen (2005) notes that the term 'context' can be interpreted in two different ways: the learning environment and the characteristic of the task presented to students. In this study, the learning environment is one where the learners are encouraged to discuss and interact with each other and with me, in line with a social constructivist approach. The characteristic of the task presented to the learners, is defined by van den Heuvel-Panhuizen (2005, p.2) as 'referring either to the words and pictures that help the students to understand the task, or concerning the situation or event in which the task is situated.'

As already established, Davydov (1992) argues that the theoretical concepts of number and the multiplicative relationship can be taught through measures contexts involving continuous quantities. Examples provided by Davydov also have some story around the measures (e.g., feeding rabbits). Van den Heuvel-Panhuizen (2005) explains that, in the context of Realistic Mathematics Education, developed in the Netherlands from Freudenthal's views of mathematics learning (discussed in Chapters 2 and 4) the term 'realistic' should be interpreted as 'imaginable' to the learners involved; this can include fantasy or even the formal world of mathematics, but the aim is to make mathematics 'real' in the mind of the learners and ensure learners can experience the mathematics as real for themselves.

The words and pictures used to introduce a task can, however, also obscure mathematical purpose (Clarke and Roche, 2018). For example, van den Heuvel-Panhuizen (2005) suggests that real situations can be simplified in a way that makes them unrealistic, or that a situation might be used that is culturally unfamiliar to students. Furthermore, Boaler (1993, p.14) suggests that 'real world' problems should be those that arise out of learners' interactions with the environment, rather than being problems that have been oversimplified to make them seem real, or that have been extracted from an adult's world.

As noted in the design principles, all contexts for tasks were measures contexts. For some tasks, further contexts were used to support a possible reason for undertaking the task (e.g., to give an imaginable reason for wanting to know how many little cups will be contained within a big jug of liquid).

5.4 DEVELOPMENT OF TASKS: PHASE 1 SCHOOL-BASED RESEARCH

Alongside literature, the development of tasks was informed by two main sources of initial research activity (see Table 4, p91) :

- Phase 1 research in school: Observation of learning environment and focus group interview with practitioners: information and discussion
- Pre-assessment activity with learners

Phase 1 research, introduced in Chapter 4, was exploratory, with the aim of finding out how teachers planned for the teaching of both multiplicative reasoning and measures, to consider what learners within the school might typically experience. This included a 'learning walk' (Estyn, 2021) style observation of the Foundation Phase (Nursery to Year 2) learning environment over a morning when mathematics was taking place in all classes and a semi-structured group interview with four practitioners, all teachers, who worked in the Foundation Phase setting (Nursery to Year 2).

As discussed in Section 4.7, within the learning walk observation, there was no intention to observe or judge specific mathematics lessons, rather to consider what learners might typically be used to in terms of mathematics provision. A structured observation schedule was used (Appendix C). The observation of the learning environment took place a few days before a face-to-face semi-structured group interview, which is discussed later in this section.

It is important to consider Phase 1 in relation to the curriculum context at that point. The Foundation Phase Framework (revised version WG, 2015a), statutory since 2010 and up to

2022, set out requirements for learning for 3-7 years old in Wales. It outlined the need for 'a balance between structured learning through child-initiated activities and those directed by practitioners' and emphasised the role of play as a 'serious business' (WG, 2015a, p.4). Furthermore, there was an emphasis on experiential learning and the use of stimulating environments both indoor and outdoor (WG, 2015a, p.3). In terms of mathematical development, the framework outlined that children 'develop their skills, knowledge and understanding of mathematics through oral, practical and play activities. They enjoy using and applying mathematics in practical tasks, in real-life problems, and within mathematics itself.' (WG, 2015a, p.27). The framework was revised in 2015, to include yearly expectations for mathematics and numeracy, as part of a national programme to raise standards in these areas. An example of statements related to the multiplication and division in Year 2 is provided in Table 1 (p.8).

The observation of the learning environment suggested that learners' experiences of mathematics very much reflected Foundation Phase pedagogic approaches and curricular expectations for mathematics and numeracy at the time; the Foundation Phase environment was organised with direct access to outdoor provision from each classroom and this included areas for play with water and sand. Mathematics manipulatives such as Numicon, Unifix and Base 10 equipment were readily available within classrooms. Teachers also used physical resources such as the counting stick and digital resources (such as the hundred square on an Interactive Whiteboard) to support counting in equal steps. There was evidence of learners using the outdoor and indoor environment for structured mathematical measuring activity and free play; for example, learners in Year 2 had been using the water tray and measuring cylinders to measure capacity, learners in a Year 1 class had been challenged to find out the height of superhero images in Unifix cubes, whilst an interactive display in the reception class focussed on placing 'long, longer, longest snakes' in positions. Such activity reflected curricular expectations of measures as a development from direct comparisons to use of non-standard units then progressing into using standard units (WG, 2015a, p.33). In lessons observed, learners typically experienced a whole class introduction, focusing on language development for mathematics or rehearsal and development of key teaching ideas (e.g., using the Interactive Whiteboard) and then subsequently worked in small groups (e.g., 6-8 learners) for mathematics, supported by a practitioner (teacher or teaching assistant). Mathematical discussion could be heard taking place (e.g., between practitioner and learners in whole class discussions around counting in steps using a counting stick and between learners in Reception using Unifix to measure).

The interview, which took place a few days later, allowed for further consideration of what was observed and specific exploration of teaching approaches, including the teaching of number, the multiplicative relationship and measures. Appendix J contains the first part of the semi-structured interview that took place with practitioners. The remaining conversation, though transcribed, is not included because it involves organisational aspects around date setting for pre-assessment and how resources would be used. In extracts of the interview discussed within this chapter, the following key (Figure 14) can be used to identify those speaking.

Key:

I – interviewer (researcher)

T1 – teacher 1

T2 – teacher 2

T3 – teacher 3

T4 – teacher 4

(teachers numbered according to order in which they introduced themselves around table)

FIGURE 14: KEY FOR PRACTITIONER INTERVIEW

The interview with teachers reinforced that the way in which the multiplicative relationship and measures were planned reflected the curricular expectations at the time. The approach to the multiplicative relationship, and in particular multiplication, reflected the view of multiplication as repeated addition with recognition of the role of resources such as

Numicon and coins, and contexts such as pairing socks or counting animal legs, in supporting this. One teacher focused on the word 'commutative' as being important. Although this term was not mentioned in the Foundation Phase (or indeed the Key Stage 2) curriculum at the time (e.g., WG, 2015a), it was mentioned in draft descriptions of learning for the Mathematics and Numeracy Area of Learning and Experience (AoLE) within the Curriculum for Wales (WG, 2021), being re-developed and shared for consultation at the time, as discussed in Section 1.3. This teacher, as the mathematics co-ordinator, might have been involved in professional development and/or feedback activity related to this.

Discussion around division suggested a focus on the partitive approach (discussed in Section 2.6), with the word such 'sharing' being emphasised as key:

T1: *Sharing, you always say for division, sharing…*

However, it should be noted that, though this partitive perspective of division (division as 'sharing') seemed to be emphasised by one practitioner, it is not assumed to be the only perspective of division experienced by learners. Preceding T1's comment (above), the following exchange occurs.

I: *And lots of you mentioned earlier…*

T4: *Sets of…*

I: *Sets of*

T2: *Sets of, groups of, piles of, I just, why I say it in so many different ways..*

The idea of grouping applies to multiplication but can also apply to a quotitive perspective of division (e.g., how many groups of four are in twelve, as discussed in Section 2.6) and the Year 2 Foundation Phase Framework (WG, 2015a, p.31) requires that learners 'begin to link multiplication with simple division, e.g., grouping and sharing in 2s, 5s and 10s'. Therefore, it is reasonable to conclude that learners may typically be introduced to division through sharing contexts (partitive nature of division) and then explore the quotitive aspect of division through making links with multiplication.

Discussion around measures also reflected curricular expectations in which measuring is seen as an important skill to be mastered, moving from comparative measures to understanding non-standard units and then understanding standard units.

In relation to approaches to teaching mathematics, one of the Foundation Phase teachers, also the mathematics co-ordinator, discussed how the school used a 'Concrete, Visual, Abstract' approach; this is another term for the 'Concrete - Pictorial - Abstract' approach discussed in Section 2.3. This was reinforced by the other teachers with one commenting how it was important to offer equipment such as Numicon and Base 10 (Dienes) to all learners. Thus, there was reference to specific concrete materials that might support learners in understanding mathematical structure. In addition, teachers also reinforced the importance of 'real-life' contexts; this, again, reflects curricular expectations, with numeracy seen as 'the application of the skills learned in mathematics in across-curricular, real-world way, and not purely about the skills themselves' (WG, 2013, p.20).

In relation to mathematical 'behaviour' that might be praised, one teacher referred to praise for effort, 'having a go' and not being afraid to make a mistake, whilst another discussed the use of vocabulary. When asked about possible learner reactions to collaborative challenges, one teacher suggested learners enjoyed working together in mixed attainment groups, whilst one teacher reflected:

T2: It's odd that I do tend to focus, if it's a number problem, they're in their sets but if it's when we do time and measure and everything they're put into mixed ability, until I know that 'right who can go this far with the clock' so they may become sectioned to push that…I hadn't really thought about it.

The comment above suggests a possible view that a context such as measures could be taught in mixed attaining groups, until the mathematics that may evolve from it needs additional support or challenge.

In addition, this teacher also referred to the approach to organising the learning in Year 2:

T2: *Year 2 is more class based. They will be taught and then the work is differentiated and I go around them all, but I focus on my less able unless I need to focus on other groups depending on what they're doing.*

It is not clear from the comment whether the intention of support is influenced by particular mathematical knowledge needing to be taught, or a need to support certain learners should certain aspects of a task become difficult and I did not explore this further within the interview.

It is noteworthy that conversations around approaches in Year 2 seemed, to some extent, to be influenced by an external influence to meet standards. This reflects the national context at the time, and a drive to raise standards in mathematics and numeracy through the introduction of a Literacy and Numeracy Framework (LNF) and national tests in literacy and numeracy for Years 2 – 9 (WG, 2013). For example, in the comment below the teacher discusses the need to reinforce ideas and presents an approach to support understanding of what standard units are used for different measure contexts.

T2: One of the Year 2 questions, and it's because of the LNF. It's not because of the LNF, but that is why we have the major push is…Today we were doing measurement, we're doing measurement this week, but the children need to learn if I'm measuring water, it means I need a measuring jug and I measure in litres and millilitres, if I'm measuring time I need a clock or a stopwatch and they get them so muddled up because the language is so so similar. Centimetre, millimetre, millilitre and it's so the drumming drumming drumming and that continual…Mrs G, she's killer G* and so they know if you're weighing killer G* always weighs to just get that K G because the language is so similar for them, it's very very difficult, but then they've got to have the practical to know that Mrs G always weighs whether we're cooking and doing real things or measuring dinosaurs or plastic animals or what not.*

T2: If they know quarter past, half past, quarter to they can't keep on doing it so they've got to go on to all the past times and then…

Measures as a context for learning the multiplicative relationship was not raised explicitly by the practitioners, although one teacher did suggest that measure can be used to focus on number.

T4: With measure as well, you know, we'd use things like Duplo, you know, to measure length initially and giving them the choice as well, so you know saying we need superhero capes, what do you want to use to measure, and if the cubes are smaller, well let's see what the difference is, and just getting them to use lots of non-standard units first of all .

The comment by this teacher recognises how the relationship between the size of a unit and the resultant number (referent) used to quantify a measure can be part of teaching and learning activity.

As discussed in Section 4.8, it is possible that interviewees were saying what they thought I may have wanted to hear, however, what was said seems to confirm what was seen in the learning walk observation; for example, there was evidence of learners being asked to measure using a range of non-standard units.

To conclude, the Phase 1 activity confirmed that I could expect to be planning for Year 2 learners who were typically likely to have:

- experienced measuring with non-standard and standard units in a range of contexts, with an emphasis on standard measures in their Year 2 experience
- experienced multiplication as repeated addition and have some awareness of the commutative nature of multiplication
- experienced division as sharing, though are also likely to have some experience of grouping
- some experience of working in groups to approach problems (although the experience of mixed attainment groups may be less familiar as they progress through Foundation Phase)

but who may be unlikely to have:

- experienced measures as a context for learning number relationships, and in particular, the multiplicative relationship
- experienced the multiplicative relationship as involving a 'change in unit'

It was not entirely clear how familiar learners would be with collaborative challenges in mathematics, in which learners might need to debate different suggestions, or situations in which familiar approaches were not efficient or practical, requiring a new approach to be found. On reflection, this is an area I could have explored further within the interview.

5.5 PRE-ASSESSMENT TASKS

The main aims of the pre-assessment tasks were:

- to explore learners' familiarity with the concept of a unit; as discussed in Sections 2.3 and 2.8, this is identified by Davydov (1990; 1992) as being central to the concept of number, with the multiplicative relationship involving a change in the unit. It would therefore be important to establish learners' familiarity with units within quantification.
- to explore learners' experiences of multiplicative reasoning and their application of multiplicative relationships; though the observation and interview provided some useful information about what I might expect, the pre-assessment would provide more insight into learners' experiences of specific multiplicative relationships and aspects such as their ability to use repeated addition to support solutions.

- to explore learners' reactions to tasks which are set up as problems in which they would be invited to share and discuss ideas.

Pre-assessment tasks were informed by Davydov's work (1990, pp.67-68) in which he outlines 'assignments' presented to learners to establish their concept of number as a relationship between quantity and unit (see Section 2.3). Also in Section 2.3, it was noted that Moxhay (2008) trialled a Davydov and Elkonin curriculum in Maine, US. Moxhay's (2008) paper presents results of assessments undertaken with learners (aged 6 to 8), informed by the curriculum of Davydov and Elkonin, and designed to assess learners' scientific, or theoretical, concept of number. As Moxhay was a translator for Davydov's (2008) book, it is likely he could access the Davydov and Elkonin curriculum through his expertise in the Russian language. As noted in Section 2.3, much of the work of Davydov and Elkonin appears unavailable in English and so Moxhay's (2008) paper was particularly useful for the pre-assessment tasks because the tasks in the paper focused on assessing ideas around units, and the size of a unit and its referent number being used in quantification, with similar aged learners.

The pre-assessment tasks were implemented with the eight Year 2 learners who had been identified, with support from their class teacher, as having parental consent for the research in Cycle 1. They were described as being 'average to high attaining' in mathematics by their teacher. The group included a mix of genders, a learner with a diagnosis of autism and two learners with English as an additional language.

The pre-assessment tasks were implemented a week before the main tasks were to be introduced, in the space used for all work with learners throughout the study. The space used was a central area from which all Foundation Phase classrooms could be accessed directly, which had open areas and group tables for breakout/small group work outside of the classrooms.

An overview of the pre-assessment tasks, their purposes and a summary of their implementation is provided in Table 8.

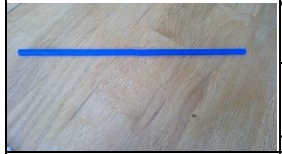
Pre-assessment task	Summary of task	Design notes	To explore	Notes on implementation
PA1 	Measuring length of a heavy object with a restricted number of plastic rods so that its length could be reproduced in a different area of school.	Adapted from tasks in Moxhay (2008, p.7) and Davydov (1990, p.67)	Understanding of need for a unit to be of equal size and repeated without gaps	Completed in pairs. Learners did not consistently recognise the need to use the same unit repeatedly (1 of 4 pairs iterated), 3 pairs attempted to use different units.
PA2 	Reproducing the amount of liquid in one container to ensure the same amount of liquid in different container (with the use of a small or large cup)	Adapted from task in Moxhay (2008, p.10)	Understanding of units and equality of liquids	All groups completed the task successfully, repeating use of larger cup showing understanding of unit being used (e.g., 2 and half large cups).
PA3 	Measuring the same length (how far a car travels along floor) twice using different sized straws, where one straw is twice the length of the other	Adapted from task in Davydov (1990, p.68)	Understanding of relationship between number and the size of unit	Learners used both straws independently of their relationship
PA4 	Identifying how many of a set of 4 Unifix cubes would be equal to a length of 20 Unifix cubes, writing an associated calculation and predicting another calculation	Adapted from task in Davydov (1990, p.67)	Understanding of unit	This was scheduled to take place as individual learner interviews. Time restraints meant that this occurred with only 4 learners. All 4 learners approached task initially using addition (e.g., counting all 20, counting 4 then counting another 4 etc. to get to 20) – suggesting they saw the unit as the Unifix cube rather than the set of 4

TABLE 8: PRE-ASSESSMENT TASKS

Data collection in the pre-assessment was affected by two main factors: the space in which the pre-assessment occurred and time. Due to the large open space being used, and learners spread in this space for the first three tasks, audio quality was variable. A time restriction meant that not all learners could be individually interviewed. Nevertheless, the reflective notes recorded straight after the pre-assessment, with audio recording as a cross-reference, allowed for conclusions to be drawn that could inform the task design and implementation.

Through the pre-assessment tasks, it was evident that learners showed some understanding of the concept of a unit, though this varied across tasks. In tasks PA2 and PA3, learners were successful in using units to measure length and capacity. However, understanding of the necessity for equality in the unit was less secure when the units available for measure were more restricted, as indicated in the results of PA1. In this situation, only one pair iterated (repeatedly used) the restricted unit; the other pairs chose to try using a different unit from other objects available within the space and used objects that were unequal in size. This suggested a need to reinforce the idea that units being used needed to be equal.

In tasks PA1 and PA2, the requirement to replicate a quantity needed learners to consider the magnitude of the quantity, yet replication can occur with one-to-one matching rather than explicit measuring; for example, in PA1 learners using objects of unequal size and then moving these objects to where the length was to be replicated by one-to-one matching could achieve replication without needing to communicate a specific measurement in any unit. Similarly, in PA2 capacity could be replicated through one-to-one matching of quantities poured out into cups, even if the quantity in different cups was not equal. For example, Figure 15 (below) shows how a pair of learners replicated quantities.

FIGURE 15: REPLICATION OF QUANTITY

In both tasks, learners typically communicated in terms of units (e.g., in PA2 they talked of the liquid being ‘two and a half big cups’, or in PA1 ‘7 book lengths’) but the task design and implementation could have been improved through restricting the potential units available for measure and including a requirement for the learners to communicate the measurement prior to replication. Nevertheless, the learners’ responses to the tasks indicated that the concept of a unit within measure, and the need for it to be equal, required reinforcement.

Both tasks PA2 (in which learners had choice of a small cup or a large cup) and PA3 (which required learners to measure a length using two different length straws, where one straw was half the length of the others) indicated learners had some understanding of the size of the unit being inversely proportional to the referent number (e.g. Learner 7 was heard to say that using the small cup would take longer than using the big cup in PA2). However, it is noteworthy that learners did not make explicit links with the half-double relationship between the resultant measurements. Task PA3, in particular, could be further developed for incorporation into the future activity, with an explicit focus on this particular multiplicative relationship, whilst also reinforcing the relationship between unit size and the referent number in a measure.

In PA4, which was carried out with half of the group, all learners used the additive relationship at first. When asked how many of the smaller tower (a block of 4 Unifix cubes) would fit into the larger tower (a block of 20 Unifix cubes), they counted the Unifix towers, working out that there was a difference of 16 and then tried to work out how many groups of 4 would be within the 16. This suggested that learners were seeing the unit as the Unifix cube (rather than treating the group of 4 Unifix cubes being the unit). It could be argued the use of Unifix cubes impacts the choice of working in that size unit, though care was taken to use the same colour Unifix cubes within each tower to try to support learners to consider a unit as being 4 Unifix cubes.

The pre-assessment tasks showed that learners had experience of using non-standard units in measure and were showing awareness of the use of standard units of measure (e.g., learners discussed how they might use rulers and tape measures for PA1 and measuring jugs for PA2). Learners appeared less confident in situations where unit usage was restricted, preventing one-to-one counting.

Learners were enthusiastic and positive in working together and reported that they had enjoyed the tasks. They were particularly enthusiastic about the task involving liquids, commenting that they liked working with liquids and it made them think. Although it is possible that learners were reporting what they believed I may want to hear, their excitement at working with the materials was visible and audible in the way they reacted, for example, making exclamations when liquids were produced.

Overall, the pre-assessment results confirmed the expectations of learners' experiences identified from Phase 1 activity. The concept of a change in unit was not explicitly explored within the pre-assessment tasks, and though this would be the focus of planned tasks for implementation in Phase 2, aspects such as relationship between unit size and referent number (as in PA3) could be further incorporated into tasks, to support multiplicative reasoning within the tasks being developed.

5.6 PHASE 2: TASK DESIGN AND IMPLEMENTATION

In addition to using the design principles (p.122) as a guide, the main source of reference for tasks developed for Phase 2 was Davydov's (1992, pp. 20 -38) description of tasks to support the concept of multiplication as a change in the unit. The Phase 1 school based research suggested that, within measure tasks, there needed to be reinforcement of ideas around units themselves and their relationship with referent numbers. Furthermore, I wished to incorporate a range of measure contexts in the tasks; in particular, measure experiences involving length, capacity and mass, as these were the contexts with which learners were likely to be most familiar, based on the curriculum expectations and Phase 1 exploration.

Most tasks were adaptations of tasks outlined in Davydov's (1992) work, with the addition of warm-up tasks to reinforce ideas about units and measures. Schmittau and Morris (2004) discuss implementation of a Davydov and Elkonin curriculum in the US with grade 1 (aged 6 – 7) learners, with a particular focus on algebraic thinking and additive reasoning. They note tasks involving Cuisenaire rods, using their metric lengths, and this discussion informed the use of Cuisenaire rods for some tasks (Schmittau and Morris, 2004, p.73). There was little available literature on use of mass and weighing scales as a context, and so I designed the mass task independently.

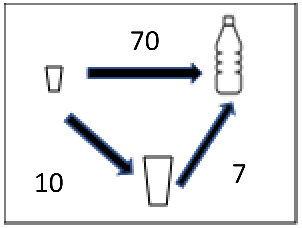
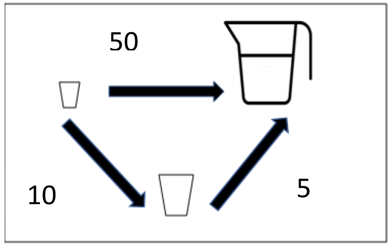
An overview of each task and the sequence in which they were used is provided in Table 9. Tasks are numbered 1-4 according to the day on which they took place and lettered a-d according to the order that day. As discussed in Section 5.3, Burkhardt and Swan's (2017, p.181) aspects of task difficulty have been used in consideration of the tasks; these aspects were considered at the point of task design, but are noted explicitly in a retrospective manner, following task implementation. The schematic, discussed by Schmittau (2004), and introduced in Section 2.8, has been used to support reader understanding of multiplicative relationships involved. The schematic was used in tasks, but this was modelled by me as the teacher and was used for display, rather than an expectation of it being used independently by the learners. Given the limited time I would be working with the learners, and because they had been introduced to multiplication and division notation previously, I chose not to introduce an additional expectation for recording, though some learners were invited to try to use the schematic, if it seemed appropriate.

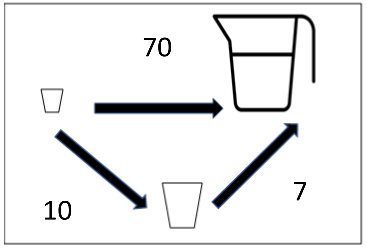
The tasks took place at the same time each day, over four consecutive days. Tasks were implemented in one part of an open plan area outside the main classroom, with the eight participant learners (Learners 1 to 8). This area had tables and chairs which could be rearranged into groups or paired working spaces and was close to a source of water. As discussed in Section 5.5, the area was a familiar space to the learners, frequently used for small group work outside the Foundation Phase classes. On two of the four days, other nearby parts of the open plan area were used by teaching assistants and groups of learners, but this did not appear to distract the learners. Also, on two of the four days, student-teachers were visiting the school conducting observations; on occasions the student-teachers observed the tasks and spoke to the learners. This, again, did not appear to distract the learners. As learners were in close proximity to each other (even when spread across the area), though several dictaphones were used, only one dictaphone was typically needed to provide complete audio. The audio recording picked up some background conversation (i.e., other users of the space) but most conversations were audible and distinguishable. However, a disadvantage of using one recording device was that it could be difficult to distinguish all conversations when learners worked in pairs, with those furthest away being more difficult to hear. Occasionally it was also difficult to distinguish particular learners talking.

Each task began with the group of eight learners, with learners splitting into pairs when they had agreed approaches. As teacher and researcher, I made reflective notes each day about the tasks that had been implemented.

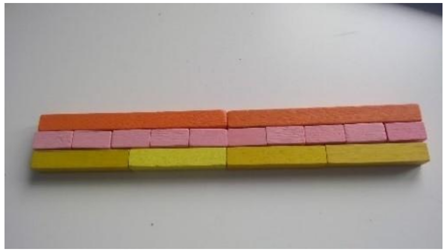
On Day 5, learners were interviewed individually, in the same space in which the tasks took place. They were shown photographs of the tasks they had undertaken and asked questions (see Appendix K for an overview of questions) to explore their perceptions of the tasks and what they think they might have learnt.

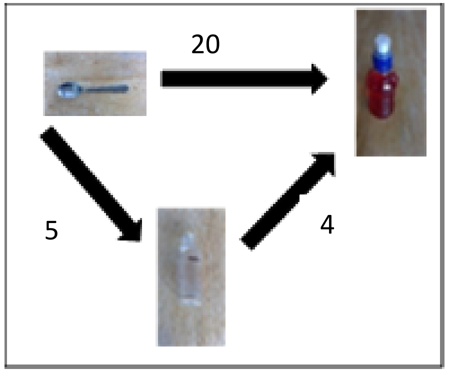
Task	Summary of expected learner activity	Purpose	Design notes
C1.1a Big and little cups Question: Look at these two cups and this container of liquid. When I used one of these cups, I needed to use it 12 times to make the same amount of liquid and when I used another of the cups, I needed to use it twice to make that amount of liquid. Which cup do you think I needed to use 12 times, and why? 	Predicting and explaining to each other, in pairs, which cup would be used 12 times and which cup would be used 2 times to measure a given capacity of liquid. Discussing reasoning as a group.	To reinforce that the size of the unit is inversely proportional to the referent number in a resulting measurement.	**Warm-up task** Although this notion was incorporated into pre-assessment tasks PA2 and PA3, learners had not been asked to explain the relationship between unit size and referent number in a measure. Complexity: Not complex (Un)familiarity: Building on pre-assessment and interview with practitioners Technical Difficulty (level of maths): Not difficult Autonomy: Teacher and group, with talking partners
C1.1b Straws Question 1: I have got some string and some straws. If I told you I needed to use the orange stripey straw 5 times to make this length of string, how many times do you think I would need to use the green stripey straw? Question 2: If I give you this string and these green straws, can you predict how many orange straws you would need? 	Measuring a length of string using two coloured straws, where one straw is half the size of the other. Predicting the number of small (half straws) needed when given the number of larger straws needed.	To reinforce that when a unit is changed the referent number in a measure changes. To establish that if there is a multiplicative relationship between the units there will be the same relationship between the resultant measurements.	**Warm-up task** to build on findings of pre-assessment. Although learners had undertaken a similar task for pre-assessment (PA3) the multiplicative relationship between small and large straws needed reinforcement. Complexity: Not complex (Un)familiarity:Building on pre-assessment Technical Difficulty (level of maths):Not difficult Autonomy: Teacher and group, with talking partners

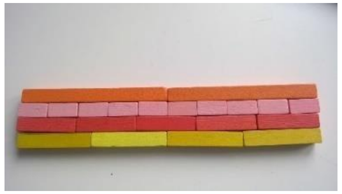
Task	Summary of expected learner activity	Purpose	Design notes
C1.1c Bottle and cups Question: If I tell you that one rabbit needs this amount of water (pointing to a tiny cup) each day, how many rabbits could I feed with this amount of water (pointing to a bottle)? Could you find an efficient way of working this out? 	Capacity: Identifying how many of a very small container make up a large jug, with the introduction of an intermediate larger cup	To introduce an intermediate unit To introduce a multiplicative relationship through an intermediate unit: 	**Key task** First explicit introduction of an intermediate unit. Task based on Davydov's (1992) discussion of introduction of multiplication as a change in unit. Complexity: Introduction of idea of intermediate unit and the multiplicative relationship (Un)familiarity: Unfamiliar – task involves introduction of new idea Technical Difficulty (level of maths): Multiplicative relationship 10 x 7 = 70 Autonomy: Teacher led demonstration
C1.1d Jug and cups (i) Question: If I tell you that one rabbit needs this amount of water (tiny cup) each day, how many rabbits could I feed with this amount of water (in jug)? 	Capacity: Identifying how many of a very small container make up a large jug, with the introduction of an intermediate larger cup	To reinforce use of intermediate unit (same relationship between intermediate and small unit as 1c) 	**Key task:** To reinforce notion of change in units where counting in ones is restricted. Complexity: Introduction of idea, change in large container to 1.1c (Un)familiarity: Builds on task 1.1d Technical Difficulty (level of maths): Multiplicative relationship 10 x 5 = 50 Autonomy: Learner exploration in pairs

Task	Summary of expected learner activity	Purpose	Design notes
C1.2a Jug and cups (ii) Question: How could you find out many of these tiny cups would fill this jug? 	Capacity: Identifying how many of a very small container make up a large jug, with the introduction of an intermediate larger cup	To establish and reinforce the use of an intermediate unit (different container used to 1d). 	**Key task:** Similar to tasks 1c and 1d but with different units. *Based on reflective notes from T1c and T1d:* *-markers were placed on the units to support with ensuring equal sized units.* *-in the initial establishing of relationship, the actual number of little cups was available to support visualisation of relationship.* *-the relationship diagram included images* Complexity: Reinforcement of idea of intermediate unit and the multiplicative relationship (Un)familiarity: Familiar – building on 1.1c and 1.1d Technical Difficulty (level of maths): Multiplicative relationship 10 x 7 = 70 Autonomy: Initial introduction followed by paired exploration to establish relationship between intermediate unit and jug

Task	Summary of expected learner activity	Purpose	Design notes
C1.2b Pancakes Question: Here is a recipe for pancakes. If one cup of flour makes six pancakes, how could you find out how many pancakes could be made from this amount of flour? 	Volume: Finding how many pancakes could be made from a quantity of flour if one cup could make a particular amount	To reinforce that a unit can represent a number other than 1 (composite unit)	**Key task**: The task was designed to reinforce the notion of a composite unit, using a material other than water. One-to-one counting would not be possible. The context of pancakes fitted the day on which this was taking place. Complexity: New mode of presentation (no visible little unit to count) (Un)familiarity: Likely to be unfamiliar, though context of pancakes may be familiar. Technical Difficulty (level of maths): Multiplicative relationship 6 x 4 = 24, 6 x 5 = 30 Autonomy: Initial introduction followed by paired exploration, and another bag

Task	Summary of expected learner activity	Purpose	Design notes
C1.3a Cuisenaire (i) Question 1: If this rod measures 1cm, what do you think these rods measure (2cm, 10cm, 5cm)? Question 2: Can you find out how many 2cm, 5cm and 10cm rods make up 20cm? 	Length: Finding how many different sized Cuisenaire rods made up a fixed length	To bridge between standard units (cm) and multiplicative relationships. Essentially, learners were being asked: 20cm = 10cm x ? 20cm = 5cm x ? 20cm = 2cm x ?	**Warm-up task:** As rods do correspond to cm measurements, it was incorporated to support the transition into multiplicative reasoning using standard units of measure. The 1cm rod was only available for Question 1. Complexity: New mode of presentation of metric units, 3 different lengths (10cm, 2cm, 5cm) (Un)familiarity: Teacher informed me that learners were unfamiliar with Cuisenaire Technical Difficulty (level of maths): Not technically difficult, although 'division' style question. 20cm = 10cm x ?, 20cm = 2cm x ?, 20 cm = 5cm x ?, Autonomy: Initial introduction followed by individual exploration

Task	Summary of expected learner activity	Purpose	Design notes
C1.3b Spoons of medicine Question: If my dog needs one 10ml spoon of medicine each day, how can I find out how many days' worth of medicine I have in this bottle? 	Capacity: Finding how many spoonfuls of a liquid would be contained in a bottle, with the introduction of an intermediate measure.	To incorporate standard units (ml) into tasks involving intermediate units. 	**Key task:** Similar to 1c and 1d but with different objects. The smallest unit was a spoon (10ml), an intermediate unit was a small bottle (50ml). Complexity: Introduction of standard units adds to complexity as there are two multiplicative relationships (Un)familiarity: Similar approach to previous tasks, with the introduction of standard units Technical Difficulty (level of maths): Not technically difficult although two multiplicative relationships Autonomy: Initial introduction followed by paired exploration to establish relationships

Task	Summary of expected learner activity	Purpose	Design notes
C1.4a Cuisenaire (ii) Question 1: What do you think of this rod (4cm) might be? How could you check? Question 2: Can you find out how many 2cm, 4cm, 5cm and 10cm rods make up 20cm? Question 3: How could you predict how many 2cm, 4cm, 5cm and 10cm rods would make up 40cm? 	Length: Finding how many different sized Cuisenaire rods made up a fixed length(with introduction of new composite units)	To bridge between standard units (cm) and multiplicative relationships To establish and reinforce multiplicative relationships that would be relevant to T4b 20cm = 10cm x ? 20cm = 5cm x ? 20cm = 2cm x ? 20cm = 4cm x ? And to predict 40cm = 10cm x ? 40cm = 5cm x ? 40cm = 2cm x ? 40 cm = 4cm x ?	**Warm-up task** Similar to 3a but with the introduction of the 4cm rod and a 40cm length. No 1cm rods were available. Complexity: Introduction of 4cm (Un)familiarity: Building on 1.3a Technical Difficulty (level of maths): Not technically difficult although division style questions and use of relationships (e.g., 4cm = 2cm x 2) incorporated Autonomy: Initial introduction followed by individual exploration

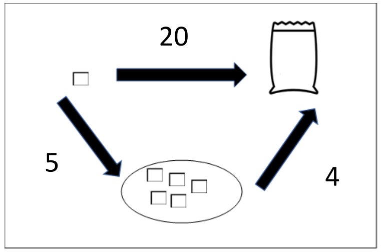
Task	Summary of expected learner activity	Purpose	Design notes
C1.4b Sugar cubes Question: Using this pan balance, how could you make up a bag of sugar weighing 80g, if you know one sugar cubes weighs 4g? Can you find an efficient way of doing it? 	Weight/Mass: Finding out the weight of a bag of sugar using a pan balance by comparing with 4g sugar cubes. Learners are encouraged to consider an intermediate unit, a bag of 5 sugar cubes, which weighs 20g.	To use weight/mass in a multiplicative context. To use standard units. To use an intermediate unit (bags of sugar cubes). 	**Key task:** This task was designed to explore weight/mass as a measure context for exploring the multiplicative relationship. Sugar cubes were chosen as a readily available manipulative which could be handled easily. Complexity: Introduction of standard units adds complexity as there are two multiplicative relationships (Un)familiarity: Whilst utilising relationships established in 1.4a, the task would be unfamiliar Technical Difficulty (level of maths): Technically difficult as learners needed to work in multiples of 20g Autonomy: Initial introduction followed by paired exploration

TABLE 9: TASKS IN CYCLE

5.7 APPROACH TO ANALYSIS OF DATA FROM TASK IMPLEMENTATION

In Cycle 1, data gathered related to the implementation of tasks include:

- audio data from the tasks (learning and teaching episodes)
- reflective notes
- interview with learners (see Appendix K for semi-structured interview questions)

Analysis focused on exploring the learning and teaching of multiplicative reasoning through measures, in relation to the following sub-questions:

S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?

S4: What is the impact of learning multiplicative reasoning through measures on learners?

S5: What are teachers' and learners' views on teaching/learning multiplicative reasoning through measures using the materials developed?

As noted in Section 4.4, Shavelson *et al*. (2003) suggest that, in design research, the use of narrative accounts is problematic. Indeed, they comment that 'although narrative accounts purport to be true, there is nothing in the narrative form that guarantees veracity' (Shavelson *et al*., 2003, p.25). As discussed in Section 4.6, I attempt to be transparent in my approach to data analysis and so I have chosen to adopt a narrative account to discuss the approach to how data were analysed. Hence, I discuss the analysis approach in the order of data collection.

The most immediate data were my reflective notes. Discussed in Section 4.11, I made reflective notes as soon as possible after each day of task implementation and an example of this is provided in in Appendix D. The reflective notes provide my account of the task and, in particular, how they were implemented and how I felt the learners responded. The notes also guided how I implemented tasks on subsequent days; as noted by Bakker (2018), the reflective component of design research allows changes to be made from lesson to lesson. This process also facilitates ethical practice, with a guiding principle to support learners. An example of this day-to-day reflection and target setting can be seen in reflective notes from Day 1 (Appendix D), where I consider the need to mark objects to support understanding of the need for units to be equal; subsequent tasks on Days 2-4 involved marking of containers.

My reflective notes and initial feelings about the tasks after implementation were often quite self-critical and cautious. For example, in Appendix D, I make comments like 'fairly well' and 'I made the mistake of putting the bigger cups out earlier'. Another comment in my reflective notes was:

'*My questions could be considered leading (need to check the recording) and I felt that I was doing a lot of the talking. This is something I need to consider further'.*

Hence, my reflective notes also guided me to cross-reference with other data sources. As discussed in Section 4.11, a non-linear approach is needed within qualitative data analysis, with a need to move between data analysis and interpretation (Cohen, Manion and Morrison, 2018). I not only used the reflective notes immediately after implementation but also used them, as part of the analysis, in cross-referencing with the other data sources, including the comments from learners about the tasks, collected on Day 5, and the audio data from task implementation collected each day.

Learners' responses to the tasks were sought on Day 5 through interviews (see Appendix K). I also reflected on these in my reflective diary, e.g.

'I found the interviews fascinating. Having the pictures of the tasks definitely helped. Learners were able to point to particular things to recall.'

And

'The children clearly seemed to enjoy some of the activities and could comment on what they think they learned.'

'It was fascinating that, in many cases, they reflected on issues similar to my reflections – for example many of the learners selected the Cuisenaire rods as a useful activity.'

I later transcribed learner responses to the questions, and collated notes about responses in a table (Appendix L), to act as a summary of the responses provided.

In the weeks following the task implementation, I listened to all the audio data of tasks, making notes as I did so (see Appendix M). These notes were later added into the data analysis package NVivo as memos (see example of memo in Appendix E). This initial engagement with the data was an attempt at sense making (Cohen, Manion and Morrison, 2018), which involved listening to all the audio data before making any decisions about organisation.

Whilst listening to the audio data, I believed that what was captured on audio could be considered in terms of three broad categories: behaviour (what was being done), emotion (what might have been felt) and awareness (what awareness might be developing). These categories derive from a framework for task design, Behaviour-Emotion-Awareness, discussed by Mason and Johnston-Wilder (2006). Though I had applied this framework previously as part of my study of a master's module in research into mathematics learning, it was only in this initial sense making phase of becoming familiar and trying to make sense of the data that I considered using it to support data analysis. Through using the three broad categories of behaviour, emotion and awareness as organisers, I developed a framework for coding the audio data gathered during task implementation.

Coding is a common approach within qualitative data analysis, and the process of coding allows the researcher to categorise or label data (Cohen, Manion and Morrison, 2018). It allows text to be considered in terms of smaller units (Saldaña, 2016). As noted by Cohen, Manion and Morrison (2018), possible limitations of coding are that it can fragment data, resulting in too many codes, and coding can mislead a researcher into searching for non-existent patterns. However, I felt that coding the transcribed audio data from task implementation in this study would allow the aspects of behaviour, emotion, and awareness to be considered and compared across the tasks, supporting analysis in relation to research sub-questions. As learner and teacher perspectives (including my own researcher reflections) were also being gathered, it was felt the holistic view would not be lost and a risk of over fragmenting data would be reduced.

The codebook can be seen in Table 10 below. Notes justifying the use of the codes, written at the time the codes were developed are included in the table. Following Table 10 is further explanation of the application of codes, with examples.

Code	Definition for applying code	Link to Research Question and notes/justification
Behaviour codes	**-behaviours through audible actions**	**S3**
Teacher behaviour Codes (below)	**-teacher behaviours through audible actions**	**S3 –The analysis of teacher actions in implementation, could allow for analysis of relationship between teacher actions and learner responses to inform effective/ineffective actions to support tasks. The analysis also could improve behaviours in future cycles**
Teacher instruction TI	-teacher gives instruction	S3 Instances when I instruct may be important to consider in terms of development of tasks
Teacher reiteration TR	-teacher reiterates point	S3 To distinguish between introduction of an idea/point and reiteration
Teacher suggests idea TSI	-teacher suggests an idea/approach	S3 Analysis of how many times I suggest an idea may allow me to consider the extent to which learners are coming up with ideas of their own: Davydov (1990) suggested making something problematic so that previous approaches are more difficult; if I am making the suggestions then I may need to consider how I develop situations so that learners have more opportunity to do so.
Teacher question: focussing TQF	-teacher asks a question to focus learners' attention on a relationship/pattern/effect	S3 Mason and Johnston-Wilder (2006) categorise questions: Teacher may know the answer but attempts to focus learners (rather than telling) e.g. Can you see a pattern?
Teacher question: rehearsing TQR	-teacher asks a question to rehearse or check knowledge	S3
Teacher question: Enquiring TQE	-teacher asks a question to genuinely enquire	S3 The distinguishing feature here is that the question is genuinely seeking to establish what a learner/group of learners may be thinking. I anticipate that follow up questions may be more difficult to categorise (e.g., a learner responds to a TQE and teacher follows up by asking 'What do other people think?' – if the first response is

		correct then I am checking others agree (although I would count this as enquiring)
Teacher relates to narrative	-teacher relates the task to a specific narrative	S3 Some tasks have been given a narrative to make meaningful and my reflections suggested I dropped the narrative at points
Learner behaviour codes (below)	**-learner behaviours through audible actions**	**S3 and S4 – through considering learner actions I will be able to identify whether behaviour might be consistent with multiplicative reasoning**
Learner suggests idea (LSI)	-suggests an idea/approach	S3/S4 Linked to comments on TSI – to what extent are learners bring given opportunities to suggest approaches and ideas? This would not be in response to a TQR question – it would have to be in response to a TQE
Learner counts in ones (LCO)	-learner counts in ones	S3/S4 The extent to which learners are counting in ones is worthy of analysis. Counting in ones might be considered as not demonstrating multiplicative thinking. However, taking Davydov's (1992) definition of multiplication involving a change of unit, it may imply multiplicative thinking if the learner is counting a composite unit.
Learner counts in steps other than one (LCM)	-learner counts in steps that are not one	S3/S4 Counting in steps other than one (composite units) can relate to multiplicative ideas (as discussed in literature review).
Learner gives correct response (LRC)	-learner responds correctly to a (rehearsing) question	S3/S4 Although I may not need to know whether a response was correct or incorrect, it may be useful to record this
Learner gives incorrect response (LRI)	-learner responds incorrectly to a (rehearsing) question	S3/S4 Although I may not need to know whether a response was correct or incorrect, it may be useful to record this
Learner indicates agreement (LIA)	-learner audibly indicates agreement of what has been said (e.g. Mmm huh, yes)	S3/S4 Considering these (learners with each other) might allow for recognition of situations where learners engage in debate with each other; this could be linked to the merits of discourse and mathematics learning (e.g., Ryan and Williams, 2007) as well as the suggestion by Davydov (1990) and that learners should be

		engaged in debate – thinking about learning activity (e.g., Eriksson and Lindberg, 2016)
Learner indicates disagreement (LID)	-learner audibly indicates disagreement of what has been said	S3
Learner relates to a narrative	Learner relates the task to a narrative	S3 This might allow consideration of the extent to which learners may refer to narrative
*Learner asks a question	Learner asks a question	S4 This was suggested by a critical friend when undertaking interrater coding Questions may suggest: Need for clarification Curiosity in terms of mathematics Curiosity in terms of other aspects
Emotion codes (below)	**-interpretation of possible emotion indicated through audible response (this would include non-words but possible expressions (e.g. Yay!)**	**S5 – verbal or audible indications of emotion could support S5** Mason and Johnston-Wilder (2006, p.19) suggest that emotion is harnessable. They argue that involvement in decision making (related to comments about 'suggesting ideas' above) can support engagement. Furthermore potential enjoyment of working with materials would seem to suggest that a task offers good opportunity for engagement.
Indicator of enjoyment (IE)	-audible indication of possible enjoyment	S5
Indicator of not enjoying (INE)	-audible indication of possibly not enjoying (e.g. a moan sound)	S5 This was not evident in the first listen to Cycle 1 audio, but is included to ensure balance
Indicator of excitement/interest/surprise (II)	-audible indication of possible excitement (e.g. Yay!/a gasp)	S5 Mason and Johnston-Wilder (2006) also note that surprise can be an important feature of tasks
Indicator of boredom/lack of interest (IB)	-audible indication of boredom/lack of interest	S5 S5 This was not evident in the first listen to Cycle 1 audio, but is included to ensure balance

Awareness codes (below)	**-interpretation of possible awareness of something (e.g. awareness of a relationship, knowledge of a fact,**	**S4 – indicators of awareness will support my understanding of the possible impact of tasks on learning (or potential for learning, see discussion in paragraph above)**
Awareness of necessity of equal units (AEU) **added during transcription*	-an instance where there is recognition of need for the unit being considered to be the same size	S4 Inherent in the idea of unitisation is the idea that the unit under consideration should be of uniform size each time
Awareness of change in unit (ACU)	-an instance where there is acknowledgement that the unit is different (has changed)	S4 Davydov (1992) defines multiplication as change in unit and tasks were set up on this basis
Awareness of quantity in relation to unit (AQU)	-an instance where awareness of a relationship between quantity of something being measured and the unit/s being used to measure is indicated	S4 Recognition of needing more units might apply here There are also situations where two different sized units have been compared
Awareness of additive relationship (AAR)	-an instance where awareness of an additive relationship is suggested (e.g. working in single units with no change of unit, one variable)	S4 In a multiplicative relationship a student may approach it in an additive way. Clark and Kammii (1996) and Davydov (1996) discuss how a multiplicative relationship may be approached in an additive way usually counting out in ones. Note that a behaviour of counting in ones may not necessarily imply additive thinking – e.g. if there is awareness of the unit being composite. So these two codes are different – counting in ones may imply additive thinking but it may relate to multiplicative thinking if it relates to counting a composite unit
Awareness of a multiplicative relationship (AMR)	-an instance where a specific multiplicative relationship is acknowledged	S4 This is different to awareness of a change of unit because it could account for instances where a learner indicates knowledge of a specific multiplicative relationship (e.g., two fives would be equal to ten, or a half-double relationship). Awareness of these relationships would not necessarily imply multiplicative reasoning is being used but seeing whether specific learners demonstrate this and tracking changes in occurrence of this should be useful.

Awareness of composite units	-an instance where composite units are being acknowledged and used	S4 This may overlap with learner counting in steps other than one, though this code be necessary for situations where the learner is not demonstrating that behaviour but says something indicating that awareness (e.g., I have 6 cups and each cup represents 10 little cups) – the many to one relationship.
Awareness of standard units of measure	-an instance where a learner shows awareness of standard units	S4 This may apply when there are instances where learners show awareness of standard units. This is not necessarily relevant to S4 directly although I did develop a task later in the week that made use of standard units as part of the context.

TABLE 10: CODEBOOK BEHAVIOUR-EMOTION-AWARENESS

The codes were induced from the data iteratively in the following way:

1. A first draft was developed after initial engagement with all audio data of the tasks. The draft was informed by these notes, but also by reflective notes made at the time; for example, I reflected on the way in which I introduced tasks and the way I questioned learners, therefore I included codes related to these, explained further below and within the table notes.

2. As the audio was being transcribed (see Section 4.10 for discussion of approach to transcription taken in this study), some edits were made to the codes. These were most typically in relation to definitions of codes, to refine them. One new code was added during the transcription process: awareness of necessity of equal units (marked in Table 10 as*). This code was added to account for instances where learners were making comments that showed they recognised that units they were working with needed to be the same value.

3. As discussed in Section 4.11, Thomas (2006) notes the benefit of consistency checks when coding. Two work colleagues were introduced to the codebook and were asked to independently apply the codes to the same section of transcription. There was agreement in the codes used, particularly those being used when analysing learner comments. The main queries raised were around the use of the teacher behaviour codes and the categorisation of questions (see Table 10, p.160). These codes were used because my reflections had suggested I was possibly leading too much with the questions. The teacher question codes were defined in relation to the teacher's intentions, which was therefore difficult for others to identify. I felt that as the teacher-researcher could identify my intentions more easily. One of the work colleagues also suggested using a code for learner questions, which was introduced.

4. The transcription was reconsidered in relation to coding, to produce a final transcription for the recordings, with coding.

In the Behaviour-Emotion-Awareness framework (Mason and Johnson-Wilder, 2006), and its application in this work, behaviour is seen as what might be said or done and emotion is seen as what might be felt. Mason and Johnston-Wilder (2006) clarify that Gattegno (1987, in Mason and Johnston-Wilder, 2006, p.19) claims that 'only awareness is educable', whereas behaviour can be trained, and emotion can be harnessed. Within this work, behaviour codes are used for things I, as the teacher, or the learners say, which indicate something is being done, such as: asking a question, counting in multiple steps or relating a task to a context. Thus, behaviour codes relate to actions and in the case of audio data, actions that are indicated through speech. On reflection, though the classification of 'teacher asking a question' fits into 'behaviour', the distinction of the type of question based on *intention* is more difficult to class as behaviour. This is why work colleagues found those codes problematic, and, in retrospect, though I had felt at the time of coding within both cycles I would be able to apply the intention distinctions, these were the most difficult codes to apply. This is discussed further in Section 6.5 (Pedagogic approach to the tasks). In future work, I would reconsider question codes.

In the audio, learners could be heard counting in steps (sometimes in ones and sometimes in steps other than one) and these can be considered behaviours. Thus, I developed behaviour codes for counting in steps of one and counting in steps other than one. Examples are:

Task C1.4a when a learner is finding out how many 4cm rods make up 40cm:

Learner 7: Let's see, one, two, three, four, five, six, seven, eight, nine, ten. Ten! Ooh ten!

This was coded as 'Learner counts in ones'.

In Task C1.2a, where a relationship was established between a little cup, an intermediate cup and a jug, the following exchange could be heard:

RW: *So if there were seven full cups in that jug we'd have ten, twenty*

Learner 4: *If that was full, seventy*

Learners (collectively): *Thirty, forty, fifty, sixty, seventy*

In this exchange, the code 'learner counts in steps other than one' was applied.

It is important to recognise that a learner may count in steps other than one, but this is a trainable behaviour, an action, and therefore not necessarily an indication of knowing a multiplicative relationship. As discussed in Section 2.6, the mode of calculation can be distinguished from the way a calculation might be represented or modelled, and the mode of calculation does not necessarily relate to whether additive or multiplicative reasoning may be being used. In the examples above, the counting in ones occurs when the learner is counting a unit of 4cm, and thus the counting is of composite units. Hence, although counting was coded (either in steps of one or in steps other than one), it is not assumed either behaviour is necessarily associated with additive or multiplicative thinking, because it depends on the context in which the counting occurs. I used annotations to make note of particular contexts; I now reflect on whether codes distinguishing the context (e.g., counting in ones of single items, counting in ones for composite units) may have been more useful. However, at the time, the use of annotations across a relatively small number of tasks allowed for distinction between the contexts.

Codes for emotion, particularly codes for excitement and enjoyment were used because these arose from initial familiarisation with audio data in which learners were heard expressing enjoyment or excitement. For example:

C1.1d *'This is fun'* and C1.2b *'I love doing things like this'* were coded as indicators of enjoyment.

And:

C1.1d *'I'm excited'* and C1.3b 'Ooh! Ah!' are coded as indicators of excitement or surprise.

Whilst the use of further emotion codes would be beneficial (e.g., confusion, frustration), the use of audio meant only emotions that might be conveyed as specific utterances could be coded, rather than the possible inclusion of facial expression or body language as possible indicators of emotion. Hence the emotion codes are indicators of emotion at certain points, rather than being used to monitor all emotions through tasks.

Awareness, as Coles and Brown (2016) note, was, for Gattegno, a technical term reflecting a view that mathematics was about awareness of relationships. Mason and Johnston-Wilder (2006) note that awareness can be about recognising subtle differences. Coles and Brown (2016, p.156) explain that Gattegno saw the purpose of awareness as that which 'illuminates action'. Thus, an important point here is that awareness is not seen as necessarily the outcome of behaviour or emotion experiences; awareness may also guide or direct behaviour or emotions. The coding of awareness within this work reflects the social constructivist theoretical paradigm, discussed in Chapter 3; individuals may possess subjective, internalised knowledge which may or may not align with socially accepted 'taken as shared' knowledge. Awareness of something (e.g., a relationship) might be indicated through what is said within the audio, but this is not interpreted or claimed to be knowledge or understanding possessed by an individual.

Awareness codes were informed both by the data and the theoretical ideas around multiplicative and additive thinking discussed in Chapter 2. As Davydov (1992) saw the multiplicative relationship as involving a change in units, and tasks were designed to support development of this idea, instances where learners may be indicating some awareness of this were coded.

For example:

In Task C1.1d*, 'Miss, we didn't use any little cups'* was said by a learner, when discussing approach to measuring the amount of water within a jug (when a relationship between the intermediate cup and the little cup had already been established) suggesting awareness of a change in unit from a little cup to an intermediate cup. This was coded as 'awareness of change in unit', indicating it was a possible instance of this.

There were occasions when learners expressed a need for more units (e.g., more intermediate cups). For example, C1.1d, when a learner says:

Girls can I borrow one of your cups?

Initially I saw this as only a weakness in task design, but discussion with my supervisors in relation to the data and coding process suggested that the reflections on task design and implementation were causing me to possibly miss what some comments may be indicating. In this instance, the learner was indicating awareness that there was a need for more measuring equipment (an intermediate cup) and thus was showing awareness of a relationship between the quantity being measured and the unit being used to measure it. Therefore, the code, awareness of quantity in relation to unit was introduced.

All tasks involved particular multiplicative relationships and there were instances where learners were showing awareness of these. For example, in Task C1.1c, when it had been established that seven big cups made a jug and water and learners were asked how they could work out how many of the small cups might make up the jug, learners showed awareness of multiples of seven, with predictions such as these:

Learner 7: Twenty-one and, later, *Learner 6: Forty-two*

Thus, these learners could be showing awareness of multiples of sevens, and these were coded as 'awareness of a multiplicative relationship'.

Following transcription and application of the coding of data derived from task implementation, the three main sources of data (reflective notes, audio data from task implementation, learner interview data) were triangulated to consider points of learning from Cycle 1. These are discussed in the next section.

5.8 POINTS OF LEARNING: CYCLE 1

Points of learning are organised under four foci, related to sub-questions outlined in Section 5.1 of this chapter.

1. **Task efficacy**: The extent to which tasks may support multiplicative reasoning. Tasks were developed to support multiplicative reasoning through measures and therefore a focus on their efficacy in relation to this supports S3 and S4.

2. **Task difficulty**: As discussed previously, Burkhardt and Swan (2017, p.181) identify features that can guide considerations around task difficulty in design research.

These are: complexity (aspects such as number of variables, modes of presentation of information) ; (un)familiarity (similarity to a task that might have been practised previously); technical demand (the level of mathematics required) and student autonomy (the level of guidance from teacher or from structuring or scaffolding of task).

In terms of complexity, the key consideration is variables within the task. All tasks were presented to the learners in similar ways: through introducing the materials and raising a question for learners to discuss. Very little information was presented in written form, only a simple, illustrated recipe in C1.2b, and this was read with the learners because of their age. Familiarity of tasks was considered in preparing for tasks through Phase 1, and any equipment being introduced was explained. However, familiarity also needs consideration in the way tasks may be sequenced. The technical demand of tasks in terms of level of mathematics varied across tasks and can be considered a factor contributing to task difficulty. In terms of student autonomy, considerations around the guidance given, structuring and scaffolding are noted, which also links to the pedagogical approach taken. Consideration of task difficulty and the factors outlined above will support research question S3.

3. **Learners' emotional and evaluative responses to tasks**: Consideration of learners' responses to tasks being implemented (through coding what was said) and their comments on tasks in the interviews supports research question S5.

4. **Pedagogic approach to tasks:** As discussed in Chapter 4, Tabak (2004, p.227) applies two constructs 'exogenous design' and 'endogenous design' in relation to context of tasks within design research, where 'exogenous design' refers to the materials, strategies and activity structures that have been developed for the research and the term 'endogenous design' refers to materials and practices that are in place in a local setting, including the way in which teachers and students may engage in enactment

of materials. As the researcher and the teacher, it is important to consider both the exogenous design of tasks (aspects such as efficacy, difficulty and autonomy) whilst also considering the way in which I, as the teacher, engaged with the enactment of the materials, part of the endogenous design. In discussing the pedagogic approach to tasks, I consider the way in which I enacted the tasks and how, or whether, this aspect might impact on learning or future design, thus supporting research questions S3 and S4.

TASK EFFICACY: TO WHAT EXTENT DID TASKS SUPPORT LEARNING OF THE MULTIPLICATIVE RELATIONSHIP?

Analysis of coding and analysis of learner responses to semi-structured interview suggest that tasks did support awareness of the multiplicative relationship though, unsurprisingly, this varied across tasks, both in terms of incidence and type of awareness shown. Awareness codes that related to the multiplicative relationship are shown in the table below:

Code	**Definition for applying code**	**Notes**
Awareness of change in unit (ACU)	-an instance where there is acknowledgement that the unit is different (has changed)	Davydov (1992) defines multiplication as change in unit and tasks were set up on this basis
Awareness of a multiplicative relationship (AMR)	-an instance where a specific multiplicative relationship is acknowledged	This is different to awareness of a change of unit because it could account for instances where a learner indicates knowledge of a specific multiplicative relationship (e.g. two fives would be equal to ten, or a half-double relationship). Awareness of these relationships would not necessarily imply multiplicative reasoning is being used but seeing whether specific learners demonstrate this and tracking

		changes in occurrence of this should be useful.
Awareness of composite units (ACM)	-an instance where composite units are being acknowledged and used	This may overlap with learner counting in steps other than one, though this code be necessary for situations where the learner is not demonstrating that behaviour but says something indicating that awareness (e.g. I have 6 cups and each cup represents 10 little cups) – the many to one relationship.

TABLE 11: AWARENESS CODES

Task C1.1d had the highest incidence of coding for learner awareness of a change in unit. Task C1.1d built on the more teacher-led episode C1.1c, which introduced the notion of a change in unit. In Task C1.1d learners worked in pairs to discover how many little cups would contain the same amount of water as a bigger container, using an intermediate unit. Within this task, comments such as *'use the other cup'* (Learner 5), *'big, you could use the bigger cup'* (Learner 3) were taken as possible indications that learners were aware of a change in unit of measure. Working in pairs provided more opportunity for learner discussion which might account for a greater incidence of awareness than in Task C1.1c; although Task C1.1c did indicate some awareness, in the initial suggestion by Learner 2 *'I know another way. What if we use the small cup, tip it in the water and put it into the big cup'*, greater incidence was evident when learners were working in smaller groups.

Other tasks in which learners were coded for awareness of a change in unit were Task C1.2a (again similar to Tasks C1.1c and C1.1d), and Task C1.4a, using Cuisenaire, in which two learners discussed and compared the different sized rods in relation to how many times they might occur.

As discussed in in Section 5.7, it should be noted that comments such as those shared above are not taken to indicate understanding of multiplication involving a change in unit; rather they are indicators of possible awareness of this. Learners typically discussed many aspects of their tasks and related activity and comments noted above were made during their 'free flow' conversation rather than in response to a specific question that explored their understanding of change in unit. This caused me to reflect on the need to provide more explicit opportunity to articulate and reflect on awareness at different points during the task (rather than mainly at the beginning) in Cycle 2. However, such action should be planned to support learning rather than to simply improve results of research.

In contrast to awareness of change in unit occurring in particular tasks, awareness of specific multiplicative relationships occurred across all tasks. Again, this is unsurprising because all tasks involved specific multiplicative relationships. Highest incidence of this code occurred in Tasks C1.1a, C1.1b, C1.3a and C1.4a. Notably these tasks involved specific discussion of the multiplicative relationships involved. For example, C1.1b involved straws which were half the size of another straws and learners were asked to discuss their different measurements. C1.3a and C1.4a involved the use of Cuisenaire and different length rods, with the generation of multiplicative relationships in a visual way. It is noteworthy that all but one learner suggested, in the semi-structured interviews, that tasks involving Cuisenaire helped them learn the most mathematics (see Appendix L). This may be because the task may be more in line with what they might associate maths to be; learners used the resource to explore how many different length rods could make 20cm and then 40cm and, through doing so, generated multiple multiplication calculations. Although the teacher (who is also the maths co-ordinator) told me they have not used this resource before, the generation, and noting, of multiplication relationship might have been associated with 'more mathematics'. One learner (Learner 8) related this resource explicitly to the learning of multiplication:

RW: What maths do you think you have learnt from these activities?

Learner 8: Time-sing

RW: Time-sing have you? You have learnt time-sing have you, multiplication?

Learner 8 then indicated Cuisenaire (from picture of tasks) as a resource that has particularly helped with this.

Awareness of composite units occurred in Tasks C1.1d, C1.2b, C1.3a, C1.4a and C1.4b. This occurred when learners were showing awareness of the need to work with units other than one. For example, in Task C1.2b, in which learners were asked to consider how many pancakes could be made from a bag of flour if they knew one cupful of flour could make 6 pancakes, Learner 4 says:

Learner 4: 'A quarter (referring to a quarter of the bag) would make about six and we could count in sixes then'.

Later, when using the cup and flour, the learner repeats *'Once we have done this, we can just count in sixes'*

Thus, the learner shows awareness, through the task, that the cup represents 6 pancakes, i.e., that the cup can be seen as a composite unit.

Tasks C1.3a and C1.3b, involving Cuisenaire, show high incidence of awareness of composite units. In these tasks learners frequently refer to the rods as composite units (e.g., a two, three or five). Task C1.4c, the most complex task, also involved multiple composite units and thus incidence within this task was also high. In all these tasks, awareness of composite units was sometimes noted at times when learners were also counting in steps other than one (which is noted as a behaviour code); though the behaviour itself is not seen as indicative of recognising a composite unit, it can be a potential indicator of it.

Hence, tasks do seem to support learning of the multiplicative relationship and, as noted above, different tasks seemed to support different aspects of this (in line with what they were planned to do). However, it is difficult to establish how effective tasks might have been for particular learners; originally post-task follow up had been planned but this was not possible due to school closure. As discussed in Section 1.3, and outlined in Appendix A, Phase 2 of Research Cycle 1 took place just before school closures due to the COVID-19 pandemic. The semi-structured interviews with learners explored what learners felt they had learned, and when particular tasks were discussed by learners, I asked questions to try to establish their understanding of those tasks. During the tasks, audio recording using only one device limited the extent to which all learner comments could be identified and tracked, which would have enabled more focus on individual learner experiences, and this could be cross-referenced with learner comments about their experiences in the semi-structured interviews. Whilst it is fair to conclude the tasks provide some opportunity, as intended, for developing awareness of the multiplicative relationship, the extent to which they do this is dependent on many other factors, most notably task difficulty.

Task difficulty

The most complex, technical and possibly the most unfamiliar task was Task 4b, involving mass. Although learner awareness of the multiplicative relationship is evident within the task, learners clearly found this task difficult when it was implemented. This was evident in the level of support needed, and from the responses from learners. For example, Learner 7 (see Appendix L) noted in the semi-structured interview *'Well it was kind of hard'* whilst Learner 6 stated *'I had to think hard of the one that we did yesterday'*. The tasks on this day

(C1.4a and C1.4b) had been sequenced in a way that specific multiplicative relationships (4 x 5=20 and considering multiples of 20) had been reinforced in the first task (C1.4a). However, the type of measure (from length in C1.4a to mass/weight in C1.4b) changed, and the visual representation of the multiplicative relationship (4 x 5 = 20) was very different in both tasks. Using Cuisenaire allows for a visual representation of equality with length, whereas with Task C1.4b, equality is signalled through the balancing of visually different objects. Hence perceiving equality in the context of this task may be more difficult and unfamiliar, and expecting learners to be able to transfer their understanding of 4cm taken 5 times as being equal to 20cm, into 4g taken 5 times as being equal to 20g should have been more explicitly considered at the task design and sequencing stage. It should be noted that Learner 5, in the semi-structured interview, responded that task C1.4b helped with learning the most maths:

RW…which activity do you think helped you learn the most maths?

Learner 5: Um

RW: That one, and you're pointing to the weighing activity there, why would you say that one?

Learner 5: Because it's easier to weigh

RW: So you could see could you?

Learner 5: Yes, if it was the same amount..

RW: OK

Learner 5:...it would go in the middle

In this episode, Learner 5 refers to weighing being *'easy'* and the signalling of equality using the pan balances, although my question *'so you could see could you'* arguably prompts this latter comment. The initial question referred to 'most maths', which can also be open to interpretation by learners, and the wording of the question was adapted for the next cycle.

It was noted that the issue of visual representation of equality should be further considered in future task design and structuring; this can be considered an aspect of task complexity. The establishment of an equality relationship when working with liquids (as in Tasks C1.1c, C1.1d, C1.2a and C1.3b) became a recurring point through my reflective diary, where I noted on Day 1 that re-using the same little cup (which had been deliberately restricted in number to encourage the notion of inefficiency) to establish a relationship with the intermediate cup resulted in the equality relationship being difficult to visualise. Furthermore, learners did not consistently fill the cup with equal amounts each time, and therefore accuracy (in relation to establishing pre-determined multiplicative relationships) was affected. This was addressed, to some extent, in tasks on subsequent days through the more explicit establishing of the relationship between unit being considered and the intermediate unit (through having sufficient little cups for this to take place during the introduction) and through the 'level' marking of the containers being used, to encourage the use of equal units of measure. This was noted by learners in Day 2 tasks, but also mentioned by Learner 2, in the semi-structured interview: *'I liked it when you had to pour to the line'* (in a discussion of why the learner had chosen Task C1.3b as the 'favourite' task). Furthermore Learner 7, in the semi-structured interview, commented (in relation to Task C1.3b) '*all of us had four out and we tipped in the flour'*, which might suggest that having the correct amount of measuring containers supported the learner in this task. The issue of having sufficient measuring equipment was also referred to by Learner 5 in the semi-structured interviews *'we didn't have enough straws for both our lines'*; it is likely here that the learner is referring to the pre-assessment task (very similar to Task C1.1b) where it had been anticipated that learners might iterate the unit. In any case, the comment demonstrates that restricting the availability of units of measure, which can be a useful strategy to support understanding of measurement, can impact on learners' awareness of relationships.

The establishment of multiplicative relationships as part of task complexity should not only be considered in relation to the type of measure and available containers but also the type of materials being used. Liquids, especially with young learners, are prone to spillage and therefore accuracy (in establishing pre-determined relationships) can be affected. It should be noted that this bothered me, but possibly not the learners. Learner 6, in the semi-structured interview, commented that *'I found the liquids easy… we had a lot of water…and it was quite easy to make up like stuff'*. The use of flour was less problematic in terms of accuracy though some spillage did occur. In length related tasks, the use of a wool like string was also problematic. Learner 8, in the semi-structured interview suggested this when discussing Task C1.1b : *'because when you stretched it out, it always goes smaller again'*. Hence it was noted that the physical resources being used needed to be considered more carefully; if they are being used to explore specific multiplicative relationships then the extent to which they can do this is key consideration. For example, materials that are easy to pour but less likely to spill (such as lentils) could be considered in place of water. Materials being used for length measurement might be more rigid.

Recognising that the multiplicative relationship involves of change of unit of measure was the focus of tasks, and the extent to which materials used supported this being achieved is an important consideration. Tasks over the first two days (C1.1a, C1.1b, C1.1c, C1.1d, C1.2a, C1.2b) did not involve standard units of measure, even though learners did refer to them. Tasks over the next two days (C1.3a, C1.3b, C1.4a, C1.4b) incorporated standard measures. Sourcing 'everyday' materials for measuring (e.g., containers for capacity tasks, items for weighing) that would support exploration of particular multiplicative relationships was very time-consuming. In contrast, Cuisenaire, a commercially available and structured resource, enabled exploration of specific multiplicative relationships in a highly visual way. It is noteworthy that all but one learner referred to this resource as supporting their learning of mathematics. Exploiting relationships between commercially available standard measure objects (e.g., medical containers), or those designed for use in educational settings (e.g., measuring jugs or masses for pan balances) might not only be less time-consuming but could also support establishment of relationships between intermediate units, and, in line with curricular guidelines and the findings from the semi-structured interview with practitioners, would build on learning taking place in the year group.

LEARNERS' EMOTIONAL AND EVALUATIVE RESPONSE TO TASKS

Learners responded well to the tasks; this can be concluded from their comments within the tasks and their responses to the semi-structured interview.

The following codes were used to consider learner emotion.

Code	Definition for applying code	Notes
Indicator of enjoyment (IE)	-audible indication of possible enjoyment	Mason and Johnston-Wilder (2006) suggest that emotion is harnessable. They argue that involvement in decision making can support engagement. Furthermore, potential enjoyment of working with materials would suggest that a task offers good opportunity for engagement.
Indicator of not enjoying (INE)	-audible indication of possibly not enjoying (e.g. a moan sound)	Included as an opposite to above
Indicator of excitement/interest/surprise (II)	-audible indication of possible excitement (e.g. Yay!/a gasp)	I may need to further distinguish between these but there are occasions when learners can be heard to indicate excitement (e.g. when liquids are revealed/mentioned in relation to forthcoming tasks). Mason and Johnston-Wilder (2006) also mention that surprise can be an important feature of tasks – not sure if I will have examples.
Indicator of boredom/lack of interest (IB)	-audible indication of boredom/lack of interest	Included as an opposite to above

TABLE 12: EMOTION CODES

Fifteen possible indicators of enjoyment, from all learners, were coded over the first two days with comments such as *'I can't wait'*, '*This is fun'*, *'That's satisfying'* (the latter in relation to working with flour), *'This is really fun'* and *'I love doing things like this'* being identified. It is notable that these comments occurred in the first two days and therefore it could be argued that the novelty of working with a new person and in a novel context might account for such comments. However, semi-structured interviews with learners (Appendices K and L) also suggested tasks within the latter days were seen as enjoyable. Furthermore, learners indicated excitement and surprise across all days, particularly through audible sounds such as gasping or comments like *'Woah!'* and *'Wow!'* . All the incidences of enjoyment, excitement or surprise were coded at points in which tasks and the materials involved in them were being introduced or engaged with, rather than at points, for example, when multiplicative relationships were being discovered. Although codes for indicators of boredom or lack of interest were used, there was only one instance of this code being used; when a learner asked whether I liked her hair during an initial introduction. Though enjoyment of a task does not guarantee learning, learners certainly appeared enthusiastic and engaged, particularly by the materials, and it is therefore fair to conclude that the measure contexts can provide engaging opportunities to support multiplicative reasoning.

The semi-structured interviews with learners provided fascinating insight into their perceptions of the tasks. As noted previously, an overview of these results is provided in Appendix L. Learners were asked whether there were tasks they disliked or found confusing, but no tasks were identified as being disliked. Furthermore, learners identified aspects that I had also identified within my reflective notes; for example, Learner 5 referred to the ordering of pouring between containers (as in Task C1.3b) *'I think we need to use the bottle first and then pour it into the spoon'*. This was a particularly interesting comment because the learner seems to be referring to establishing a relationship between the unit of measure and an intermediate unit, whilst showing awareness this could be achieved in a different way.

Learners' responses to the tasks did not appear to be affected by whether a particular 'story' context was provided with them. Tasks C1.1c, C1.1d, C1.2a, C1.2b and C1.3a were given 'story' style contexts to try to provide reason for such activity taking place (how many rabbits can I feed, daughters weighing out flour, how much medicine for a dog). Interestingly only one learner, Learner 7, appeared to frequently make reference to such contexts, both in the tasks (identified through behaviour coding) and in the semi-structured interview. In discussion of task C1.1d in the semi-structured interview, the following exchange occurred:

Learner 7: And I found this one helped because we had to feed a lot of rabbits

RW: So that's the jug and the little cup, that's the water jug isn't it

Learner 7: And I want, in my future, I want to have a rabbit

RW: (laughs)

Learner 7: And I want to see much water I have to feed my rabbit

RW: So you had to see how much water was going to feed the rabbit there

Learner 7: Mmm huh

RW: How do you think that might have helped you with your maths?

Learner 7: Well because you told us we had to see how much rabbits we had to feed and it helped me see how I had to get the amount in that bottle

RW: Ah, so you had to get the same amount of cups in the bottle

Learner 7: We couldn't put this in, we can only put these in, we poured tiny bits in there and then we poured them inside the bottle.

For the other learners, the 'story' style context was not referred to in the semi-structured interviews and they made few references within the tasks. The 'story' context that was used to introduce the tasks was frequently dropped by the learners once they seemed to understand what mathematics would be involved. It could be that the measurement contexts themselves provided sufficient motivation for the mathematical tasks and that the ‘story’ contexts were unnecessary for some learners. Furthermore, an attempt to make some tasks more authentic may have appeared contrived. For example, the mention in the introduction of task C1.3b of daughters weighing flour for pancakes caused Learner 7 to ask *‘do they work in a flour factory or something?’* and later, when asked to discuss with her partner how they might find out how many pancakes they could make from a bag of flour the learner asked *‘why do your daughters have to bring home flour?’*. As discussed previously, van den Huevel-Panhuizen (2005) suggests that real situations can be simplified in a way that makes them unrealistic and it could be that in this example the reason for having bags of flour was unnecessary (and arguably distracting), though the context ‘how many pancakes could be made from this bag’ seemed to support the accessibility of the task. As Boaler (1993) notes, ‘real world’ problems can be those that arise out of learners’ interactions with the environment; it seems that for the majority of learners, the context of measures could have been sufficient for providing ‘real world’ experience, and further reason to make the task imaginable may have been either too contrived or unnecessary.

PEDAGOGIC APPROACH TO TASKS

Davydov’s (1990) focus on using measure contexts for the learning of mathematics proposed a distinctive pedagogic approach. The approach involved the posing of a problem for which previously used actions and tools are either impossible, inefficient or unsuitable. In order to solve the problems, learners would need to explore new modes of actions. In doing so, learners might need to argue for particular solutions. Matusov (2001, in Eriksson and Lindberg, 2016) argues that, in Davydov’s approach, the knowledge students are expected to develop is predetermined and therefore any debate is not democratic. As Eriksson and Lindberg (2016) highlight, the tension in Davydov’s approach can be considered in relation to student agency and epistemology. They argue that, from a learner's perspective, knowledge being developed is new and learners can develop agency in order to establish activity in order to solve the problem.

The extent to which genuine debate, as proposed by Davydov (1990; 1992) and Eriksson and Lindberg (2016), took place in any of the tasks is certainly questionable. Problems were posed and, in all tasks, learners were asked to discuss how they might approach finding solutions. These small group discussions generally seemed to be where the richest conversation could be heard. For example, when asked how they might find out how many pancakes could be made from a bag of flour, the following conversation occurred:

Learner 4: If we look at the bag maybe we can guess how much. Well if we split it in half

Learner 3: Yes

Learner 4: From quarters to halves

Learner 3: Yes

Learner 4: If we have a quarter we could make one pancake

Learner 3: But

Learner 4: A quarter would make about six and we could count in sixes then

Learner 3: But we can't, we can't cut it, we've already cut it

In this conversation Learner 4 shows multiplicative reasoning; the learner seems to be equating a quarter of the bag to one cup of flour (which was correct as the bag contained 4 cupfuls of flour). Learner 3, in stating 'we can't cut it' could be showing awareness that a measure of some sort needs to be used.

The conversation above highlights tensions in pedagogic approaches that I frequently reflected on, initially (in my reflective notes) and within memos and annotations in NVivo. In the example above, after introducing the context and establishing that one cup made 6 pancakes, I meant to ask 'How might you find out how many pancakes you could make with this bag of flour?' but I actually asked 'How many, can you find out how many pancakes I can make with that much flour?. Thus, the focus on 'how many' encouraged estimation of numbers rather than discussion of a general approach. I was able to draw the learners back to that question, by referring to the cup, but the initial wording of questions and problems that learners are asked to discuss is clearly vital. Davydov's (1990) approach proposes that learners should transition from the general to the particular (i.e., that they should be introduced to the generalised 'scientific' concept before meeting specific 'concrete' examples of it). Through the week, there was a focus on tasks supporting learners' awareness that the multiplicative relationship involves a change in the system of units; in the example above, from considering one pancake to a cup which represented six pancakes. Yet each task involved particular multiplicative relationships for the learners to explore, and these were pre-determined by me. A lack of attention to specific relationships could result in learners struggling to find solutions to the problems set. Hence, it was important to ensure learners were able to effectively establish those relationships, and also to ensure those relationships were not too technically difficult. Many questions asked by me (coded by behaviour) were 'rehearsing' questions; questions where I seemed to be deliberately checking learners' understanding of their approach or the relationships involved. It should be noted that the frequency of enquiring questions did seem to increase through the tasks, but overall, I did seem concerned, through rehearsing and focussing questions, to support learners in focussing on pre-determined relationships and actions. This bothered me when I was reflecting on the tasks and when I was transcribing the data, because I felt that the focus should be on the theoretical concept of a change in unit rather than over-emphasis on specific instances. However, the learners needed to engage with tasks that would support them in developing this idea and therefore specific relationships needed to be included and, on occasions, reinforced.

Another example which highlights pedagogic tensions that arose is in task C1.1c, the first task to introduce an intermediate unit. In this task, learners were introduced to the problem of trying to find out how many rabbits a large container of water could feed, if one small cup was enough for one rabbit.

After the expected initial suggestion to find out how many small cups made up the container, the following conversation ensued:

RW: Learner 4, you said it would be a big number. Would you all agree that this might be a lot of cups?

Learners: collectively Yes

RW: It is going to take a long time

Learner 4: It's going to take forever

RW: It's going to take forever. Learner 7, what do you think, do you think it will take a long time to find out how many cupfuls of that water we'd need for here?

Learner 7: I think it's going take about ten minutes

RW: It'll take a long time. What do you think Learner 8?

Learner 4: I've got a good idea...

RW: Do you think it's going to take a long time?

Leaner 4: I've got a good idea

RW: Is there a quicker way?

Learner 8: We could use the bigger cups.

Learner 4: Yeh.

Learner 4: If we use the big cups then each rabbit can, uh, have the same amount as long as they've got, uh, enough water

RW: So we could use the big cup and find out how many of the big cups are in there?

Learner 6: Mmm huh

RW: Let's do that then. Let's find out how many of the big cups are in there.

Learner 7: I think we will need to use a big cup for that

In this example, although it was not impossible to use just the little cups, learners are led to the idea that using them may be inefficient. Interestingly Learner 4, a learner who appeared confident, was not heard at the time to say *'I've got a good idea'* (as each time it was said it was at the same time as I was talking) but the later agreement with Learner 8 suggests it was the same. In this episode learners are involved in a discussion but not a debate. Furthermore, the suggestion by Learner 8, and agreement by Learner 4, is accepted by me even though Learner 4's last comment does not suggest that the cup may be seen as a way of finding out how many small cups there may be.

At that point, the group worked to discover how many intermediate cups were in the container. This decision was taken partly to respond to learners' suggestions (and thus promote student agency) but also to try to ensure the learners were active. After establishing that 7 intermediate cups made up the container, I then tried to draw the conversation back to little cups, when a learner suggested there would be lots of the little cups:

RW: Are we agreed there will be more of these?

Learners: Yes

RW: There'll be more. But we don't know how many yet do we?

Learners subsequently make guesses (with a few guesses being multiples of 7) and some appear to count imaginary little cups, when Learner 4 interjects:

Learner 4: Wait, I have an idea, this could also be like a measuring one… because if you put it there you can see how many cups you are going to need …

RW: Let's have a look, let's see, what you were saying Leaner 4 is that if we find out how many of the little cup is the same size as that cup then we would be able to work out how many of the little cup would be the same amount

Learner 4: Yeh

Learner 7: If we, if we counted the cup and all of them are the same then we could count them and then you would know what the answer is

Learner 6: Maybe forty two

Learner 7: And then that could be how much the amount of water is

Learner 6: Maybe forty two

As noted in the section on task difficulty, the decision to establish the relationship between the intermediate unit and the large container first in this task possibly impacted on learners' understanding of the requirement to efficiently establish how many little cups would be equivalent to the large container using the intermediate unit, and thus impacted on the establishment of the relationship between the three objects. Hence this episode reflects tensions that exist between promoting student agency and involvement, whilst also ensuring a task is structured and experienced in a such a way that the important relationships are recognised.

Although coding showed there were many instances of learners suggesting ideas across the tasks (more, in fact than the teacher doing so), the analysis and reflection on episodes such as those above, suggest that Erisksson and Lindberg's (2016) reconciliation of the tension between epistemology and student agency in relation to Davydov's approach is one that could be applied more fully in Cycle 2. It would be important to recognise where student agency could be exploited and to ensure learners would be given the opportunity to explore and suggest the activity undertaken. Furthermore, structuring within tasks to establish key relationships would need to be considered with student agency in mind.

5.9 DESIGN PRINCIPLES REVISITED

After analysis of Cycle 1 data, it was concluded that design principles did not need to undergo major change, rather the way in which they were applied needed greater clarification and focus. These are discussed below, followed by revised design principles for Cycle 2.

1.*Through the context of measure, the task should support the development of the theoretical concept of multiplication involving a change in the system of units.*

This design principle is a fundamental principle of Davydov's (1992) view of multiplication. For Cycle 2, it would still be a fundamental vision that multiplication is seen as a change in the system of units, though this need not necessarily refer to the unit of measure itself; rather the unit being 'counted'. Although this was implicit in Cycle 1, it would be explicitly considered in Cycle 2, because a conclusion from Cycle 1 was that the use of standard measures could be exploited more; this would support understanding of measures and could mean that commercially available containers/objects with standard measures could be exploited. Hence, although the tasks should support the development of the theoretical concept of multiplication as a change in the system of the unit being considered, the units being 'counted' might be standard units or multiples of these.

2. *The task should be set up as a problem, where counting in ones is restricted, inefficient or impossible.*

The restriction of counting in ones certainly seemed to necessitate the need for a change in a system of units. However, whilst counting in ones should be restricted to solve the overall problem, counting in ones would need to be considered explicitly as part of the setting up of a relationship with an intermediate unit; Cycle 1 analysis suggests that tasks were most successful when learners were able to perceive an equality relationship and the way in which this equality relationship is established would need to be considered explicitly as part of the planning process in relation to the particular measure context. The structuring of how an intermediate relationship might be established within a task would also need to be considered, both in relation to the supporting of student understanding of an equality relationship and in relation to student agency.

3. *The problem, with the facilitation of the teacher, should invite social interaction, discussion and possible debate in order to suggest possible approaches to finding a solution.*

Again, the application of this principle needed to be considered more explicitly. Students would need to be invited to consider approaches to solving the problem and would be asked to discuss these in pairs and as a group.

4.*The task, with the facilitation of the teacher, should encourage transfer between the theoretical concept of multiplication as a change in units, and particular instances of this.*

To facilitate transfer between the theoretical concept of multiplication as a change in units and particular instances of this, the incorporation of key reflective questions would need to be considered. There would need to be facilitation of learner generalisation which should support learners to generalise a particular approach which is then experienced through particular examples.

5. *The task should be able to unfold in a range of possible directions, according to learner agency and teacher facilitation.*

The possible unfolding of tasks in different directions did not occur a lot in Cycle 1, possibly due to time and resource restrictions, but could be encouraged more within Cycle 2. Fundamentally, learners could be asked more questions such as 'what could you find out next?'.

6. *The tasks should involve a range of measures contexts.*

As noted in point 2, the measure context would be considered explicitly in relation to how an equality relationship could be established.

Revised design principles for Cycle 2, with changes in bold, are shown in Figure 16 below:

- Through the context of measure, the task should support the development of the theoretical concept of multiplication involving a change in the system of units under consideration; **this may involve standard or non-standard units.**
- The task should be set up as a problem, where counting in ones is restricted, inefficient or impossible, **though counting in ones may be necessary initially to establish an equality relationship.**
- The problem, with the facilitation of the teacher, should invite social interaction, discussion and possible debate in order to suggest possible approaches to finding a solution.
- The task, with the facilitation of the teacher, should encourage transfer between the theoretical concept of multiplication as a change in units, and particular instances of this.
- The task should be able to unfold in a range of possible directions, according to learner agency and teacher facilitation.
- The tasks should involve a range of measures contexts, **with explicit consideration of how equality is experienced**.

FIGURE 16: DESIGN PRINCIPLES RESEARCH CYCLE 2

CHAPTER 6: RESEARCH CYCLE 2: FROM DESIGN PRINCIPLES TO POINTS OF LEARNING

6.1 INTRODUCTION TO RESEARCH CYCLE 2

The focus of this chapter is the second cycle of design, from principles to implementation, including analysis of, and reflection on, the tasks to support the learning of the multiplicative relationship through measures.

Research Cycle 2 built on the findings from Cycle 1, to consider the same research questions as for Cycle 1:

Exploring the learning and teaching of multiplicative reasoning through measures: A design-based research project, in particular:

S1: What are teachers' and learners' prior experiences of teaching and learning multiplicative reasoning?

S2: What are learners' prior experiences of learning number and measures?

S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?

S4: What is the impact of learning multiplicative reasoning through measures on learners?

S5: What are teachers' and learners' views on teaching/learning multiplicative reasoning through measures using the materials developed?

Though research questions were the same, the research activity needs to be considered in light of contextual factors specific to Cycle 2, discussed further below.

Research Cycle 2 took place in the same primary school as for Cycle 1 and, again, with Year 2 (ages 6-7) learners. As discussed in Section 4.13, the school had expressed interest in being part of the complete research study and had an established research relationship with the university. Given the desire to seek teacher views on tasks and consider learners' prior experiences, working within the same school setting supported continuity. However, as noted in Section 1.3 and as illustrated in Appendix A, due to national and local lockdowns, and subsequent contingency measures within schools, there was a two-year gap between Cycle 1 and Cycle 2.

Cycle 2 began with a practitioner interview (two Year 2 teachers) to consider Year 2 learners' prior experiences; this was particularly important given the two-year gap with disruption to schooling caused by COVID-19 lockdowns and the local authority mitigations in place when schools re-opened for all learners. One of the Year 2 teachers was new to the study and the other had been a Year 2 teacher in Cycle 1. Whilst continuity with one of the participants was beneficial to seek perspectives on developments between Cycle 1 and Cycle 2, a new participant practitioner in Cycle 2 could also provide new perspectives. As noted in Section 4.13, ethical considerations within a study are ongoing and these practitioners were asked for informed consent (see Appendix G as an example). As in Cycle 1, participants in Cycle 2 were identified through seeking informed consent from parents (see Appendix H for an example).

The COVID-19 contingency arrangements still in place at the time of Cycle 2 affected the ways in which research could be conducted. For example, spaces were designated for use by certain year groups, and the timetable of the school day needed to account for year groups being kept together with limited mixing of adults and learners. This imposed some time restrictions on research activity. Furthermore, during Cycle 2 research, a short-notice school closure due to inclement weather affected the implementation of some tasks, and the planned semi-structured interviews with learners. This meant that learner responses were explored through a group interview, undertaken at short notice when the closure was announced for the following day. This block of research, conducted over four days, is discussed as Cycle 2a.

A subsequent short cycle of research, Cycle 2b, was undertaken to follow up on the implementation of some tasks and to consider learner and teacher responses. Due to staff absence and the re-implementation of some tasks, this research took place with a different group of learners. Time restrictions meant that semi-structured interviews in Cycle 2b also took place with a group of learners. Though two different groups of learners were participants within Cycle 2, this can be viewed as a benefit; this design research involves exploring tasks in authentic contexts and thus, a variety of learners can support this authenticity and can bring a wider range of perspectives and responses to the tasks. The actual learning of individuals has not been a focus of the research, rather the range of effects it may have on learning, and thus a variety of learners being involved can be seen as an advantage.

After Cycles 2a and 2b, a follow up interview took place with the Year 2 teacher who had been involved in both Cycles 1 and Cycle 2. Though perspectives from the other Year 2 teacher would have been beneficial, this was not possible due to illness.

Table 13 below provides an overview of research activity and participants in Cycle 2.

Research Question	**Method of exploration in Cycle 2**	**Data collected**
S1: What are teachers' and learners' prior experiences of teaching and learning multiplicative reasoning?	Semi-structured interview with Year 2 practitioners (n = 2)	Reflective notes. Audio recording of interview
S2: What are learners' prior experiences of learning number and measures?	Semi-structured interview with Year 2 practitioners (n = 2)	Reflective notes. Audio recording of interview
S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?	Implementation of tasks involving learner (n = 8 Cycle 2a, n = 5 Cycle 2b) and teacher feedback (n = 1).	Audio recording of tasks Reflective notes Audio recording of semi-structured interviews with learners
S4: What is the impact of learning multiplicative reasoning through measures on learners	Implementation of tasks and learner feedback through semi-structured interview (n = 8 Cycle 2a, n = 5 Cycle 2b)	Audio recording of tasks Reflective notes Audio recording of semi-structured interviews with learners
S5: What are teachers' and learners' views on teaching/learning multiplicative reasoning through measures using the materials developed?	Interview with Year 2 teacher (n = 1) Learner feedback through semi-structured interviews (n = 8 Cycle 2a, n = 5 Cycle 2b)	Audio recording of interviews with practitioner Audio recording of semi-structured interviews with learners

TABLE 13: CYCLE 2 RESEARCH ACTIVITY

Participants in Cycle 2 were:

Two Year 2 teachers, who are identified as Teachers 1 and 2 (T1 and T2) in the semi-structured interviews. Teacher 1 was also interviewed at the end of Cycle 2 and had been involved in Cycle 1. Teacher 2 was not available for interview at the end of Cycle 2 due to illness.

Thirteen learners in Year 2 (ages 6-7). These learners were learners for whom parental consent had been obtained and were considered by the class teachers to be a mixed attaining group in mathematics. The group included a learner with autism, a learner with emotional and behavioural needs due to poor speech and language and a learner with English as an additional language.

Learners have been identified as Learners 1-13 in all discussion. Learners 1-8 took part in Cycle 2a, and Learners 9-13 took part in Cycle 2b.

6.2 DEVELOPMENT OF TASKS: ITERATION FROM CYCLE 1 AND INTERVIEW WITH PRACTITIONERS

As discussed in Section 5.9, design principles for Cycle 2 were developed from analysis of Cycle 1. The design principles for Cycle 2 can be seen in Figure 16 (p.187).

Analysis from Cycle 1 suggested that a focus on standard units of measure could be exploited to support multiplicative reasoning and that greater focus on establishing equality relationships may support learning. Hence tasks were developed to take account of these principles.

An interview with Year 2 practitioners was also used to inform the development of tasks; this was particularly important as the tasks were to be implemented after the Year 2 learners in Cycle 2 had experienced a disrupted two years of education, due to lockdowns and subsequent contingency arrangements in place when schools re-opened for all learners.

The interview with Year 2 practitioners can be found in Appendix O. The interview informed task design, particularly as the Year 2 teachers suggested that the typical expectations for Year 2 would need to be adjusted. For example, when asked what factors may need to be considered when planning tasks, teachers responded as below:

T1: *And I just think they are not where they should be, they are not.*

T2: *The basics skills, mmm*

T1: *The basic skills, yes, which I didn't feel would happen because I thought we had that term back and we would have caught up, but I certainly don't think so, in maths and reading.*

T1: *I'd say they wouldn't be where they should be, where they would have been. We wouldn't have done so much on capacity…we weren't allowed to cook…*

The comments above suggest that learners' experiences during the pandemic had not only affected their basic skills in mathematics (with basic skills referring to aspects such as knowledge of multiplication facts or counting in steps other than one) but also experiences with equipment such as through cooking or water trays.

Further comments by teachers suggested a need to consider the number relationships learners might be working with. For example:

T1: …certainly not at the level…I mean you've got your obvious highflyers and they've been able to push, and we've gone onto our five times table now, but even last years, able as they *were, and they could, they had strategies for their tables. But learning them, and the Year 3 teacher has said they don't know them. They've got the strategies but I don't think they had the daily drilling so I do think that we're still covering the basic skills…*

And:

T1: inverse operations and things… The commutative law, they love learning the commutative law and these lot I'm finding they haven't, even my more able couldn't…we were doing arrays as we were doing the two times table and they couldn't see it…as much as they would have in other years. I think there has been two terms out, the younger the children, the more of an impact COVID has had and I think maths has definitely taken a hit.

Teachers also commented that learners' approach to problem solving might have been affected. For example:

T1: I don't know. I don't feel mine are offering. I keep saying just tell me anything… be creative in your thinking because they've got to be creative, and I'm… it's always the same group of children but I am not having as much…

T2: I think there are quite a few children, and whether this is COVID related or not I don't know, they are frightened of having a go.

T1: Yeah

T2: They feel it has to be right, and we're forever saying 'This is why we're here, I'm learning constantly and it doesn't matter if it's right, wrong, we're here to talk about it' and a lot are still frightened. And, as I said, whether it is COVID and they've been at home and things have been done for them I don't know, but we try that a lot in school 'Let's have a go', well 'Let's try it your way, let's have a go your way' and get them to say. I have got some children that will, I'm quite confident, and will come out with ideas and they don't mind if it's right or wrong, but some will hold back and I think you get that in any year group anyway.

In addition to this, teachers commented on learners’ approach to working together, including sharing of resources and taking turns with equipment:

T2: So I've had quite a bit of problems in my class where sharing has been, so we've had to talk a lot more than I ever have done before and actually had to physically show them how we share, how we take turns and yeah I've not had that before. I think they are slowly getting there.

Thus, in planning tasks it would be important to account for the learners’ potential lack of experiences with measuring equipment and I should expect their knowledge of counting in steps other than one may be more limited to counting in twos, fives or tens. It would also be important to build confidence in sharing ideas and equipment.

In terms of the way the school approached the multiplicative relationship, visual images such as arrays were mentioned, and the concept of grouping was reinforced in comments made by the teachers. Comments from one teacher suggested multiplication was seen as a skill to be taught, with the application of this coming afterwards. For example, when asked about whether there had been any differences in the approach to mathematics teaching since the previous visit, this teacher responded:

T1: I’d say reasoning. We’re very much still concrete visual abstract, very much real life wherever possible I mean you still need to teach multiplication before you can apply it, but I'd say everything is more or less the same. The reasoning I think we're far more aware of trying to get a reasoning problem into any situation, be it maths or whatever really. And that’s our core purposes and everything.

And later:

T1: And reasoning problems after doing, you know while you're doing the two times table, and you throw in a reasoning problem it's well 'Woah', it's like you're doing another language, 'what you doing now, you were doing times table a minute ago', not realising the connection at all.

The comments above possibly suggest a belief that a concept should be taught before being applied within a problem-solving situation. It was concluded that learners' experiences of problem solving and reasoning situations may have been affected by their experiences of learning during school closures. Hence, introducing a concept through a problem-solving scenario may be unfamiliar to the learners.

To conclude, the interview suggested that I could expect to be planning for Year 2 learners who were typically likely to have:

- some experience of measuring with non-standard and standard units, though the experiences may be limited due to constraints caused by lockdowns and subsequent school mitigations
- experienced multiplication as repeated addition and may have some awareness of the commutative nature of multiplication
- experienced division as sharing, though also likely to have some experience of division as grouping
- some experience of working in groups to approach problems

but who may be unlikely to have:

- experienced measures as a context for learning number relationships, and in particular, the multiplicative relationship
- experienced the multiplicative relationship as involving a 'change in unit'

and, further, may also be likely to need:

- support with working with measuring equipment
- support with working in problem-solving situations and sharing thoughts and ideas
- reinforcement of some number relationships

6.3 TASK DESIGN AND IMPLEMENTATION

Tasks were developed in accordance with revised design principles (Figure 16, p. 187) and informed by the interview with practitioners. Many of the tasks were adapted from tasks used in Cycle 1, with the adaptations and reasons for these adaptations recorded in Table 13 below. As discussed in Section 5.3, Burkhardt and Swan's (2013) ideas around task difficulty were also considered for each task and are noted in Table 13. Aspects of difficulty noted by Burkhardt and Swan (2013, p.181), such as technical demand (level of mathematics required) and familiarity (e.g., with equipment such as pan balances) needed to be considered following the interview with practitioners (Appendix O), discussed in Section 6.2. Furthermore, some increased use of pair work, moving into to group work was included to account for comments about learners' possible reactions to group work and the sharing of equipment and ideas.

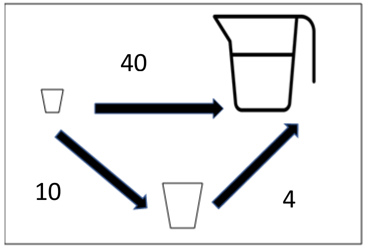
Unlike Cycle 1, there is no distinction between warm-up tasks and key tasks. Time constraints and COVID-19 mitigations meant a pre-assessment with learners had not been possible and, in Cycle 1, warm-up tasks sometimes took as long as the key tasks. Tasks in Cycle 2 were structured to incorporate key ideas considered within Cycle 1 warm-up tasks, whilst also accounting for comments from practitioners within the interview.

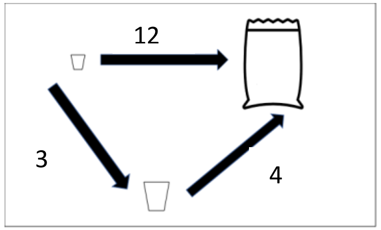
As noted in Section 6.1, due to a short-notice school closure for inclement weather, some tasks were followed up in Cycle 2b, with a different group of learners. This also gave the opportunity for the further re-development of tasks involving length and mass, particularly those that were new to Cycle 2a and affected by time constraints. Tasks are labelled according to the cycle (2a or 2b), day within cycle (1 to 4) and order (a to c) in which they took place.

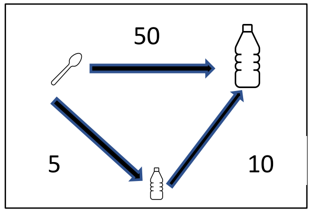
As with Cycle 1, the tasks took place in one part of an open plan area outside the main classroom. This area had tables and chairs which could be rearranged into groups or paired working spaces and was close to a source of water. The area was a familiar space to the learners, frequently used for small group work outside the Foundation Phase classes. As the area was open plan, there were often other classes or staff members walking past, though this did not appear to significantly distract the learners. Audio recording devices were used and, though an attempt was made to record different conversations as learners worked in pairs, the devices tended to record all the conversations in the space rather than isolated ones. On occasions, some conversations were difficult to distinguish even when a separate device was used.

Learners 1-8 participated in Cycle 2a and Learners 9-13 in Cycle 2b.

Task	Summary of expected learner activity	Purpose	Design notes	Development from previous implementation
C2a.1a Making same quantity of liquid Questions: Here is some red liquid and here is some yellow liquid. Can you make the same amount of yellow liquid as red liquid in this container? How will you be sure that you have the same amount of liquid 	Suggesting ideas for how they can ensure the same amount of yellow liquid as red liquid. Showing awareness of a need to quantify/measure the amount of red liquid in order to reproduce the same amount of yellow liquid.	To assess learners' understanding of concept of unit.	Acting as an assessment of learners' understanding of unit. Restriction on pouring red liquid directly into container. Different shape container for red liquid to necessitate quantification. Complexity: Not complex (Un)familiarity: Idea of unit not new but context and equipment likely to be unfamiliar. Technical Difficulty (level of maths): Not difficult Autonomy: Teacher and group, with partner work.	Learners were challenged to share ideas as a group initially. The use of a cup as a possible unit was introduced explicitly. Learners were asked to only pour red liquid first and the number of cups available was restricted to prevent replication.
C2a.1b Using straws to measure Questions: Here are some straws – red straws and yellow straws. Do you notice anything about the relationship between the straws? If you measure with the red straws and then also measure with the yellow straws, how will your answers be different? Can you measure these sticks with both the red straws and the yellow straws? Could you predict what the number of yellow straws would be if you knew the number of red straws? 	Discussing relationship between yellow and red straws. Showing awareness that the yellow straws will give a larger number than the red straws. Possibly being able to predict that the number of yellow straws will be double the number of red straws.	To assess learners' understanding of unit and the relationship between units and referent number in measure.	Acting as an assessment of learners' understanding of relationship between unit and referent number in a measure. Red straw measures 10cm and yellow straw measures 5cm. All sticks multiples of 10cm. Restrict number of red and yellow straws available to necessitate iteration and possible prediction of yellow straws. Ask one partner to use red straws and other to use yellow. Complexity: Not complex (Un)familiarity: Learners may have had limited opportunity to undertake measuring with non-standard units due to disrupted schooling Technical Difficulty (level of maths): Not difficult Autonomy: Teacher and group, with partner work.	More rigid materials were used for measuring. Partners were introduced so some learners could measure with one colour straw and the other could measure with the other colour to encourage learners to apply multiplicative relationship between the two units.

Task	Summary of expected learner activity	Purpose	Design notes	Development from previous implementation
C2a.1c How many flapjacks? Questions: Here is recipe for flapjacks. One cup makes five flapjacks. How can I find out how many flapjacks I can make from this container of oats? 	Discussing how to find out how many flapjacks can be made from the bag. Showing awareness that each cup represents 5 flapjacks	To assess learners' understanding of a composite unit.	Acting as an assessment of understanding of a composite unit. Have enough cups so that each cup can be filled for visual representation. Complexity: Not complex (Un)familiarity: Learners may have had limited experience of recipes Technical Difficulty (level of maths): Not difficult, the composite unit 5 was chosen rather than 6 in Cycle 1 Autonomy: Teacher and group, with partner work.	Adapted from Pancakes task using a less messy material. The unit size was chosen to be 5 to account for a range of learners and based on teacher interviews of learner experiences. In structuring, there would be a greater focus on equality relationship through establishing initial relationship and through having sufficient cups for measuring task
C2a.2a How many rabbits? Questions: This little cup is enough water for one rabbit for a day. I want to find out how many rabbits I can feed with this amount of water (in the jug)? How could I do this? Is there a quicker way? 	Discussing how to find out how many rabbits can be fed and whether there may be a quicker way of finding out. Recognising (with support) an equality relationship between a little cup and an intermediate unit. Using the composite unit to find out how many rabbits can be fed from a jug.	To introduce notion of intermediate unit 	10 little cups = 1 big cup. This relationship will be established together, by counting the number of times the little cup needs to be filled and how many cups fill the intermediate unit. Learners are then asked to find out how many rabbits can be fed with a jug of water but they are not given access to the little cup. Complexity: Not complex (Un)familiarity: A new idea being introduced Technical Difficulty (level of maths): Not difficult, the relationship was chosen to be less technically difficult Autonomy: Teacher and group, with partner work.	The establishing of the relationship between one little cup and the intermediate cup was reinforced. The little cup was removed. A relationship of ten little cups being equal to one large cup was chosen to account for learners' experience with number relationships.

Task	Summary of expected learner activity	Purpose	Design notes	Development from previous implementation
C2a.2b How much porridge? Questions: This container contains enough oats for one person to have a portion of porridge. How many portions of porridge are in this bag? Is there a quicker way of finding out (intermediate unit cup available)? 	Discussing how to find out how many portions of porridge are in the bag, building on 2a. Using intermediate unit to work out how many portions of porridge there are.	To reinforce concept of intermediate unit 	3 little containers = 1 cup. The relationship is established together as a group. Learners are then asked to find out how many portions of porridge are in a bag (12 portions). Complexity: A little more complex (Un)familiarity: Building on C2a.2a Technical Difficulty (level of maths): Not difficult, the relationship was chosen to be less technically difficult Autonomy: Teacher and group, with partner work.	A new task building on C2a.2a and C2a.1c. In Cycle 1, learners reported enjoyment of capacity tasks and this allowed for the reinforcement of the concept of an intermediate unit.
C2a.3a Making lengths without using single centimetres. Questions: If this is 1cm, what might these lengths be (show straws)? How do you know? If I make a line 20cm long, how many 2cm will I need? How many 5cm will I need? What if you make a line 40cm long, or 60cm long? 	Discussing and establishing lengths of green, yellow and red straws. Exploring relationship between 20cm, 10cm, 5cm and 2cm. Making lines 20cm, 40cm and 60cm long and finding out how many 2cm, 5cm and 10cm straws are equal to these lengths.	Reinforcing the use of composite unit, restriction of counting in single unit. To make links with standard units of measure.	Making lengths as multiples of 10cm encourages consideration of multiplicative relationship as learners will need to establish how many red straws are needed. They are then asked to work out how many yellow and green straws are needed to make the same lengths. For the longer lengths (40cm and 60cm, availability of green and yellow straws restricted) Complexity: Complexity building with multiple (though familiar) relationships (Un)familiarity: Building on 2a.1b Technical Difficulty (level of maths): Familiar relationships Autonomy: Teacher and group, with individual work	Learners in Cycle 1 showed good engagement with warm-up tasks involving Cuisenaire. To avoid learners associating Cuisenaire rods as specific measurements, straws were used, and this built on 2a.1b as the same straws were used (red = 10cm, yellow = 5cm

Task	Summary of expected learner activity	Purpose	Design notes	Development from previous implementation
C2a.3b How much medicine? Questions: My dog needs 10 millilitres of medicine each day. This spoon is worth 10 millilitres. I want to find out how many spoons worth of medicine is in this bottle. How could I do that? Is there a quicker way than counting spoons? 	Suggesting ideas for a quicker way of counting spoons. Recognising an intermediate unit (bottle) could help. Finding out how many spoons worth are in the bottle by using the intermediate unit	To reinforce the notion of an intermediate unit. To make links with standard units of measure.  	5 10ml spoon = 1 50ml bottle. There are 10 50ml bottles worth in the big bottle. Relationship between spoon and little bottle is established as a group. Learners are then asked to find how many spoons worth are in the big bottle, but the use of the spoon is restricted, necessitating counting the bottle as equal to 5 spoons. Complexity: Complexity building with multiple (though familiar) relationships (Un)familiarity: Building on 2a.2a and 2a.2b. Technical Difficulty (level of maths): Familiar relationships Autonomy: Teacher and group, with paired work	A 50ml bottle is used to support 5 x 10ml and thus a composite unit representing 5. Reinforcement of equality relationship structured into task.
C2a.4a Exploring relationships between different masses Question: How many 1g weights are the same as these weights? How many 5g weights are the same as these (10g, 20g). 	Exploring relationship between 5g, 10g and 20g masses. Recognising that it is easier to weigh in multiples of 5g, 10g or 20g.	To reinforce use of composite unit. To establish relationship between 1g, 5g, 10g and 20g.	1g will only be used to introduce what 1g feels like as a mass/weight. Once the relationship between 1g and other weights is established, its use will be restricted. Complexity: Complexity building with multiple (though familiar) relationships (Un)familiarity: Learners may not be familiar with pan balances. Technical Difficulty (level of maths): Familiar relationships Autonomy: Teacher and group, with paired work	A new task to support learner experience of and understanding of working with different masses and the use of the pan balance. Establishing equality relationship is a key focus.

Task	Summary of expected learner activity	Purpose	Design notes	Development from previous implementation
C2a.4b How many portions of pasta? Questions: 10g of pasta is needed for one portion of pasta soup. How many portions of pasta soup could be made from these bags? How could you find out? 	Suggesting ways to find out how many portions of pasta can be found. Recognising that the mass can be established through use of a composite unit.	To use composite units as a measure.	The use of 1g restricted. Learners are restricted to using one particular composite unit (5g, 10g or 20g each time). Bag A = 40g of pasta Bag B = 60g of pasta Complexity: Complexity building with multiple (though familiar) relationships (Un)familiarity: Learners may not be familiar with pan balances. Technical Difficulty (level of maths): Familiar relationships Autonomy: Teacher and group, with paired work	A new task to support learners in using 5g, 10g, 20g and the composite unit. Establishing equality relationship a focus.
C2b.1a Using straws to measure (Similar to C2a1b) Questions: Here are some straws – red straws and yellow straws. Do you notice anything about the relationship between the straws? If you measure with the red straws and then also measure with the yellow straws, how will your answers be different? Can you measure these sticks with both the red straws and the yellow straws? Could you predict what the number of yellow straws would be if you knew the number of red straws? 	Discussing relationship between yellow and red straws. Showing awareness that the yellow straws will give a larger number than the red straws. Possibly being able to predict that the number of yellow straws will be double the number of red straws.	To assess learners' understanding of concept of unit.	Acting as an assessment of learners' understanding of relationship between unit and referent number. Red straw 10cm, yellow straw 5cm. All sticks multiples of 10cm. Restrict red and yellow straws available to necessitate iteration and possible prediction of yellow straws. Ask one partner to use red straws and other to use yellow. Complexity: Not complex though partner work (not seeing each other's stick) adds complexity (Un)familiarity: May be limited experience with non-standard measuring Technical Difficulty (level of maths): Familiar relationships Autonomy: Teacher and group, with paired work	Developed from C2a1b. Rigid lengths labelled to support partner communication. Partners moved between area as part of task – to support communication and working with the multiplicative relationship. Straws flattened to prevent rolling.

Task	Summary of expected learner activity	Purpose	Design notes	Development from previous implementation
C2b.1b Making lengths without using single centimetres. (Similar to 2a3a) Questions: If this is 1cm, what might these lengths be (show straws)? How do you know? If I make a line 20cm long, how many 2cm will I need? How many 5cm will I need? What if you make a line 40cm long, or 60cm long? 	Discussing and establishing lengths of green, yellow and red straws. Exploring relationship between 20cm, 10cm, 5cm and 2cm. Making lines 20cm, 40cm and 60cm long and finding out how many 2cm, 5cm and 10cm straws are equal to these lengths	Reinforcing the use of composite unit, restriction of counting in single unit. To make links with standard units of measure.	Making lengths as multiples of 10cm encourages consideration of multiplicative relationship as learners will need to establish how many red straws are needed. They are then asked to work out how many yellow and green straws are needed to make the same lengths. For the longer lengths (40cm and 60cm, insufficient numbers of green and yellow straws are available so learners will need to predict). Complexity: Complexity building with multiple (though familiar) relationships (Un)familiarity: Building on 2a.1b Technical Difficulty (level of maths): Familiar relationships Autonomy: Teacher and group, with individual work	Developed from C2a.3a. Task sequenced to follow on from C2b.1a. Straws flattened to prevent rolling. The 1cm straw introduced as a unit but restricted thereafter. Learners encouraged to recognise that use of smaller units inefficient when measuring longer lengths.
C2b.2a Exploring relationships between different masses (Similar to 2a4a) Question: How many 1g weights are the same as these weights? How many 5g weights are the same as these (10g, 20g). Can you identify multiplicative relationships between them? 	Exploring relationship between 5g, 10g and 20g masses. Recognising that it is easier to weigh in multiples of 5g, 10g or 20g	To reinforce use of composite unit. To establish relationship between 1g, 5g, 10g and 20g.	1g will only be used to introduce what 1g feels like as a mass/weight. Once the relationship between 1g and other weights is established, its use will be restricted. Complexity: Complexity building with multiple (though familiar) relationships (Un)familiarity: Learners may not be familiar with pan balances. Technical Difficulty (level of maths): Familiar relationships Autonomy: Teacher and group, with paired work	Developed from C2a.4a. The 1g mass introduced to establish its mass and to reinforce equality relationship with 5g. Thereafter 1g restricted. Time given for learners to explore relationships, e.g. 60g in 5g, 10g and 20g.

Task	Summary of expected learner activity	Purpose	Design notes	Development from previous implementation
C2b.2b How many portions of pasta? Questions: 10g of pasta is needed for one portion of pasta for a baby. How many babies could be fed from these bags? How could you find out? 	Suggesting ways to find out how many portions of pasta can be found. Recognising that the weight can be established through use of a composite unit.	To use composite units as a measure	The use of 1g restricted. Learners are restricted to using one particular composite unit (5g, 10g or 20g each time). Complexity: Complexity building with multiple (though familiar) relationships (Un)familiarity: Learners may not be familiar with pan balances. Technical Difficulty (level of maths): Familiar relationships Autonomy: Teacher and group, with paired work	Development from C2a.4b. Time given to establish equality relationship. Context of pasta soup changed. Smaller pasta pieces used to discourage suggestion of counting pasta pieces.

TABLE 14: TASKS IN CYCLE 2

6.4 APPROACH TO ANALYSIS OF DATA FROM TASK IMPLEMENTATION

Like Cycle 1, data from Cycle 2 consisted of:

- audio data from the tasks (learning and teaching episodes)
- reflective notes
- interviews with learners (see Appendix K for semi-structured interview questions; note that, as discussed in Section 6.1, these interviews took place with groups of learners rather than with individuals, though the same main questions were used as in Cycle 1)

In addition to this, as discussed in Section 6.1, an interview was undertaken with a practitioner who had been involved in Cycles 1 and 2.

As in Cycle 1, analysis focused on exploring the learning and teaching of multiplicative reasoning through measures, in relation to the following sub-questions:

S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?

S4: What is the impact of learning multiplicative reasoning through measures on learners?

S5: What are teachers' and learners’ views on teaching/learning multiplicative reasoning through measures using the materials developed?

In consultation with the supervision team, I decided that data analysis of the tasks would be approached in the same way as Cycle 1 (see Section 5.7): audio data from task implementation were transcribed with the Behaviour, Emotion, Awareness coding framework, and memoing in NVivo was also used for reflexive notes. No changes were made to the data analysis process because it was felt that the analysis in Cycle 1 had allowed for points of learning to be identified, both to the task development, implementation and to the pedagogic approach to tasks.

Learner evaluative responses to tasks were collated and tabulated across Cycle 2a and Cycle 2b (see Appendix P).

6.5 POINTS OF LEARNING: CYCLE 2

In this section the points of learning from Cycle 2 are discussed. Points of learning derive from triangulating data from the teaching and learning episodes, reflective notes and learner responses to tasks. The same four foci used in Cycle 1 (discussed in Section 5.8) are used:

1. Task efficacy (S3 and S4)
2. Task difficulty (S3)
3. Learners' emotional and evaluative responses to tasks (S5)
4. Pedagogic approach to tasks (S3 and S4)

In addition, a further focus is used to consider practitioner response to tasks, supporting research question S5, with data from the practitioner interview (see Appendix Q for interview questions and responses).

5. Practitioner response to tasks (S5).

TASK EFFICACY: TO WHAT EXTENT DID TASKS SUPPORT LEARNING OF THE MULTIPLICATIVE RELATIONSHIP?

The awareness codes used in Cycle 1 were also applied in Cycle 2 to identify instances that might indicate multiplicative reasoning. Like Cycle 1, some tasks showed higher incidences of certain aspects of multiplicative reasoning than others.

The highest incidence of awareness of a change in unit occurred in Task C2a.2a. In this task, comments such as '*I know! I know one. We could keep on filling the big cup until all the water's gone'* (Learner 4) suggest awareness of a change in unit. Similarly, a comment by Learner 2 shows awareness that a change in unit is being established:

Learner 2: Bring the water up to the black line

RW: Ah!

Learner 2: And then you'll know it's worth the same

This highest incidence of awareness of change in unit reflects findings in Cycle 1 and is, perhaps, unsurprising because the 'rabbits' task is a task used by Davydov (1992) to introduce the idea of a change in unit and was used in this way within this study in both cycles.

Also similar to findings in Cycle 1, Task C2a.3b, the 'dog's medicine' task reflects awareness of change in unit as learners established a bottle of medicine was worth the same as five spoons of medicine.

Another similar finding to Cycle 1, was that those tasks involving length with a restriction in the use of single centimetres, also showed a higher incidence of awareness of change in unit. For example, in Task C2a.3a, Learner 6 recognises the change in unit being considered:

Learner 6: You don't need the one centimetres, you only need the two centimetres

A further finding similar to Cycle 1, was that awareness of multiplicative relationships appeared across all tasks in Cycle 2. Again, this is unsurprising as all tasks involved specific relationships. In Cycle 2, highest incidence of awareness of multiplicative relationships occurred in C2a.3a and C2a.4a, and in C2b.1b and C2b.2b. These tasks were similar across Cycles 2a and 2b (C2a.3a 'length' similar to C2b.1b and C2a.4a 'mass' similar to C2b.2b); the tasks involved exploration of multiple multiplicative relationships, so it is not surprising that learners showed awareness of such relationships in their comments. One noteworthy contrast is that, in Cycle 1, learners referred to C1.3a and C1.4a (tasks involving Cuisenaire) as being tasks that helped with mathematics and 'lots of mathematics'; this was not as evident in Cycle 2. This may be for several reasons. One possible reason is that learners were not asked to record their findings in the same way as in Cycle 1, because of time constraints and because there was possibly less familiarity with methods of recording multiplication calculations, due to loss of schooling. Another possible reason is that learners did not use Cuisenaire, but straws, which moved more easily than Cuisenaire. In both Cycles 2a and 2b, learners referred to frustration in this, e.g.:

Cycle 2a *Learner 1*: *With straws, no, because it is too, because it is super annoying*

Cycle 2b *Learner 11: Um this one was hard because it was super hard to get those things to make into a straight line*

In Cycle 1, there was evidence of awareness of multiplicative relationships involving fractions (e.g., the suggestion that a cup of flour was a quarter of a bag of flour as discussed in Section 5.8). This was not as evident in Cycle 2. However, there was awareness of half and double relationships, e.g., in C2a.1b in which learners were discussing a relationship between straws, learners made the following comments:

Learner 4: They can be the same size if they touch

Learner 2: There's a relationship…because if you add four on, there's eight

Learner 2's comment shows awareness of the half and double relationship, even though it suggests additive reasoning is used to express this.

Furthermore, in C2b.1b, a learner made the following comments when trying to summarise the relationship found with using both coloured straws to measure:

Learner 10: So, this is four, because if that was eight, this would be four… If that was...six, that would be three and if that was ten, that would be five. Half of everything is the yellow straw.

Awareness of composite units occurred most in tasks C2a.1c, C2a.2b, C2a.3a, C2a.3b and in tasks C2b.1b and C2b.2a. This is not surprising because these tasks involved the establishing of composite units and the use of at least one composite unit to measure. These tasks are similar tasks to those which demonstrated higher incidence of awareness of composite units in Cycle 1. Like task C1.2b in Cycle 1, in task C2a.1c, learners were asked to consider how they might find out how many flapjacks could be made from a container of oats if they knew one cup of oats could make five flapjacks. Although learners were not asked to make estimates, learners suggested multiples of five, for example:

RW: Talk to the people next to you. How can you find out how many flapjacks you can make?

Learner 4: I think about forty.

RW: One cup makes five, but how can I find out how many flapjacks can I make from these oats? Is there more than one cup here?

Learner 4: I know. I think like 60 or 50.

RW: You're making a guess. How could you find out?

Learner 3?: I think we can make twenty.

Learner 4?: Thirty.

RW: Learner 5? Learner 6?

Learner 5: You can mmm...

RW: Any ideas Learner 8?

Learner 1?: Wow!

Learner 4: I know! We can...get lots of cups

RW: Listen to Learner 4

Learner 4: We could get lots of cups and then we could fill it up until we get all of the oats...

Interestingly, the learners tended to suggest estimates (all multiples of five), even though the initial question asked them to suggest how they might find out how many flapjacks could be made. As I had reflected on the wording of some questions in Cycle 1, I had been more aware of this in Cycle 2 and had tried to avoid asking 'How many' when I wanted to discover ideas. In the interview with Year 2 teachers prior to implementing the tasks, Teacher 1 had mentioned that estimation had been a focus of some tasks *'We've been doing estimating today'*. Although the tasks were implemented several weeks after this interview, it is possible learners were drawing on such experiences.

Overall, across the tasks, the coding incidences mirrored the incidences in Cycle 2 suggesting that particular tasks encouraged particular aspects of multiplicative reasoning, in line with what they were designed to do.

Task difficulty

After Cycle 1 analysis, the most technically difficult and complex task C1.4b was adapted. Unlike Cycle 1, where task difficulty seemed more apparent in particular tasks, in Cycle 2, learners being challenged by tasks was apparent across a greater range of tasks. Although learners' prior experiences (or possible lack of these) had been accounted for in the choice of relationships used within tasks and was considered when introducing materials such as pan balances (see Table 14, p.205 for notes in relation to each task), it is possible that the learners were still affected by having missed play experiences involving water, sand or balances. For example, in task C2a.4a, learners spent a lot of time exploring the pan balances and masses; although this was built into the task, time constraints that day (due to announcement of closure of school the subsequent day) and the interest in exploring the new equipment, meant that the task was repeated in Cycle 2b. Hence, the familiarity of materials and equipment to learners, though accounted for in initial task design in Cycle 2a, was still a factor in the difficulty of this task. More focus was given in C2b.2a to exploring how the pan balance worked, and to introducing the masses and to exploring equivalencies within masses in a more structured way. In discussion of Cycle 2b, learners commented that they had learned about weight/mass, e.g., when asked whether they thought the activities helped with mathematics, the following responses occurred:

Learner 9: Uh strength

Leaner 10: What!

Learner 9: I mean weight…and how much one gramme is like, two, uh, twenty grammes is like the…

Learner 12: The heaviest. It's not that heavy though

Although learners showed awareness of multiplicative relationships within this task, it appears the key learning for some may have been what masses felt like. This does, perhaps, reflect the difficulty of the task, because familiarity needed to have been considered even more at the planning stage, particularly as teachers had indicated this in the interview:

T1: I'd say they wouldn't be where they should be, where they would have been. We wouldn't have done so much on capacity…we weren't allowed to cook…

T2: Yes, we weren't allowed certain things for a long time…

T1: So weighing…

T2: Until recently

T1: Yes, equipment…

Also reflecting practitioner comments in the interview, learners did need encouragement in sharing ideas. For example, here is an extract from task C2a.1a:

RW: Right, talk to the people next to you. Learner 6, talk to Learner 5. How could you use these to help you?

Learner 1: I'm talking with Learner 6 and Learner 4

RW: That's OK, Learner 5, Learner 6 and Learner 4 can talk. How could you use these to help you?

Learner 4: I don't know...

It could be that this was an early task and learners were getting used to the approach being used. Furthermore, as discussed later, this task could also have been difficult because of the way it was introduced. As tasks developed there appeared to be less encouragement needed to share ideas, although at the end of Cycle 2a, learners did comment on finding it hard not knowing initially what to do:

Learner 1: Because when me and Learner 8 were partnered up, we didn't know what to do and that was the most maths

RW: That was the most maths because you didn't know what to do, and you had to think

Learner 1: Yes, but then I told Learner 8 to tell you, so yes

RW: OK and Learner 7, what were you going to say?

Learner 7: I was going to say the liquids because at the start when I had my partner, my partner, I didn't know what to do but then my partner helped

Thus, the familiarity of approach (experience of problem solving and not knowing initially what to do) may have contributed to task difficulty in Cycle 2.

Of note in both Cycles 2a and 2b, was learners' response to the pasta tasks (C2a.4b and C2b.2b). Although it had been established what 10g of pasta looked like, in both tasks learners suggested finding out how many portions by either counting pasta or by portioning into single portions (and they suggested strategies for doing this) rather than finding the mass.

For example, in task C2a.4b, where A is used to refer to a bag of pasta, learners were asked to discuss how they might establish how many portions of pasta there may be, if they knew 10 grammes was one portion, learners started to count:

For example:

RW: So if you, how could you find out much pasta this is enough for

Learner 6: Count out...(starts counting)...one, two, three, four, five...

RW: You don't need to count the pasta pieces

Learner 6: But I want to

And:

Learner 7: Count A

Learner 1: OK so I am going to make

Learner 6: One, two, three, four, five, six, seven, eight

It may be that the pieces of pasta, being more discrete, invited counting. Another material, such as lentils, may have reduced the temptation to count.

In relation to portioning, in task C2b.2b, the following extract provides another example:

Learner 9: And then we could put like, try and put one portion there, and portion there, so we know that's one portion for one baby and the other portion for another baby and then we could like keep on doing that

RW: Ah, you could keep on doing it

Learner 9: And count all of the bags

At first, it seemed surprising that learners suggested portioning rather than establishing the mass of the bags and working out how many ten gramme portions of pasta for which that would be equivalent, but the actions in all other tasks involved portioning and therefore it is understandable that learners suggested this strategy. Establishing the number of portions without having to portion can be seen as a progression within multiplicative reasoning; whilst this was possible within this task, learners needed support in this strategy, and the inefficiency for avoiding direct portioning was perhaps not sufficiently evident, due to relatively small bags of pasta being used. The technical demand of the tasks was a key consideration in Cycle 2 and within tasks C2a.4b and C2b.2b, and though counting individual pieces of pasta was set-up to be inefficient, the masses of bags of pasta used were quite small. Presenting tasks so that previous approaches are rendered inefficient is a key part of the design principles (see Figure 16, p. 187) and within these tasks, they could have been set up to make the of counting individual portions more inefficient.

The sequencing of tasks affected the task difficulty in task C2a.2b. This task was designed to build on task C2a.2a, supporting the notion of a change in unit, with an intermediate unit being used, although with a different material (oats). However, although the smallest units being considered were different in tasks C2a.2a and C2a.2b, the intermediate units being used were the same. In task C2a.2b, although the establishing of a relationship between the smallest unit and the intermediate unit was included as a group demonstration, learners seemed to associate the intermediate unit with the relationship established in task C2a.2a.

For example, this conversation can be heard between Learners 1 and 8 in C2a.2b:

Learner 1: That's one full cup

Learner 8: We're counting in tens

And later, with another learner:

RW: Can you remember how much a cup is worth, is enough for how many people?

Learner 3: Ten

Hence the difficulty of task C2a.2b seems to be caused by the sequencing and the choice of use of the same intermediate unit.

In Cycle 1 analysis, it was identified that the choice of materials to be measured affected task difficulty. For example, certain materials were prone to spillage, threatening the accuracy of pre-established relationships. Despite the materials being considered carefully for Cycle 2 in relation to potential spillage and related task difficulty, learners did, understandably, still spill materials. They did, however, account for the spillage in their comments, showing awareness of quantities in relation to the units of measure being used.

For example, in task C2a.2a, in which ten little cups were equivalent to one bigger cup the following occurred when discussing how many rabbits might be fed with a particular volume of water, where nearly three cups had been filled:

RW: So if we had make an estimate, a good guess at how many rabbits this will feed, what could we say? Ten...
Learners: Ten, twenty
Learner 2: Twenty-nine

Learner 2 was accounting for one less rabbit due to the cup being nearly, but not completely, full.

A little later, when working with the same units and relationship, where more spillage had occurred, the following conversation occurred:

Learner 1: Ah, I think we had ten, twenty
Learner 8: Twenty
Learner 1: Twenty ah
Learner 4: Five
Learner 1: Twenty-five and ah!
Learner 8: Twenty ah!
RW: You spillt some as well didn't you!
Learner 8: Spilled a lot!
RW: So at the moment...
Learner 7: Yeh! I'm wiping it up!
RW: You're saying twenty...Why are you saying five there?
Learner 1: Because that's not a lot
RW: Oh, because it's half of it, it's about half of that cup, you're saying is it
Learner 1: Yes

Such conversations suggest that learners were able to account for spillage when working with materials and that, indeed, spillage could become part of reinforcing relationships. Though in the example above I use the term ‘half’, Learner 1 had originally said ‘five’ when a cup was half full, suggesting awareness of it being half of the quantity being considered.

To conclude this section, Cycle 1 analysis suggested that difficulty was mainly affected by structuring within tasks and sometimes the availability of items used to establish relationships. This had been considered more carefully within the design of the tasks for Cycle 2, but difficulty in Cycle 2 seemed to have been affected more by the sequencing between tasks and the way in which some materials were used.

Learners' emotional and evaluative responses to tasks

As in Cycle 1, learners frequently expressed enjoyment and surprise when working on the tasks. Like Cycle 1, many expressions of excitement or surprise were coded in the early tasks and occurred when materials were being introduced. Nevertheless, learners appeared to respond positively to latter tasks too. Conversations such as the one below, from C2a.4b, is an example of conversations which occurred frequently throughout tasks where learners appeared to enjoy the experience of working with the materials:

Learner 1: That's too much more

Learner 2: OK, that's too much so let's take that out

Learner 1: Can we do this (sounds of moving pasta)

Learner 2: OK, look, so we made ten, now we're going to drop that back in (sounds of dropping), satisfying, very satisfying

Learner 1 (laughs)

Learners also expressed 'satisfaction' when working on capacity tasks, for example:

Learner 2: I liked the liquids

RW: You liked the liquids, why did you like the liquids?

Learner 2: Because it was satisfying

Learner 7: Me too

RW: Satisfying

Learner 2: Satisfying when you poured it out

Learner 7: I feel the same as well, I'm the same

RW (to Learner 7): You're the same, you liked the liquids

Learner 7: Yes

Indeed, as with Cycle 1, working with liquids was frequently discussed as being enjoyable:

Learner 1: I liked the liquids…

RW: Oh, OK. You liked the liquids as well. Were the liquids your favourite?

Learner 1: Yes

RW: Why were the liquids your favourite?

Learner 6: I know my favourite

Learner 1: Because I liked pouring them in cups

RW: Pouring them, you liked the pouring. What about you Learner 6? What was your favourite?

Learner 6: The straws, the straws because you were helping

It is particularly noteworthy that Learner 6 expressed enjoyment of working with straws because this is a task, discussed earlier, in which learners (including Learner 6), particularly in Cycle 2a, expressed frustration. In C2a.3b, Learner 6 frequently expressed annoyance at working with straws because they moved about, Learner 6 also noted, several times *'I hate those little ones!'* (referring to 2 centimetre length straws). Though Learner 6 did not articulate learning that using bigger units can be beneficial, it could be that this is recognised when Learner 6 referred to this task being enjoyable.

The group interviews in Cycle 2, necessitated by time constraints, resulted in fewer learner individual insights into tasks in comparison to Cycle 1. Appendix P provides an overview of comments made by learners when discussing tasks in a group setting. Learner 6, for example, notes 'everything' as helping with learning. Although these interviews gave a sense of what learners had enjoyed or found difficult or 'annoying', it was not possible to explore individual insights and learners tended to refer to tasks more generally. However, one benefit of a group interview in Cycle 2b was that, after a comment by one learner, learners discussed how the tasks may relate to things in their lives. For example, in discussing task C2b.2b the following comment was made:

Learner 9: I think this one as well because people don't need to do weighing, like if they are driving like a taxi and then they are like 'I need this type of taxi' you don't like put them in a weighing scale…And you need length to know how much..

RW: Ah, so...

Learner 9: Or like...

RW: so you think you need to use length more in your everyday life?

Learner 9: Or how much the bed needs to be

Learner 12: Yes because you need how much length petrol needs to be

Learner 9: So suppose this is how long my room is, I need a length to...

Learner 12:...make the carpet

Learner 11: That's how long your bed is

Later in the conversation, the point is raised by Learner 12:

Learner 12: So, these (pointing to tasks C2b.2a and C2b.2c), I think these are useless.

RW: They are useless?

Learner 10: No, they are not useless!

Learners: No!

RW: How interesting

Learner 11: I'm with Learner 12.

Learner 12: You can't just put a bed in there and then a something else in there to weigh a bed. You need straws to length a bed.

RW: So, you think length is really important to understand.

Learner 10: No, it isn't

Learner 12: It's useless

Learner 10: No, it isn't

RW: Learner 13 and Learner 10 disagree. Tell them what you think.

Learner 13: These are actually really good because when you are like trying to measure how much sugar you need and how much like...

Learner 10: Yes

Learner 13:...things for a cake you need you need to measure it

This debate, about the perceived usefulness of different types of measures, whilst not informing task development directly, gives some insight into learners' experiences of measures in different contexts. When considered in relation to practitioner interviews, it is perhaps not surprising that learners volunteer insight into the relevance of tasks when learning in authentic contexts is a pedagogical principle of Curriculum for Wales (see WG, 2023). Teacher 1 alluded to this:

T1: I'd say reasoning. We're very much still concrete visual abstract, very much real life wherever possible

Furthermore, the discussion of weighing can be further considered in relation to teacher comments regarding learner experiences:

T1: And unless they have done the weighing at home, making pancakes and fairy cakes and what not…

T2: But also they don't cook at home so they've not been exposed to that language, of that we need a certain amount of food. Very few I find have cooked.

T1: And they haven't had the experience in school of us doing regular baking

T2: Because we're not allowed to cook, so of course they're not shown scales so I mean I would just take into account everything is very basic.

Thus, it could be that the relevance of understanding mass and a need for weighing was not experienced as much by these learners. It should be noted, that although Learner 12 suggests tasks C2b.2a and C2b.2b are 'useless', the learner also expressed enjoyment of these tasks through saying 'I liked weighting' [sic].

For Cycle 2, it was decided to exploit standard measures in task design as much as possible; both through the purchase of containers with known relationships (e.g., empty bottles of specified capacity) and through the incorporation of standard measures more explicitly in some tasks. Whilst it has not been possible to consider the extent to which this has supported the learners in developing multiplicative reasoning, there was, understandably, a higher incidence of 'awareness of standard unit' code. The use of standard units certainly supported the development of the tasks and the preparation of materials. Furthermore, some learners indicated developing knowledge of standard measures, for example:

Learner 9 (in discussing C2b.2b) noted: *'strength…I mean weight… And how much one gramme is really like, two, uh, twenty grammes is like..'.*

Hence comments such as that above, suggest that the use of standard units in some tasks supported learners in developing awareness of standard measures and their 'size' as well as potentially supporting multiplicative awareness.

Analysis in Cycle 1 suggested that some contexts given to tasks, to try to make them imaginable, appeared unnecessary. For tasks in which the contexts seemed to work well in Cycle 1, these were kept. In Cycle 1, the context given to task C1.2b appeared unnecessary and in similar tasks in Cycle 2a (C2a.1c and C2a.2b) the 'story' context was omitted, and the tasks involved simply finding out how many portions could be made. Learners appeared to respond well, with incidences of codes relating to multiplicative reasoning appearing evident in a similar way to Cycle 1.

Task C2a.1a, in which a story context was not provided, in which learners were asked to reproduce a quantity of red liquid using yellow liquid (as an assessment of understanding of unit) was quite difficult to explain to learners. Learners' initial responses suggested they found it hard to understand, for example, learners, when asked for suggestions, initially said *'I don't know'* or when asked for ideas, said *'I haven't (got any)'.* My reflective notes, after listening to the audio noted:

'I also feel the task would benefit from a context and the use of partners could be exploited (e.g., a barrier type challenge) as discussed in Moxhay (2008) – this was not possible due to space restrictions. However, a relatable reason for having the same amount of two different coloured liquids could be beneficial.'

Setting a context of someone needing to know the volume of liquid without being able to see the original volume and only having available a unit to communicate about the liquid would support the explanation and implementation of the task and, thus, would support the 'presentation' element of task difficulty as noted by Burkhardt and Swan (2017, p.181).

To conclude this section, learners demonstrated similar enthusiasm for the tasks as noted in Cycle 1, demonstrated through emotional codes and through comments, with particular enthusiasm for tasks involving liquids and mass. Their comments also suggest learning about standard measures. It might be, however, that such responses have been evident in Cycle 2 because of the more limited experiences learners had due to their disrupted schooling, as noted by their teachers.

Pedagogic approach to tasks

As discussed in Section 5.8, a tension can exist between the structuring of tasks, for example the establishing of an equality relationship, and the genuine debate that may be possible if learners are asked to suggest and debate ideas to solve problems, which is a pedagogic approach suggested by Davydov (1990). When structuring tasks to include focus on equality relationships between intermediate units, learner suggestions may not be always acted on, possibly reducing student agency. In most Cycle 2 tasks, I was asking questions about learners' ideas for possible approaches when I had already planned actions, and these actions seemed necessary to support understanding. I planned to pay more attention to the way in which the equality relationship would be established (as noted in the design principles, Figure 16, p.187). Though I felt student agency was a tension in Cycle 1, further scrutiny of Davydov's (1992) tasks to support multiplicative reasoning, in preparation for Cycle 2, suggested the establishing of the equality relationships with teacher demonstration, with pre-determined relationships and actions, were part of Davydov's approach. Furthermore, from information available to me, it is not clear whether learners undertaking tasks related to multiplicative reasoning through measures in a Davydov approach (e.g., Davydov, 1992) engaged in paired or individual follow up work with materials, particularly when working in capacity and mass contexts. All tasks I planned incorporated learner involvement with materials, through demonstration and then through follow up work. The tensions discussed in Cycle 1 resolved to some extent, because I believed that student involvement and interaction, through making and discussing suggestions, alongside working with the materials in small groups and pairs, was the key pedagogic approach, as reflected in the design principles. Indeed, social interaction was being facilitated through small group discussion followed by paired work, also involving discussion.

For example, in task C2a.2a, when learners had been asked about finding out how many small cups would be worth the same as a jug of water the following discussion took place:

RW: One cup. One of these tiny cups holds enough water for one rabbit to have for one day

Learner 1: Ah ha

RW: I want to find out how many rabbits I can feed from here

Learner 1: One

RW: What would I need to do to find out

Learner 4: Keep on doing it

RW: Keep on doing what, what would I have to keep on doing?

Learner 4: Pouring it. You have to have lots of little cups and fill them up to the top.

RW: So I'd have to fill up lots of little cups

Learner 1: To the line

RW: To the line and find out how many there are in that

Learners: Yes

RW: Do you think that's going to take me a long time?

Learner 7: No

RW: What do you think Learner 3? Do you think that would take me a long time to find out, how many of these cups were the same amount as this

Learner 3: No, it's going to take forever

RW: It's going to take longer than if I used that cup

Learners: Yes

RW: But I want to find out how many of this cup are in that, how much water would fill these cups, this little cup...Is there anything I can do?

Learner 2: The things we did yesterday

RW: Tell me what you mean

Learner 2: Pour water into the little cups and count how many

RW: I could do that but do you know what, I've only got one cup

Learner 1: Oh!

Learner 3: Yikes

RW: I've only got one little cup. I've got lots of big cups but I've only got one little cup. Any ideas? Talk to the people next to you.

Learner 3?: Um, how are we going to do this?

Learner 4: We could use some of the big ones

The discussion above is typical of the approach used to sharing ideas whilst leading into establishing a relationship between the small unit (cup) and an intermediate unit (larger cup). It is noteworthy that Learner 7 commented it would not take too long to fill all the little cups and count. This action was not impossible, but could be seen as inefficient, and I had deliberately chosen to have only one cup available during the task, to reinforce that this could be inefficient. Learner 3's reaction 'Yikes!' to this, could possibly reinforce that this idea of inefficiency was accepted. As part of the task design and presentation, I was trying to establish a mathematical way of thinking, or norm, to seek more efficient approaches.

Later in the task, when the equality relationship between the little cup and the intermediate unit had been established, and before finding out how many of the intermediate unit were in the jug, the following exchange took place:

Learner 1: I've got an idea…

RW: What's your idea Learner 2?

Learner 2: What you can do is...

RW: Listen, listen to Learner 2's idea

Learner: Basically because you've got the black line there, what you can do is...

Learner 4: Measure

Learner 2: Yes, no. Bring the water up to the black line

RW: Ah!

Learner 2: And then you'll know that it's worth the same

RW: Ah,OK so if we pour another cup, we've got another ten?

Learner 7: Yes we've got twenty

Learner 3: Yes twenty rabbits

In the above exchange, even though the mode of approach had already been established earlier, Learner 2 was articulating awareness of the approach being taken; it may have been that the learner had not fully understood this previously, or it may have been that, in sharing the ‘idea’, the learner was articulating understanding of the approach being taken and it was accepted and agreed by other learners.

Restricting the amount of measuring equipment was a deliberate pedagogic choice in some tasks, as it would support the notion of inefficiency and encourage the idea of efficiency using composite units. In contrast, task C2a.1c did not involve such restriction; in this task learners were asked to find out how many flapjacks could be made from a bag of oats when they knew one cup of oats would make five flapjacks. Here, counting in single flapjacks was impossible, and so learners were given multiple cups to use for measuring. However, in this task, the following exchange occurred:

Learner 6: Miss we don't have enough oats

RW: Not enough oats for what?

Learner 6: For these

RW: Ah, but do you need all the cups, I wanted you to tell me how many flapjacks you can make, it doesn't matter if you can't use all the cups

In this particular example, the learner had thought there was a need to use all the cups available to them. This caused me to reflect on the availability of units even more carefully in subsequent tasks, ensuring that when choices were made in relation to availability, I was clear about this. For example, in task 2a.2a the following day, as learners undertook their paired work with a jug and a cup, in which the relationship of one medium cup being equivalent to ten little cups had been established, they were reminded of this:

RW: Now then, I'm going to put some cups on...yesterday...Yesterday I gave you cups but you didn't need all the cups did you
Learner 1: No

RW: And it doesn't mean you have to fill all the cups just because they are there. So today I'm going to put cups in the middle of your table and then you just take a cup when you need one, OK, but share with the people on your table.
Learner 3: Sharing is caring.

The incident in task 2a.2a drew my attention to the need to consider more carefully, as part of the task design, the potential misunderstandings learners may develop and that I might account for when planning for task introductions.

In Cycle 1, I recognised my initial reflective notes focused a lot on the way in which I questioned learners, and I was concerned that I asked too many rehearsing questions. In my analysis I also identified the value of providing more opportunity for reflection. The extract below (C2a.1a) is an example of the approach to incorporating reflective and enquiring questions.

RW: Let's have a think about my question. How can we be sure that you had the same amount of yellow liquid as red liquid?

Learner 4: We actually didn't have the right amount, because when we poured all of them in, when we poured the red in we didn't have a full cup and when we poured the yellow in we could fill all the cups

RW: Ah
Learner 4: But there was still some left, so it wasn't even
RW: You're not convinced it was even? How would you know it was the same amount? How would you know? Learner 2?
Learner 2: If you have another cup, two other cups, pour the red liquid into one cup, pour the orange, yellow into the other and then switch them into the cup that the yellow was in, switch the red to the yellow and the yellow to the red cup and you'll see if they was the same size as they were in the cup.

I did intentionally use questions to focus attention or rehearse, to allow me to consider the learners' awareness. For example, in C2b.2a:

RW: So remind me again why aren't we using the little one gramme weights?

Learner 9: Because they're too small

My focus on questioning as part of the coding process was useful to reassure myself that I was using a balance of questions in Cycles 1 and 2, rather than too many rehearsing questions, which I perceived to be negative. However, it was difficult to code questions, even though I was guided, as discussed in Section 5.7, by my intentions. The question above 'So remind me again why aren't we using the little one gramme weights', could be coded as 'rehearsing' or 'focussing' depending on the context, and in retrospect, even I was not sure of my intention. I believe I asked the question to focus the learner's attention on the use of composite units (five grammes, ten grammes etc.) as being more efficient, but it can be argued it appears to be checking awareness. As noted by Mason and Johnston-Wilder (2006, p.105) 'it is hard to imagine not asking questions that stimulate learners to rehearse and reconstruct what they know' and through the cycles, I have concluded that the approach to questioning within the tasks should be to invite ideas and develop awareness about how to search for efficient solutions, whatever the type of question used. In future cycles, I would reconsider my coding for teacher questioning.

PRACTITIONER RESPONSE TO TASKS

Cycle 2 ended with a semi-structured interview with a Year 2 teacher (see Appendix Q for initial questions, followed by transcription of the interview). As noted in Section 6.1, this interview took place with a teacher who had been involved in both cycles, and though the views of other teachers involved would have been beneficial, this was not possible. The teacher interviewed was also the co-ordinator for Mathematics and Numeracy within the school, which meant that there could be valuable and informative discussion about the potential of the tasks being used and adapted in a range of different ages.

Because of time constraints, it was not practical to discuss each task in both cycles in detail, hence within the interview the teacher was provided with a condensed overview of tasks used in Cycle 2 (see Appendix R), many of which were developed from tasks used in Cycle 1. Once the teacher had been asked general views on the tasks, Appendix S was shared, providing an overview of learner responses to the tasks. Although this was a lot of information to share, time was given for the teacher to read through them, and they were also left with the teacher should there be any follow up comments or questions. No follow up comments or questions were received.

Of note in the practitioner interview is a positive reaction towards the tasks and what they might offer. For example:

RW: …So what I was going to ask you first of all is if you were using tasks like this what would you think possible benefits would be?

P1: I would be able to see who understands the multiplicative rule and just about capacity and all the different mathematical concepts because I think a lot of them are taught in isolation. Alright, we are doing this, and it needs to be underpinning everything. You know the maths isn't just about, um, numbers, we have got to be using it and applying it, and real life, and all of these are real life, very good real life problems, actually.

Here the teacher suggests these tasks are valuable because of their contexts, but also because they would allow assessment of understanding of 'the multiplicative rule'. Later the teacher, after suggesting some limitations, notes:

P1: But in term of the actual problems, they are lovely problems but they would have to be adapted, differentiated, but yes, time, space, money.

And later, another suggestion of the benefit of the tasks:

P1: I can really see that these will make it more meaningful. *For my more able,* maybe my middles. *My lowers...I... don't know.*

Throughout the interview, and seen in the comments above, there are suggestions that generally the tasks may be challenging, especially for learners without a certain level of understanding or known facts. For example, when looking at (I believe) task C1.4b, the practitioner says the following (note the words in brackets are what is inferred because it is whispered):

P1: *Oh that's tricky*, (with weights?)

Later, after discussing limitations, the teacher notes the following:

P1: But if the children haven't, don't know, their multiplication, you know, they've got to be at a certain level to understand it anyway. That's the main thing.

And there is also discussion about learners who may struggle:

P1:...never mind I've got the coins out, or the Numicon, or we've got two dots on each finger, it's using and applying, so that would be the main problem.

Such comments are examples of suggestions of a view that tasks such as these should come after an understanding of the multiplicative relationship is developed, rather than as vehicles for developing concepts themselves. This is further suggested when the teacher says:

P1: So, definitely I would like to pass some of these on (laughs) to some of my colleagues to use, as our reasoning problems...

It may be argued that the teacher was generally positive about the tasks because it was believed this would be what I might want to hear. As discussed in Sections 4.8 and 5.4, I recognise this as a possible criticism of interviews. Nevertheless, I did reinforce, as can be seen in Appendix Q, that I was seeking genuine perspectives so that tasks could be developed. Furthermore, whilst there is enthusiasm for the tasks, there is also caution in their use with younger learners, or learners who may be seen as lower attaining than peers, suggesting the teacher does provide genuine feedback.

Overall, there seems to be view that the tasks are contextually rich, and useful for assessing understanding but that they may not be seen as suitable for *introducing* the multiplicative relationship, and ideas around multiplication and division. It could be argued that I did not pursue this point with the teacher sufficiently to explore the view of the potential of tasks in introducing, rather than applying or reinforcing, concepts, although this view does tend to appear in several places within the interview.

There is a suggestion that the teacher reflects on pedagogic approaches within the interview, for example, in this comment:

P1: I just like the way you've applied it to real life, maybe I have not made them enough of a reasoning problem. I've just said how many cups do you think this will hold and then we'll do it practically, were we right, who was nearest, write down everyone's in that groups trial, and try and refine so they are very different in that they are so applicable.

As King, Horrocks and Brooks (2019) note, meanings and ideas can be co-constructed in interviews, and in the comment above the teacher seems to re-consider previously used approaches and recognises the problem-solving 'reasoning' approach taken through asking learners' ideas.

As discussed in Sections 4.4 and 5.8, Tabak (2004, p.227) applies two constructs 'exogenous design' and 'endogenous design' in relation to context of tasks within design research. Exogenous design refers to the tasks and related materials and endogenous design refers to the way tasks may be enacted locally. A possible conclusion from the interview is that the teacher liked the exogenous design of the tasks but may adapt the endogenous design, through using them to reinforce rather than to introduce the multiplicative relationship and through suggesting the use of the tasks with older learners.

It must be acknowledged that the concept of multiplication and division were not completely new to the learners I was working with, and therefore I was also using the tasks to extend ideas about the multiplicative relationship. Learners were asked to draw on prior knowledge of counting in steps other than one, or known facts, to solve problems but, as reinforced in the interview discussion, solving the problem with an answer, however satisfying, was not a main aim; rather the tasks were designed to develop multiplicative reasoning and the notion of a change in unit using particular instances of this. In terms of the endogenous design of tasks, and implementation by others, it is important that the idea of a change in unit and the notion of seeking efficiency through changing the unit are preserved, and that tasks are not used primarily as rehearsal of skills of reasoning and multiplication.

6.6 DESIGN PRINCIPLES REVISITED

In this final section of the chapter, design principles are revisited to offer concluding comments on the principles, based on both cycles.

1. Through the context of measure, the task should support the development of the theoretical concept of multiplication involving a change in the system of units under consideration; this may involve standard or non-standard units.

Tasks in Cycle 2 involved greater potential for discussion of standard units because standard units had been considered in the development of most tasks. In implementing the tasks, the standard units involved were centimetres and grammes, with some discussion of millilitres. As in Cycle 1, there is a suggestion within some comments made within Cycle 2 that there might have been awareness of a change in units developing. Some comments made by learners suggested they had developed understanding standard units themselves, and also were beginning to recognise that a change in the size of the standard unit being considered could support efficiency. Standard units could have been exploited further in some tasks (e.g., C2a.3b), had time allowed, and this could be developed in any future cycles.

2. *The task should be set up as a problem, where counting in ones is restricted, inefficient or impossible, though counting in ones may be necessary initially to establish an equality relationship.*

All tasks were designed to make the counting of ones restricted, inefficient or impossible. However, the inefficiency of counting in ones, or single portions, may not have been as evident in some tasks as it could have been (e.g., C2a.2a, C2a.4b C2b.2b). Hence, there is a need to ensure the inefficiency of counting in ones is as clear as possible, whilst also accounting for the technical difficulty (e.g., relationships involved and materials being used). Whilst it is reasonable to encourage learners to look for efficient approaches as a general strategy in mathematics, it is important to ensure tasks are designed to ensure that learners identify this for themselves.

Though counting in ones was identified in the tasks, it was often composite units being counted and thus the 'one' was understood to be a different sized unit. Furthermore, when standard measures were used, counting in ones was accepted as being inefficient due to the small size of units in comparison to the quantity under consideration.

3. *The problem, with the facilitation of the teacher, should invite social interaction, discussion and possible debate in order to suggest possible approaches to finding a solution.*

In Cycle 2, there was a greater focus on asking learners how they might approach finding a solution; this was also recognised by the teacher when looking at the tasks. Despite a greater focus on asking learners how they might find a solution in Cycle 2 (rather than asking them to determine or estimate a quantity), learners did tend to initially give estimates rather than discuss approaches. Nevertheless, suggestions of approaches were made. The opportunity for social interaction was built into all the tasks, particularly as group introductions were followed up with paired or small group tasks in which learners discussed the tasks. The extent to which genuine debate can occur within tasks in which pre-determined actions have been planned (as discussed in Section 5.9) is still questionable. Although this would be possible within the tasks that were designed in Cycle 2, debate did not tend to occur; time constraints, a factor in this research but also within any typical classroom, may mean that the 'debate' aspect of this design principle needs to be reconsidered or more fully defined in any future iteration.

4. *The task, with the facilitation of the teacher, should encourage transfer between the theoretical concept of multiplication as a change in units, and particular instances of this.*

Tasks allowed for exploration of particular instances of changes in units and were designed to support the notion of a change in units supporting efficiency. As discussed above (design principle 2), in some tasks, the notion of inefficiency could have been further facilitated through the choice of relationships and/or materials. Furthermore, as noted in Section 5.9 for this principle, greater use of reflective questions could support the transfer between the theoretical concept and particular instances of it; although this was included in Cycle 2, awareness of change in units occurrence seemed similar to Cycle 1. Both Cycles 1 and 2 took place in concentrated periods, over consecutive days. The interview with the teacher (see Section 6.5) suggested that tasks may not be sequenced in that way if implemented by teachers, and thus further iterations would need to consider the sequencing of tasks for use by schools, whilst still preserving the opportunity to encourage the transfer between the theoretical notion of a change in unit, and particular instances of it.

5. The task should be able to unfold in a range of possible directions, according to learner agency and teacher facilitation.

As noted by the teacher in the interview (Appendix Q), the tasks have good potential to develop reasoning. Though the tasks offered potential to unfold in a range of directions and according to learner agency, time constraints meant this was not typically pursued. Nevertheless, there were instances when the possibility for this was evident; for example, in task C2b.2a when exploring masses and the relationships between them Learner 9 noted

'There's a pattern' when exploring 20 grammes as multiples of 10 grammes and 5 grammes and then 40 grammes and 60 grammes as multiples of 20 grammes, 10 grammes and 5 grammes. The task offered potential to unfold in a direction which would have allowed for understanding of relationships between factors of a number. In the interview with the teacher (Appendix Q), the potential of the mass set being used in such a way was recognised:

P1: Oh, I like...I'm going to have to check I have the 1 kilogramme and the five, ten and twenty kilogramme weights. My maths cupboard...

RW: Yes, that's the thing. I bought a little set of weights...

P1: This is brilliant.

RW:...the hexagon weights

P1: Yes, I can see, I saw the picture

RW: Yes, it as, as you say, trying to, sourcing things

P1: I'm putting in an order, the maths order's gone in, but...

RW: Laughs

P1:..they are...I hadn't thought of them for using multiplication before...

RW: Yes, yes

P1:...but it makes, it's using and applying and it's reasoning and there's...that is brilliant

To conclude, this design principle is important to guide choice of tasks, and further iterations could explore how the tasks could unfold in different directions.

6. *The tasks should involve a range of measures contexts, with explicit consideration of how equality is experienced.*

The careful structuring of the establishment of an equality relationship in a range of different measures contexts became a key consideration in Cycle 2, based on analysis of Cycle 1. It was not assumed learners would recognise equality in measures; situations were set up in which equality was established and verified within each measurement context. This appeared to support the learners in using the intermediate units to establish relationships, although, as discussed in Section 6.5, the consecutive sequencing of two tasks in which I used the same intermediate unit, but different small units, did cause difficulty for the learners. This was considered to reflect sequencing and choice of intermediate unit than the establishment of equality relationship.

Though not possible within the scope of this study, further development of tasks would use the design principles for Cycle 2, with a focus on the sequencing and structuring of tasks and how they might be incorporated into school use by teachers, whilst preserving and further exploring opportunities for learners to develop awareness of the multiplicative relationship involving a change in units.

The final chapter of this thesis focuses on analysis of data from both cycles in order to draw general conclusions and offer further insight into the teaching and learning of the multiplicative relationship through measures contexts.

CHAPTER 7: CONCLUDING THEMES, DISCUSSION AND CONCLUSION

7.1 INTRODUCTION

The focus in this chapter is the synthesis of analysis and reflection from Research Cycle 1 and Research Cycle 2, to discuss themes induced from exploration of the learning and teaching of multiplicative reasoning through measures; these themes form the conclusions to the thesis. In the discussion section of this chapter, I draw out the contributions made by the thesis and consider directions for future research.

The research involved the design and implementation of tasks to support the learning of multiplicative reasoning through measurement-based tasks. As discussed in Section 2.3, Nunes and Bryant (2009a and 2009b) note that if a quantity is made up of discrete elements, then measurement (assigning a numerical value to a quantity) is straightforward as it can involve counting the discrete elements. With a continuous quantity, measurement involves deciding on a unit and applying that unit iteratively to the quantity to find out how many times the unit can be fitted into the quantity. All tasks in the research cycles involved quantities to be measured, involving capacity, length or mass. Furthermore, all tasks involved finding out how many times a unit could be fitted into a quantity. Tasks were designed to necessitate a change in the system of units, where counting in smaller units was either inefficient or impossible, a key idea in developing understanding of multiplication, as noted by Davydov (1992). In establishing a change in units, an intermediate unit was introduced. This intermediate unit is a composite unit, i.e., a unit that is composed of other units (Steffe, 1994). Learners were asked to establish a relationship between the intermediate (composite) unit and the quantity under consideration.

In the tasks, learners were applying quotitive approaches to measurement. Nunes and Bryant (2009a, p.27) call the quotitive model for division 'measurement division' because the purpose is to find out how many times a given quantity is contained within the larger quantity. As discussed in Section 2.4, there are two models for division: partitive in which a quantity is shared equally into a given number of parts and quotitive in which the purpose is to establish how many times one given quantity is contained within another. In all tasks developed and implemented within Cycles 1 and 2, quotitive approaches were applied through finding out how many of the smaller unit are equal to the larger (intermediate/composite) unit or through finding out how many of the intermediate (composite) unit were in a larger quantity.

For the two research cycles, points of learning were identified, with reflection on the design principles. Within this chapter, data from both cycles of research, and points of learning, are considered further, to share and discuss concluding themes.

7.2 THE ANALYTICAL PROCESS FOR IDENTIFICATION OF CONCLUDING THEMES

Points of learning for both cycles involved discussion based around the following aspects:

- task efficacy (the extent to which the tasks may have supported multiplicative reasoning)
- task difficulty, in relation to Burkhardt and Swan's (2013) ideas of complexity, familiarity, technical demand and autonomy
- learners' emotional and evaluative response to tasks
- pedagogical approach

and, discussed in Chapter 6,

- practitioner response to tasks

As discussed in Section 4.12, the analysis of qualitative data can involve recursion and movement backwards and forwards between data, analysis and interpretation (Cohen, Manion and Morrison, 2018, p.644). In inducing concluding themes, I began by re-visiting each task from both cycles and I considered the way it had been implemented, with the points of learning from both cycles in mind. In re-visiting each task, I considered the data related to it (from learner responses, my responses and teacher responses) and compared with similar tasks and responses across both cycles. My main focus was to analyse what I was asking learners to do, what was similar about responses, what was different about responses and, importantly, why there might be similarities or differences. Though messy and recursive, involving movement back and fore between tasks and data within both cycles, the advantage of working in this way is that I was constantly immersed in the data itself, and I re-visited and re-analysed each task and responses to it. This analytical process is illustrated in Figure 17 below:

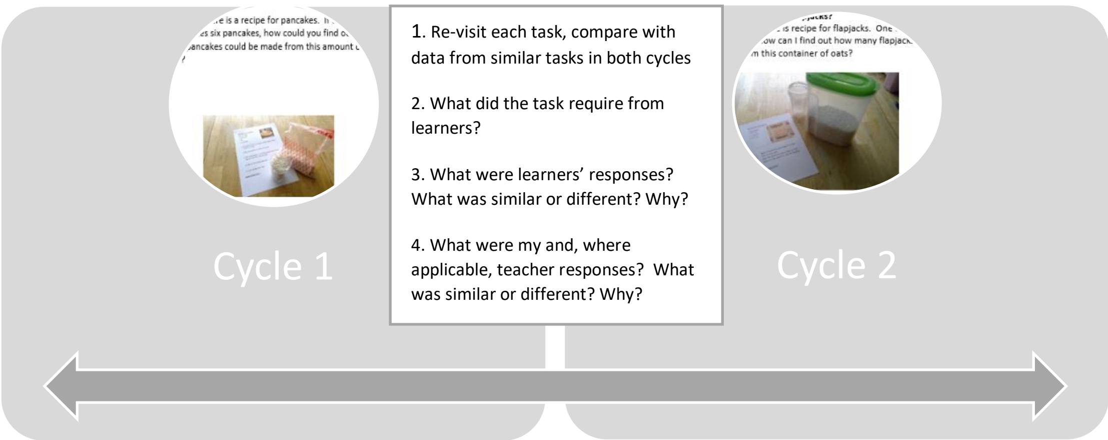


FIGURE 17: ANALYTICAL PROCESS FOR CONCLUDING THEMES

Through this process, the following concluding themes have been induced for discussion:

1. The perception and demonstration of equality in quantities: variance by measure.
2. The acts of measurement when using an intermediate unit.
3. The 'discreteness' of a quantity and its measurement.

4. The benefits and limitations of introducing standard units of measure to support multiplicative reasoning.
5. The ‘hiddenness’ of the mathematical operation within tasks.

Hence, themes are induced from the data and my interaction with it; as discussed in Section 4.12, these themes are based on my interpretations of the data, and it is possible that other interpretations might be drawn from qualitative data such as these. However, in discussing the themes, I draw on data from both cycles and from across the tasks and, in doing so, show that I have considered the range of data and responses.

7.3 THEME 1: THE PERCEPTION AND DEMONSTRATION OF EQUALITY IN QUANTITIES: VARIANCE BY MEASURE

A theme central to points of learning in Cycle 1, and key to the success of all tasks across both cycles, is how the equality of quantities can be perceived and demonstrated. Analysis of the tasks and responses across both cycles led to the conclusion that it cannot be assumed that perceiving and establishing equality between continuous quantities is straightforward for learners. Furthermore, perception and demonstration of equality can differ according to the measure under consideration and the way in which units might be used to establish equality.

Firstly, I analyse the methods by which equality between quantities may be visually perceived, and then I analyse the method by which equality might be established, using units.

THE PERCEPTION OF EQUALITY RELATIONSHIPS IN QUANTITIES

I realised, through implementing and analysing the tasks, that for an equality relationship between quantities to be visually perceived, there are two methods in which this can happen. I use the terms transference and replication to describe these two methods, illustrated in a simple form in Figures 18 and 19 below, where the square represents any type of quantity.

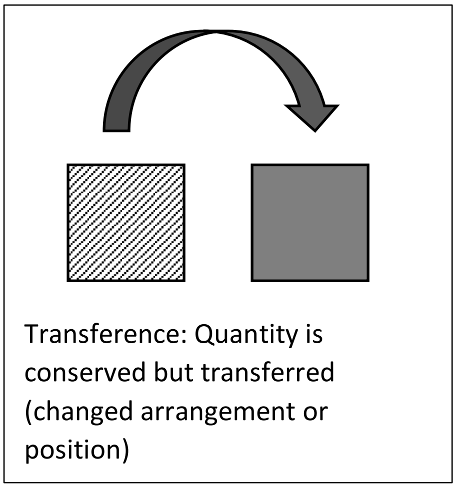


FIGURE 18: TRANSFERENCE TO SHOW EQUALITY

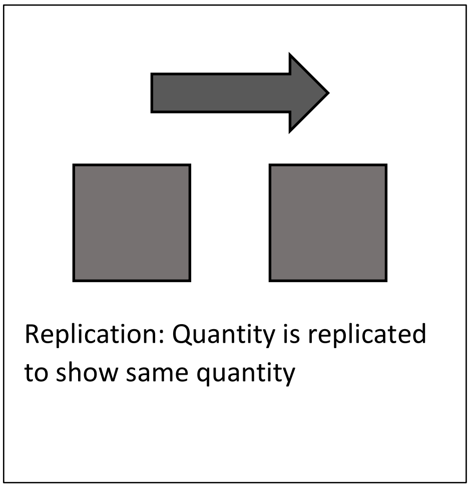


FIGURE 19: REPLICATION TO SHOW EQUALITY

Transference (Figure 18) involves transferring a quantity in a way that maintains (conserves) the amount of quantity, so that no quantity has been added or removed. In transference, it must be recognised that the same amount of quantity remains, even though it may be moved and redistributed in some way (e.g., into smaller parts). Transference relies on the concept of conservation, identified as an important aspect of cognitive development by Piaget and colleagues. Conservation was seen by Piaget and colleagues as an understanding that, though a quantity may be rearranged, provided nothing is added or removed to the quantity, it will be quantitively invariant (Bibok, Müller and Carpenter, 2009). Piaget (in Bibok, Müller and Carpenter, 2009, p.242) noted there was need for recognition, that through rearrangement, 'what the object loses in one dimension, is made up for in another'. Figure 18 illustrates the most straightforward example of transference, but transference can also occur through moving a quantity into a different arrangement. For example, a length of string will remain the same length whether laid straight or curved, or liquid may be poured from one container into another, and provided no liquid is spilled or added, the quantity of liquid will be the same even though it may look different in another container. A mass such as a piece of plasticine might be squashed but if nothing has been added or removed, its mass remains the same. Hence in transference, equality must involve understanding of conservation.

Replication (Figure 19) involves replicating the original quantity for comparative purposes to show that the quantities are the same. For replication to be used to establish equality, there needs to be an understanding of measure attributes. For example, to establish that two quantities are the same mass on a pan balance, there would need to be replication of a quantity so that the pans can balance. When establishing equality with lengths of rods or straws, learners may replicate a length so that both lines have the same length. Thus, this method relies on an understanding of a specific measure attribute being considered (e.g., lengths being the same, pans balancing or liquids being the same level in two identical containers). This method also relies on an understanding of conservation because a replicated quantity may appear to be rearranged or reorganised, but it can also be compared to the original quantity to establish equality.

Although I had been using these methods in both cycles, I had not labelled, identified or considered the implications of them explicitly; it was only through re-visiting the tasks and through considering the way in which I had asked learners to perceive the equality relationship and how they had responded to this, that I distinguished these methods. Furthermore, the most appropriate method depends on the measure context under consideration (length, capacity and mass).

As illustrated in Figure 20 below, when establishing equality in length tasks, whilst possible, transference could be considered difficult for visual comparison, and it would rely on markers being used and thus some form of measurement being introduced. In contrast, replicating a length so that the new length is the same as the original seems more intuitive and this was the method used in length tasks.

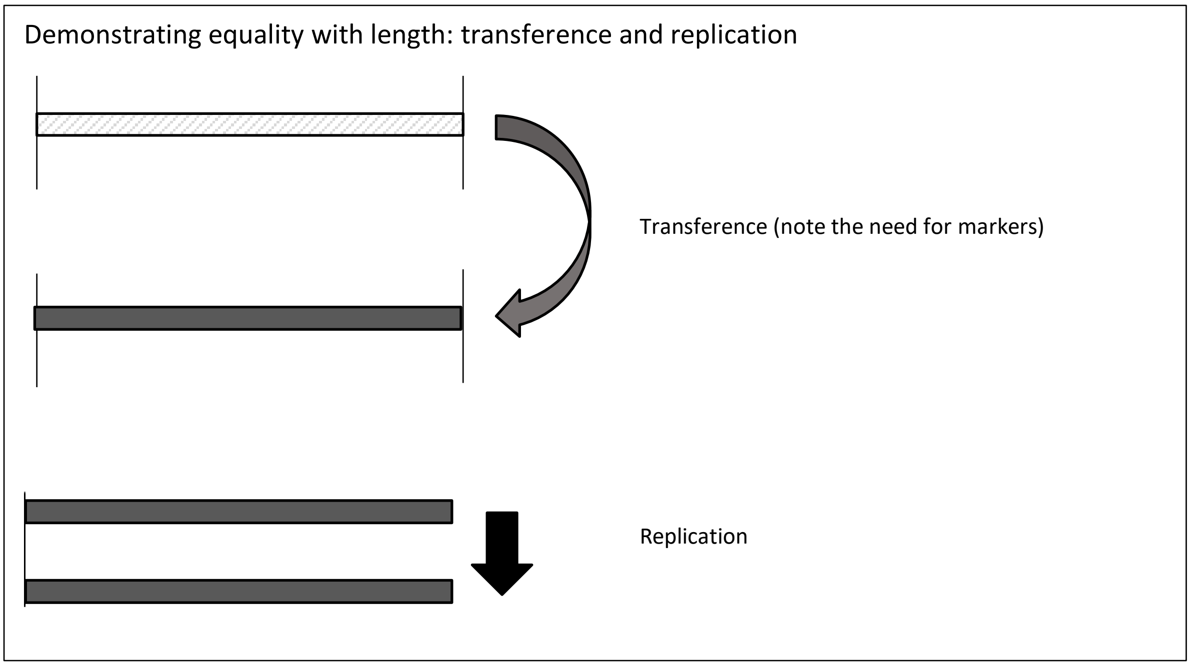


FIGURE 20: TRANSFERENCE AND REPLICATION IN LENGTH TASKS

Comparison of lengths through replication is something learners were familiar with, for example, through using non-standard measures, as observed in tasks on the initial learning walk (e.g., measuring the length of printed worms using non-standard units) and further implied by a practitioner in the first interview:

T4: With measure as well, you know, we'd use things like Duplo, you know, to measure length initially and giving them the choice as well, so you know saying we need superhero capes, what do you want to use to measure, and if the cubes are smaller, well let's see what the difference is, and just getting them to use lots of non-standard units first of all…

When establishing equality in capacity, both replication and transference are possible, as illustrated in a simple example in Figure 21 below.

Demonstrating equality in capacity tasks: transference and replication

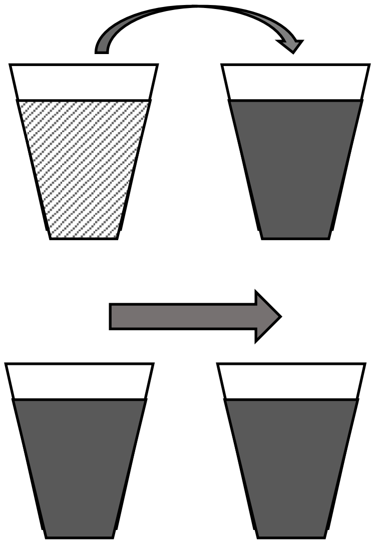


FIGURE 21: DEMONSTRATING EQUALITY IN CAPACITY TASKS

Hence, both transference and replication can be used to perceive equality in capacity tasks, although markers are useful for transference, and judging equal capacity becomes perceptually more difficult with both methods when different containers are used. In the tasks, both transference and replication were used as methods for supporting visual perception of equality in the tasks.

Replication was a requirement of some capacity tasks in both cycles (PA2 in Cycle 1 and C2a.1a in Cycle 2a), in which learners were asked to reproduce the same amount of liquid in a different container and be able to justify how they knew it was the same amount, thus assessing understanding of a unit. These tasks were adapted from tasks discussed in Moxhay (2008), developed from Davydov and Elknonin's curriculum. Indeed, such tasks are used to *necessitate* measurement using a unit, and thus reinforce the concept of number as a unit.

In Cycle 2 (C2a.1a), learners initially suggested using the level of the liquid, even though the container for replication was a different shape.

For example, in C2a.1a:

Learner 2: What we could do is put the two bottles next to each other and measure the sides.

Learners needed support in recognising that using the level of the liquid as an indication of capacity would not be possible. It is possible that the containers in Cycle 2 were not sufficiently different in dimensions to support understanding of conservation, but these learners also had less experience of comparing capacities of liquids through a loss of schooling during COVID-19 related lockdowns and restrictions placed on using materials, as noted in the practitioner interview prior to Cycle 2 (from Appendix O):

T2: I mean I do need to do more to measure and bits, but it has been hard lately

RW: Yes

T2: Staff being off as well and not being able to cross in bubbles so that has had an impact.

RW: Yes, yes.

T2: We've had to be more adaptable.

T1: We do need to be putting more of the capacity

In Cycle 1, task PA2, two of four pairs of learners used one cup repeatedly as a unit of measure, counting how many cups were in one liquid and then using this as a measure to make up the second liquid, whilst two pairs of learners used the cup, but poured out the liquid into multiple cups and did the same for the second liquid (see Figure 15, p.138). Though both approaches used the cups as units and learners could articulate that there were three cups of liquid, two of the pairs used replication through identical containers rather than iterating (repeatedly using) the cup as a unit. For Cycle 2, in the same task, the availability of cups was restricted to force iteration. Some learners appeared to find it difficult to ascertain how they could replicate the capacity of a red liquid using a cup to make an equal capacity of yellow liquid.

Learner 4: This is really hard.

Learner 2: I'm not sure

Learner 1 (whispering): Put the red liquid in the cups and put the yellow liquid, in the cups and have the same amount in both

Learner 2: And then you'll get two of the same cups and pour that back in and then you try to remember how many cups that liquid was

Learner 2: And then you can pour that much liquid for the yellow

Later, when asked to articulate how they knew they had equal amounts of red and yellow liquid, Learner 2 implied a preference for replication using identical containers as a way of perceiving equality:

Learner 2: If you have another cup, two other cups, pour the red liquid into one cup, pour the orange, yellow into the other and then switch them into the cup that the yellow was in, switch the red to the yellow and the yellow to the red cup and you'll see if they was the same size as they were in the cups.

In all other capacity tasks, transference was the method used and learners appeared to accept this method, particularly as it was reinforced through demonstration involving the learners taking it in turns to repeatedly transfer liquid between containers, counting the number of smaller units within the larger container, using markers to support.

To visually demonstrate equality in mass, *only* replication is possible, using a pan balance, as illustrated in Figure 22 below.

Perceiving equality in mass tasks: replication

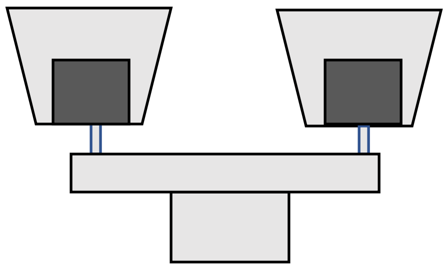


FIGURE 22: DEMONSTRATING EQUALITY IN MASS TASKS

In both cycles, learners' comments relating to mass tasks implied awareness of this way of establishing equality.

For example, in reference to the mass task (C1.4b):

Learner 5: Yes, if it was the same amount..

RW: OK

Learner 5:...it would go in the middle

And in C2a.4a:

Learner 6: Oh I want to do this...now I do this (sound of masses going it)...that makes it balance

Hence, learners used both transference and replication for capacity tasks, but only replication for length and mass tasks. However, in the tasks, learners were typically establishing or verifying equality between quantities using units. Furthermore, the application and availability of units can affect the method of establishing an equality relationship, and this is discussed further below.

THE APPLICATION OF UNITS TO DEMONSTRATE A QUOTITIVE EQUALITY RELATIONSHIP

In all tasks, across both cycles, learners were required to use a composite unit, a unit composed of other units (Steffe, 1992) and therefore they typically needed to recognise or establish an equality relationship, through finding out how many times one quantity (unit) fitted into another quantity (unit), thus applying quotitive measurement.

In the analysis of tasks and responses, just as I realised there can be different methods for establishing equality between quantities, I also realised that there are two possible approaches to demonstrating or establishing quotitive relationships through the application of units, relating to the availability of units under consideration. The option of use of these methods is also dependent on the type of measure under consideration. I call the two approaches ‘many-making-one’ and ‘many-into-one’, and these are explained further below.

Many-making-one: Availability of small unit not restricted.

The many-making-one approach involves demonstrating that a quantity can be equally distributed amongst smaller units, or vice versa. In a many-making-one demonstration, the quantity could be transferred into the smaller units, or the actual number of smaller units could be used to replicate original quantity.

Many-into-one: Availability of small unit restricted.

The many-into-one approach involves repeating (iterating) the use of the same smaller unit a number of times to demonstrate equality, either to replicate the original quantity or to transfer into a different arrangement.

In the context of capacity tasks, an equality relationship can be established between units in these two different ways, illustrated below in Figures 23 and 24 and explained further below.

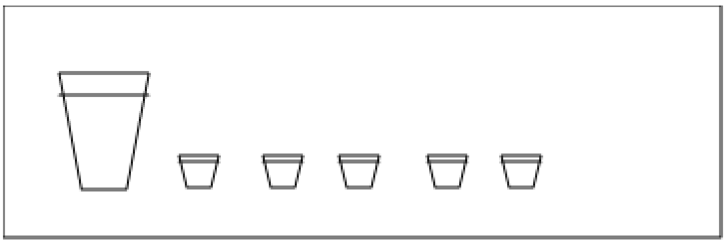


FIGURE 23: MANY-MAKING-ONE APPROACH TO DEMONSTRATE EQUALITY IN A CAPACITY TASK

Many-making-one (availability of small unit not restricted). This involves demonstrating that the volume of quantity in one large container can be equally distributed amongst smaller units, or vice versa. For example, starting with a larger cup and filling smaller cups until the quantity has been equally distributed. Note that the level of water in the cups has been pre-determined and so this cannot be considered a partitive 'sharing' approach because the purpose is to find out how many of the smaller cup is needed (Figure 23).

FIGURE 24: MANY-INTO-ONE APPROACH TO DEMONSTRATE EQUALITY IN A CAPACITY TASK

Many-into-one (availability of small unit restricted). This involves repeating the use of the smaller unit a number of times. For example, filling the small cup repeatedly and pouring into the larger cup until it is full to required level (Figure 24).

Both approaches were used in the capacity tasks in both cycles. In Cycle 1, I had planned to restrict the small unit throughout (and thus would only have been able to use the many-into-one approach), but my reflection on Day 1 noted this:

Trying to establish how many of the smaller cup were in the larger cup was problematic because the learners were using the cups already measured out – it struck me at this point that having sufficient little cups would have been beneficial from a visual sense. I need to think about how I use the resources to support the students in a visual and practical way. One pair of learners tried to submerge the little cup into the large cup. This could be because I had said 'how many times it fits in' or it could be because there was attempt to fill *the cup without pouring. Another learner attempted to guess how many times by moving his finger up the cup (seemingly approximating the amount of liquid).*

Thus, the many-making-one demonstration was introduced in Cycle 1, after reflection of the need to involve the learners more visually and more physically in setting up the initial relationship between smaller and intermediate unit.

After C1.2a, I noted the following:

I do feel that visually having the right (or excess) amount of little cups and large cups helped.

For Cycle 2, I planned the ‘many-making-one’ approach in the initial tasks and then used the ‘many-into-one’ approach in subsequent tasks. The many-into-one approach could be seen to reinforce the task context, restricting the smaller unit by only having one available. Furthermore, the repeated pouring into the intermediate unit to establish a relationship supports the notion of measuring involving iteration of units.

The approach taken also depends on the equipment available. In both cycles, the ‘Dog’s Medicine’ task (C1.3b and C2a.3b), in which one spoon of medicine was 10 millilitres, and an intermediate unit (small bottle) was worth the same as 5 spoons (50 millilitres), the spoon could not be used efficiently in a many-making-one demonstration, and learners responded well to a many-into-one demonstration by taking it in turns to add a spoon of medicine to the bottle and keeping track of how many spoons had been counted. Learners could be supported further in the many-into-one demonstration by repeatedly marking the levels as the intermediate container is filled. I chose not to take this approach (although had considered this for Cycle 2), because I wanted to restrict the possibility of learners counting single units; having the intermediate bottle labelled in units could encourage this strategy. However, it might also reinforce relationships (e.g., five ten millilitre spoons equal fifty millilitres).

In both cycles, within capacity tasks, once the equality relationship had been established, learners seemed to accept and apply the composite unit, recognising a need to work with that composite unit. Indeed, two tasks, 'Pancakes' (C1.2b) and 'Flapjacks' (C2a.1c), necessitated the use of a composite unit without providing the option of establishing an equality relationship through demonstration. In these tasks, learners were informed of a relationship (1 cup makes 6 pancakes, 1 cup makes 5 flapjacks) through a recipe, rather than establishing that relationship for themselves. It could be argued that these tasks do not reflect a change in the system of units, because the intermediate unit is pre-established, and it is composite (1 cup representing 6 pancakes, or 1 cup representing 5 flapjacks). However, these tasks were used to support awareness of a unit of measure being able to represent multiple objects and were planned into a sequence of tasks where an intermediate unit had already been introduced. In the 'Pancakes' task (C1.2b), it is noteworthy that Learner 7 says (referring to the cup) 'How would it make six pancakes?'. All other learners in both these tasks seemed to accept the established relationship and it could be that Learner 7 lacked experience with such recipes or was considering semantics regarding a cup making pancakes. Either way, this reinforces the need for the cultural relevance of the task to be considered and, furthermore, this comment provoked analysis of the nature of the task in relation to establishing a change in unit. In the context of recipes with containers representing several items, it would be very difficult to visually establish an equality relationship between a unit of food item (e.g., a pancake or a flapjack) and the composite unit. Learners needed to accept rather than establish the relationship within these tasks, which they readily seemed to do. Furthermore, as noted in Chapters 5 and 6, within both these tasks, learners seemed to quickly recognise a need to work in multiples of the composite unit:

In C1.2b ‘Pancakes’, when asked how to find out how many pancakes could be made with a bag of flour (knowing that one cup made 6 pancakes), the following responses were recorded:

Learner 5: Umm, I think we can make about 18

And a few minutes later, Learner 4 (referring to the bag of flour) said:

Learner 4: A quarter would make about six and we could count in sixes then

Similarly in C2a.1c ‘Flapjacks’, when asked how many oats could make flapjacks from a bag (knowing that one cup made 5 flapjacks), some learner responses were as follows:

Learner 2: Ten, fifteen, twenty...

Learner 4: I think about forty.

Learner 4: I know. I think like 60 or 50.

Learner ?: I think we can make twenty.

Learner ?: Thirty.

Learner 4: Five, you can count in fives

These tasks were chosen because they made counting in ones impossible, and even though an equality relationship could not be physically established, the learners seemed to readily recognise they would be working with multiples of the number that the cup represented. It is possible that these tasks were tasks which were more familiar to learners; practitioners in the initial interview gave examples of how composite units might be used to support multiplication:

T2: Understanding. If you're doing the two times tables it means that five twos is you've got five groups of two, five mountains of two…And the practical, Numicon is the best thing (sounds of agreement) and coins (sounds of agreement), applying it to money, five two pences.

T1: And we give them items that would relate to that group, so if we're doing pairs, if they were counting in twos, we'd give them pairs of socks, if we were counting in fives we'd give them gloves to have the five fingers, that kind of…

T4: It kind of depends on the topic really. Earlier on in the year when we were doing animals and we were counting in twos, did we do legs or something like that?

Hence, even though establishing an equality relationship was seen as important, in 'Pancakes' and 'Flapjacks' the cups represented items that were not visually repeated for the learners, and learners had no way of counting individual pancakes or flapjacks, but across both cycles they accepted and applied this relationship. The only exception to this was in C2a.2b 'How much porridge' where learners seemed to struggle with associating the intermediate unit (cup) with three smaller pots (where one pot was the quantity needed to make one portion of porridge). However, as discussed in Section 6.5, this task proceeded immediately from C2a.2a 'Rabbits' in which the intermediate unit was the same (but with a different smaller unit). Thus, this difficulty was likely because of a similar task using the same intermediate unit and the consecutive sequencing of these.

In length tasks, establishing equality relationships could be either by a many-making-one or a many-into-one approach, as illustrated in Figures 25 and 26 below.

FIGURE 25: THE MANY-MAKING-ONE APPROACH IN LENGTH TASKS

Many-making-one (availability of small unit not restricted). This involves demonstrating the equality relationship through laying out smaller units, with no gaps, along length of larger unit to replicate the length (Figure 25).

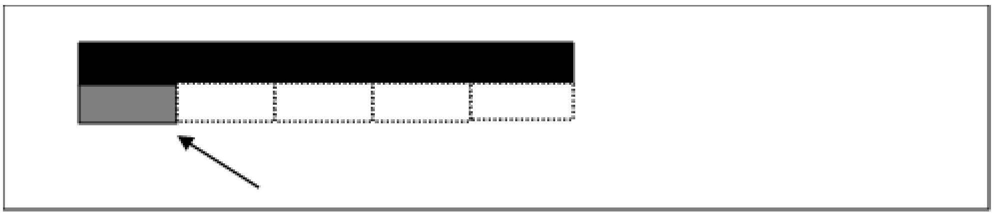


FIGURE 26: THE MANY-INTO-ONE APPROACH IN LENGTH TASKS

Many-into-one (availability of small unit restricted). In this case equality needs to be established by iterating the restricted unit along the length of larger unit, ensuring no gaps (often achieved by tipping unit over: end/start of next unit may need to be marked for accuracy). In this case, replication of the length occurs, but the replication of length needs to be visualised rather than physically reproduced (Figure 26).

In working with length, when establishing equality in units, learners appeared to use the many-making-one method with ease across length tasks and in both cycles. This included when they were working with multiple units (e.g., C1.3a Cuisenaire, C2a.1b Straws, C2a.3a Making lengths, C2b.1a Straws and C2a.3a). It is not surprising that the many-making-one equality relationship in length is familiar to learners; measuring through replication with non-standard units was mentioned by practitioners in the first interview (see earlier comment in this section).

In both cycles (C1.3a Cuisenaire, C2a.3a and C2b.1b Straws), learners were able to use the many-making-one approach confidently using multiples of 1cm, with no single centimetre units available and it was only when the many-making-one equality relationship could not be applied easily, either because of restriction of smaller unit or because of straws moving around (as in Cycle 2a.1b), that learners seemed to struggle. For example, in Cycle 1, Learner 5 referred to lack of availability of units as a reason for Task C1.1b being confusing:

Learner 5: Um, because we didn't have enough straws for both of our lines so we had to use each other's.

In Cycle 2a, C21.3a, learners got particularly frustrated because the straws moved about, preventing them from using a many-making-one representation. When discussing a similar, adapted task in Cycle 2b (C2b.1b), Learner 9 commented on the task with straws:

Learner 9: I think this one because you need to like put the exact amount

In the length tasks, the one centimetre unit was restricted, and there was also a set number (restricted) of multiple units (e.g. 2cm or 5cm rods or straws) but learners resisted using a many-into-one approach, choosing to borrow from others if needed. This is understandable as, within the context of length, the many-into-one approach could be seen as less efficient, prone to inaccuracy and requiring a greater level of abstraction.

In the context of tasks involving mass and pan balances, establishing an initial equality relationship could *only* take the many-making-one approach. Because replication of a quantity was necessary for the pans to balance, learners needed the exact number of smaller units to equate to the larger unit. In fact, for visually establishing equality in weight, the *only* possible method is to be able to replicate the quantity demonstrating a many-making-one approach, as illustrated in Figure 27 below:

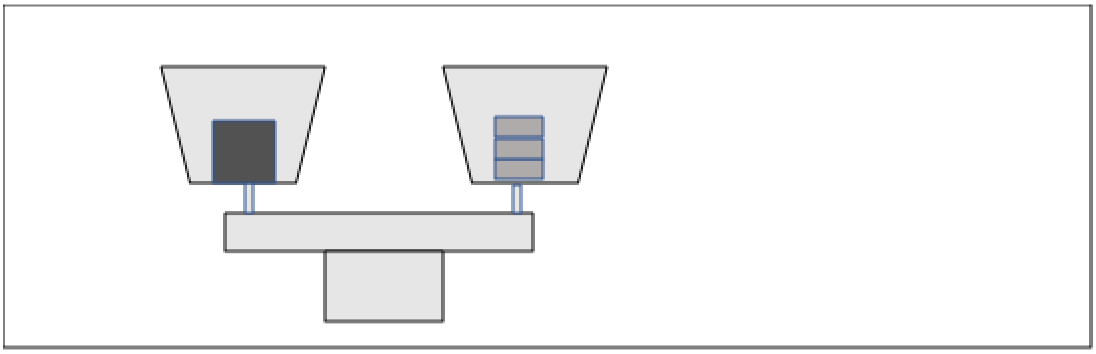


FIGURE 27: THE MANY-MAKING-ONE APPROACH IN MASS TASKS

Learners appeared to find establishing or accepting relationships involving mass and length more straightforward, particularly utilising a many-making-one approach. It is noteworthy that, in the final interview with the teacher involved in both cycles (Appendix Q), the length tasks and the mass tasks were particularly noted as being ones that could be used. For example, when asked whether any of the tasks might be used, one response was this:

Definitely. I love the straws one, because I think it's so hands on and you can give it to everyone and they can have it individually or in pairs.

Later, when considering the mass task, the following conversation occurred:

P1: Oh, I like...I'm going to have to check I have the 1 kilogramme and the five, ten and twenty kilogramme weights. My maths cupboard...

RW: Yes, that's the thing. I bought a little set of weights...

P1: This is brilliant.

RW:...the hexagon weights

P1: Yes, I can see, I saw the picture

RW: Yes, it as, as you say, trying to, sourcing things

P1: I'm putting in an order, the maths order's gone in, but...

RW: Laughs

P1:..they are...I hadn't thought of them for using multiplication before...

Although the teacher does not refer to ease of establishing equality relationships in any discussion, the tasks seem particularly appealing to the teacher, and this may be because they are more familiar in terms of previous teaching in relation to composite units.

Demonstrating or establishing equality relationships with capacity involves more options of approaches than in length or mass tasks, because transference, replication, many-making-one and many-into-one are all possible. This can be seen as both a benefit and a potential limitation; it can be a benefit because there are opportunities for learners to understand and articulate equality relationships in multiple ways, but could be a limitation if different approaches are used without recognition of the choices being made and how that may impact the learners' understanding of equality, or if learners have not experienced different ways of exploring equality through previous experiences. In Davydov and Elknonin's curriculum, learners are familiar with continuous quantities and equality relationships, as the concept of number and additive relationships will have been learnt through these contexts (e.g., Davydov, 1990; Schmittau, 2003); the learners and teachers within this context have not had those experiences and therefore it is important to consider the structuring and sequencing of establishing equality relationships with quantities in future iterations.

Hence a conclusion from this section is that establishing of equality in quantities, when demonstrating a quotitive equality relationship, needs explicit consideration within measure tasks. Furthermore, in length and capacity tasks, iteration of units can be 'forced' using a many-into-one approach, and this can support multiplicative reasoning. Indeed, I suggest that this is necessary step in understanding both measurement and the multiplicative relationship and is particularly important when using measures contexts to develop multiplicative reasoning. In future iterations, I would consider the sequencing of tasks and measures contexts very carefully to allow for progression, not just in reinforcing the idea of a change in units, but also progression in the way units are used to establish quotitive equality relationships.

7.4 THEME 2: THE ACTS OF MEASUREMENT WHEN USING AN INTERMEDIATE UNIT

As discussed in Section 6.5, in both mass tasks in Cycle2 (C2a4b and C2b2b), it was anticipated that learners would recognise they could find out how many 10g portions of pasta were in a bag by finding the mass of a bag (a multiple of 10g) and working out many portions that would be. Yet in both tasks, this was not a strategy learners suggested. In both these tasks, learners suggested portioning the quantity of pasta into 10g portions so that they could then count how many portions there were.

For example, in C2b.2b:

Learner 9: And then we could put like, try and put one portion there, and portion there, so we know that's one portion for one baby and the other portion for another baby and then we could like keep on doing that

RW: Ah, you could keep on doing it

Learner 9: And count all of the bags

RW: Ah, and you could keep on doing it. Is there another way? I see what you are saying.

Learner 13: Miss

RW: Learner 13?

Learner 13: You can take this out and then we pour some more in here because we know that's ten grammes, so wait until it gets equal again, then put that into a pile and then leave that into it and then pour a bit more in until it reaches the middle again and keep on doing that and then we could find out how much groups

RW: Ah so each time you are making one portion of pasta

At first it seemed surprising that the learners did not suggest finding the mass of the bags and then working out how many 10g bags would be equivalent. As the masses being used were multiples of 10g, using the 10g masses (after already establishing what one 10g portion looked like), would have enabled learners to efficiently establish how many 10g portions there were within a bag, without having to portion. Furthermore, working in multiples of ten should be a familiar to all learners (as discussed in practitioner interview, Appendix O, and in line with curricular expectations). However, further analysis of the tasks showed that all other tasks had involved 'portioning', that is the distribution of an intermediate unit to compose or decompose the quantity being measured (either through the many-making-one or many-into-one approach). In all tasks involving liquids (C1,1c, C1.1d, C1,2a, C1.2b, C1.3b, C2a.1b, C2a.2a, C2a.3b) learners were using intermediate units and portioning into the intermediate unit to establish how many intermediate units there were. Similarly in other capacity tasks (C1.2b, C2a.1c, C2a.2b) learners were portioning into the intermediate unit with flour or oats. In tasks involving length (C1.3a, C1.4a, C2a.1b, C2a.3a, C2b.1a, C2b.1b) learners were portioning because they were using intermediate units (2cm/5cm/10cm), which were already portioned. Thus, all other tasks involved the physical portioning into intermediate units, so it is not surprising that learners suggested this strategy. The difference with the mass task was that portioning of pasta could be avoided by using the masses themselves as the intermediate unit. Learners did achieve this:

For example: C2b.2b

RW: So how many babies will that feed?

Learner 12: Nine

RW: Do you agree Learner 11? How many babies does that feed?

Learner 11: Nine

RW: And how much does it weigh?

Learner 11: It weighs ninety grammes

Indeed, in the extract from C2a.4b below, although Learner 2 confuses gallons with grammes, the learner is able to use twenty grammes to represent two portions, enough to feed two people.

RW: What have you found out about what this pasta weighs?

Learner 2: It weighs three twenty gallons

RW: So how many people will that feed?

Learner 2: Uhh, six, it feeds six people. It will feed six people

Learner 1: Six people

RW: Well done, how did you know it would feed six people?

Learner 2: Two, four, six

Measuring without portioning is much more straightforward as a strategy when standard units of measure are being used, but it relies on known multiplicative relationships and can render the need for an intermediate unit unnecessary. For example, in the task 'How much medicine/How many spoons of medicine' (C1.3b and C2a.3b), in which the learners were trying to find out how many 10ml spoons of medicine were in a 200ml bottle, the portioning into 50ml bottle could be rendered unnecessary through measuring or knowing the capacity of the bottle of medicine and applying the relationship between 20, 10 and 200.

In future cycles, the sequencing of tasks to ensure portioning is not a predominant approach, or one to which learners become acclimatised, will need to be considered more explicitly. Making portioning more inefficient is one way of achieving this.

7.5 THEME 3: THE 'DISCRETENESS' OF A QUANTITY AND ITS MEASUREMENT

In Cycle 1, all the quantities which were required to be measured (string, water, flour, sugar) could be considered continuous because it would be very difficult to separate them into discrete elements. Although sugar and flour have grains, they are too small to handle and there are too many grains to invite counting. In Cycle 2, pasta was used as a quantity to be weighed. This was chosen because flour and sugar used in Cycle 1 could be easily spilled, threatening the accuracy of pre-established relationships. Yet the introduction of a quantity which could be considered in more discrete parts also affected the way in which learners interacted with it. In both Cycle 2a (C2a.4b) and Cycle 2b (C2b.2b), when pasta was used, some learners counted the pieces of pasta. In these tasks, learners were informed that 10g was a portion of pasta and were then asked to find how many portions of pasta were in bags of pasta. These tasks could have been completed by counting pasta if learners sought to establish a ratio relationship between how many pieces of pasta were in 10g and then apply that relationship to find how multiples of that quantity were in the other bags. Through doing so, learners would have been applying a multiplicative relationship. When learners were counting in these tasks, it did not appear to be to establish such relationships. Furthermore, when learners were asked how they might find out how many 10g portions were in a bag of pasta, establishing a relationship between the number of pasta pieces and 10g of pasta through counting was not a strategy suggested. It is possible that these learners misunderstood the task, or they had a desire or instinct to count the discrete elements even though they did not need to.

For example, in C2a.4b, leaners were asked how they might find how many 10g portions of pasta were in a bag:

RW: So if you, how could you find out much pasta this is enough for

Learner 6: Count out...(starts counting)...one, two, three, four, five...

RW: You don't need to count the pasta pieces

Learner 6: But I want to

RW: It's not going to tell you how many people you can make pasta for

Learner 7: We could do that up to ten

RW: One person needs ten grammes of pasta

Learner 7: Count A

In task 2b.2b, learners were asked how they might establish what one 10g portion of pasta looked like:

RW: How are we going to work out what ten grammes of pasta looks like?

Learner 12: Oh you go like, one, two, one, two

Learner ?: I know

RW: Have a think. How are we going to work it out?

Learner 9?: Five inside there, five inside there, five inside there, five inside there

RW: I want to make ten grammes of pasta

Learner 9: Wait, how much is one gramme?

Learner 12: Put ten here, put ten in here

RW: How will I know it's ten grammes?

Learner 10?: You count

Learner 12: You can just try it

RW: But how will I know. If I put pasta in here, how will I know it's ten grammes?

Learner 9: You could put one ten gramme in this one and one ten gramme in this one

Learner 13: You can put ten grammes there...

RW: Ah, I can use these! Right...

Learner 13:...and then see, wait until that gets to the middle

Thus, there was a tendency to suggest counting the discrete pieces of pasta. There seemed to be a relationship between the discreteness of the quantity and how it was used within measurement tasks. The more continuous the quantity, the more likely there was to be spillage (and enjoyment!), but the accuracy of the pre-established relationship was then threatened. The more discrete the quantity, the easier it was to avoid or address spillage, but it also seemed to invite counting and, in setting up situations in which it is inefficient or impossible to count (as noted by Davydov 1992), this could threaten the efficacy of the task. In future cycles, pasta could be replaced by smaller grained materials such as rice or lentils.

7.6 THEME 4: THE BENEFITS AND LIMITATIONS OF INCLUDING STANDARD UNITS OF MEASURE TO SUPPORT MULTIPLICATIVE REASONING

Standard units of measure were introduced in latter tasks, both in Cycle 1 and Cycle 2. The standard units used were centimetres, millilitres and grammes. For tasks involving length, either Cuisenaire or straws were used, with lengths of 2cm, 5cm, 10cm and, in Cycle 1 (C1.4a), 4cm. As Davydov (1992) notes, teaching materials which consist of small elements that can be grouped and replaced by other elements are especially useful, and I felt that the use of standard units of measure could be used as part of this process. Indeed, the teacher in the final interview (Appendix Q) highlighted tasks which incorporated the use of standard units and accessible materials as being tasks that would be used:

RW: So, would you use any or some of these tasks to support your learners' understanding of the multiplicative relationship?

P1: Definitely. I love the straws one, because I think it's so hands on and you can give it to everyone and they can have it individually or in pairs. The medicine one is an obvious one, because it's something, it's anything that applies to them. Next year as well I want to give the children more of the weights to use themselves. We use my electronic scales, and we measure the plastic animals, and we look for those things, but I do like the idea you know...Ideally the ten grammes of pasta...we've got the ten gramme weights.

As discussed in Section 7.3, the practitioner also identified mass tasks (and the hexagon mass set) as something that had not been considered in relation to multiplicative reasoning previously:

P1:..they are...I hadn't thought of them for using multiplication before...

RW: Yes, yes

P1:...but it makes, it's using and applying and it's reasoning and there's...that is brilliant

Tasks involving standard measures allowed learners to make links between their previous experiences of measures and multiplicative reasoning. When learners were asked about their experiences and views on tasks, some learners referred to learning about the measures themselves:

For example, In Cycle 1:

RW: Yes, you think? And you are pointing to the activity with weighing there.

Learner 1: Yes

RW: What was it about the weighing activity that helped you learn maths?

Learner 1: Uh, to measure how, uh, how heavy

RW: To measure how heavy something was

Learner 1: Yes

RW: Ah, OK, Did any of the others help you learn maths?

RW: You're pointing to the rods?

Learner 1: Yes

RW: What was it about that one?

Learner 1: Uh, to count in how many centimetres.

Also, in Cycle 1

RW: The liquids? So which of the liquids ones would you say made you think really hard?

Learner 3: Mmm, that one.

RW: So you're pointing to the medicine one there...

Learner 3: Yes

RW: ...and the little bottle and the spoon. What was it about that one that made you think hard, do you think?

Learner 3: Um, all about trying to get the...the medicine in the bottle

RW: Trying to get the medicine in the bottle

Learner 3: And how many millilitres it is

Also, in Cycle 1

RW: Why were you pointing to that one? That's the rods isn't it.

Learner 5: It helped me to um count in centimetres

RW: Right, in what way did it help you to count in centimetres because I bet you can count to twenty quite easily can't you

Learner 5: How long

RW: How long twenty centimetres looks?

Learner 5: How long in two, how long in ten centimetres

RW: Yes

Learner 5: How long in two centimetres, how long in five centimetres and how long in four centimetres

In Cycle 2b:

Learner 9: And how much one gramme is really like, two, uh, twenty grammes is like the...

Learner 12: The heaviest. It's not that heavy though.

These comments exemplify that, in using the standard measures in the tasks, learners were experiencing millilitres, grammes and centimetres (and multiples of these) and, in particular, how they looked or felt. It could be that the focus of previous experiences of measures had been *how* to measure using standard measures and related equipment.

In Cycle 1, phase 1, one of the practitioners suggested the need to focus on knowing the units of measurement and which units should be used for which type of measure situation.

T2: …the children need to learn if I'm measuring water, it means I need a measuring jug and I measure in litres and millilitres, if I'm measuring time I need a clock or a stopwatch and they get them so muddled up because the language is so so similar. Centimetre, millimetre, millilitre and it's so the drumming drumming drumming and that continual…Mrs. G, she's killer G* and so they know if you're weighing Killer G* always weighs to just get that K G because the language is so similar for them, it's very very difficult, but then they've got to have the practical to know that Mrs. G* always weighs whether we're cooking and doing real things or measuring dinosaurs or plastic animals or what not.*

Later in the interview, there was a suggestion that learners needed to understand the units they were working with, though this seemed to suggest use of language:

T3: Yeah, making sure they understand full, empty, half full and making sure they know, you know, which relates to which and obviously as they get older developing and showing them, yeah, like centimetres and things in regards to length then, and yeah, just making sure they have an understanding of what those things mean and relate to the different, you know, when they're maybe doing capacity, they're not saying things like you know it's tall, they're saying it's full, so things like that…

The comments from learners suggest that they had gained understanding of the size of the measures they were using and using multiplicative relationships with standard measures allowed this to develop further (e.g., 2cm, 5cm, 10cm, 20cm, 40cm, 5ml, 10ml, 50ml, 200ml, 10g, 20g, 50g, 100g). Furthermore, the use of standard measures facilitated task preparation because relationships between standard units could be applied to source appropriate materials for learners to use.

Nevertheless, there can be limitations to using standard units explicitly in tasks designed to support multiplicative reasoning when a focus is on using an intermediate unit. As noted in Section 7.5, if multiplicative relationships are known then this can mean the need to establish an intermediate unit can be unnecessary. Furthermore, including standard units of measure introduces another multiplicative relationship. Whilst this can be seen as a benefit, this can also be seen as a potential cause for confusion. For example, in both Cycle 1 and 2, 'Dog's medicine?' task (C1.3b and C2a.3b), learners were using a spoon (10ml), bottle (50ml) and larger bottle (200ml). This meant two different multiplicative relationships were involved depending on the unit being considered:

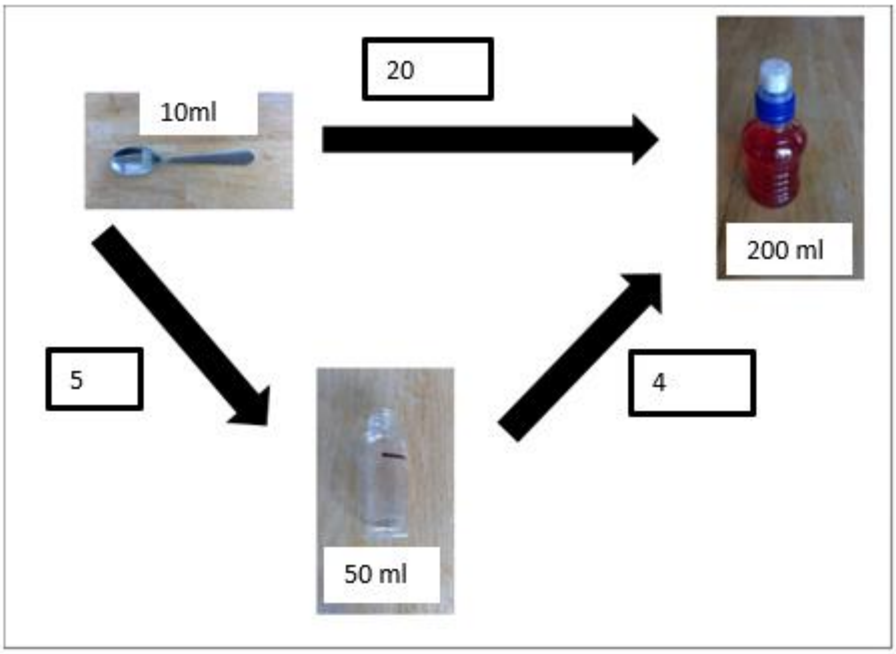


FIGURE 28: A SCHEMATIC TO SHOW THE RELATIONSHIPS IN 'DOG'S MEDICINE' TASK

As discussed in Section 2.8, Davydov used a schematic with arrows to represent quotitive relationships (Schimttau, 2010, p.269) with the direction of the arrow and number representing how many times the unit fits into the quantity.

In the case of spoons, a relationship involving 5, 4 and 20 could be discussed as noted on the arrows. Though this relationship can also be applied to the inclusion of standard units, there now become more units for learners to consider.

For example, in task C1.3b:

RW: So we said that when we have a bottle like this, we have fifty millilitres and that is worth five of those spoons. Everybody happy with that?

Learner 4: Yes

Learners: Mmm (sounds of agreement)

RW: How many of these spoons?

Learner 4: Five

RW: Five, fills that bottle there.

Learner 7: Fifty millilitres!

Later in the episode, when learners had established four little bottles were equivalent to the big bottle (notably through some learners counting in fifty millilitres), the following conversation occurred.

RW: Four bottles.

Learner 2: Two hundred.

Learners: Two hundred

RW: Four bottles. Two hundred millilitres. You are absolutely right, you were very good at counting in fifties.

RW: Four bottles. How many spoonfuls?

Learner 2 : Four

Learner 5: Eight

RW: So let's put our picture here

Learner 7: Three hundred

Learner 2: Two hundred. It looks like two hundred.

RW: So we said, remember these arrows here , so we said five of the spoons make the bottle and four of the bottles make that liquid. You're right, Learner 2, that it's two hundred millilitres, because it is four lots of fifty millilitres, but how many spoonfuls are in there, how many days can I feed my dog...give my dog the medicine?

Learner 7: Every day

Learner 4: Two hundred

The focus on spoons, and the introduction of the standard unit, meant there were two relationships to consider, and learners understandably needed reminding about which unit was being considered (millilitres or spoons).

In the final interview with a practitioner, when discussing the medicine task, the following discussion took place, indicating that the task might be viewed as too difficult for the learners.

P1: *So I would definitely use it in Key Stage 2, lower down, no. They're not ready, obviously, they...unless you've got a very able child, or if you simplify it but then you're simplifying it and you're not getting the multiplicative, you're just doing the capacity. So, definitely I would like to pass some of these on (laughs) to some of my colleagues to use, as our reasoning problems...*

RW: And I found that things like the, um, the medicine one, for example, in that one, I used, I starting using towards the end of the, the latter tasks, I started using the, the standard units but because I was focusing on their understanding of counting in a different unit, I didn't take that as far as I could have, but you could use the same problem...

P1: Yes, yes.

RW:...but bring out more in terms of the relationships because there's so many relationships there...

P1: Yes.

RW:... within, within, the standard units as well.

P1: Oh, Year 3 and Year 4, definitely.

RW: Yes.

P1: And I would start off at that level...

RW: Yes.

P1:...and see where you can go with it.

The task was more challenging than some other tasks, yet learners managed it with support in Cycle 1, and in Cycle 2 (C2a.3b), the focus on spoons as the unit was emphasised, without the exploration of the relationship between millilitres in the intermediate unit and larger bottle, partly because of time constraints, but also because learners seemed less secure in application of counting in steps other than two, five or ten, which had also been reported by the practitioners prior to working with them (Appendix O).

Hence, the inclusion of standard units in these tasks offers opportunities for experiencing standard measures and their sizes and offers opportunities for extending tasks through exploring multiple multiplicative relationships, although the extent to which learners are asked to engage with the standard units within the tasks needs careful consideration to ensure learners are clear about the units being considered.

7.7 THEME 5: THE 'HIDDENNESS' OF MATHEMATICAL OPERATIONS WITHIN TASKS

The tasks aimed to develop multiplicative reasoning through measures. All tasks involved specific multiplicative relationships, and, in some tasks, I recorded relationships using 'arrow diagrams' (see Figure 28 above, p.279 as an example) and sometimes asked learners to do this or asked them to use notation familiar to them, particularly the multiplication symbol.

In analysis of the tasks, it can be difficult to ascertain whether the tasks could be associated with a particular operation (multiplication or division). For example, in the classic 'Rabbits' task discussed by Davydov (1992) and applied in both cycles (C1.1c, C1.1d, C1.2a, C2a.2a), learners were challenged to consider how many little cups were equivalent to a large jug of water and the notion of an intermediate unit, a larger cup, was introduced. Through identifying a relationship between the little cup, intermediate cup and jug, the learners were establishing and applying a multiplicative relationship. The order in which they do this is irrelevant; for example, they could find how many of the intermediate unit are equivalent to the jug first and then find how many of the smaller unit are equivalent to the intermediate unit, or they could find out how many of the smaller unit are equivalent to the intermediate unit and then establish how many of the intermediate unit are equivalent to the jug. For example, in this task in Cycle 1, C1.1d, the following relationship is established:

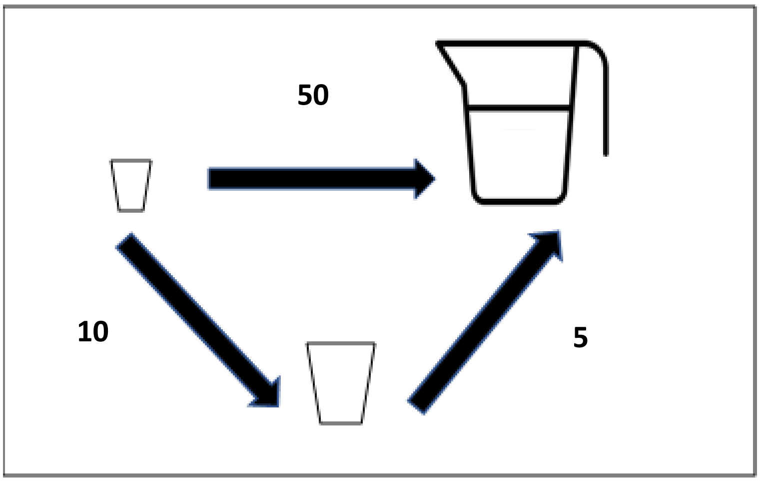


FIGURE 29: A SCHEMATIC TO SHOW A MULTIPLICATIVE RELATIONSHIP

Through finding out how many of the little cups are in the jug, the learners are undertaking multiplication, but the notion of how many of the little cup are in the big jug might suggest a division problem in a more conventional scenario.

Of course, any problem involving a multiplicative relationship could be solved by applying either multiplication or division, depending on the way it is perceived and the order in which it is approached. These tasks were not presented as either a multiplication or division problem, and learners typically showed that they could work flexibly to establish and apply multiplicative relationships.

Tasks that could have been more immediately associated with division (e.g., working out how many 10g bags portions of pasta were within a bag weighing 80g, as in C2b.4b and C2b.2b) were approached through applying multiplicative relationships, without explicit reference to division as a concept. As noted previously, learners suggested portioning the pasta into 10g portions, which demonstrates an understanding of the quotitive nature of division.

I see the 'hiddenness' of the operations as an advantage within these tasks. The introduction of the schematic allowed for a focus on a multiplicative relationship without explicit reference to multiplication and division and, if used regularly, could support flexibility in working with such relationships and could provide a basis for the introduction of symbols for multiplication and division. The schematics were not an explicit focus of this research, and their use was mainly modelled by me when relationships were being established, though in some tasks, particularly in Cycle 1, learners were invited to engage with them.

For example, in C1.2a, as a paired follow up task, learners were given a bottle of water and asked to establish how many little cups there would be within the bottle when they knew one larger cup held liquid equivalent to ten little cups.

Learners 5 and 6 worked together and quickly established there were thirty little cups:

RW: There we are. So we know that there's at least ten
Learner 6: Stop! Oh my gosh!
Learner 5: Miss, thirty

The learners were given the schematic to record the relationship, and the following conversation took place:

Learner 6: So how much

Learner 5: On the big cups?

Learner 6: So there's three big cups

Learner 5: On the big cups

Learner 6: Do a big cup like that one

Learner 5: Yeh

Learner 6: Write three

Learner 5: Three

Learner 6: And then three of the big cups, on the little cups there's going to be...

Learner 5: Thirteen

Learner 6: Thirty

Learner 5: No six

Learner 6: Thirty. Thirty. On the little cup.

Learner 5: Thirteen

Learner 6: Thirty. Oh yeh, look at that you draw a little cup and then write thirty. Don't rub all of them out

Here, Learner 6 was supporting Learner 5 in making sense of the diagram and establishing the relationship, recognising that there would be thirty little cups equivalent to the bottle. Although Learner 5 appeared to have quickly established earlier that there would be thirty little cups equivalent to the bottle, transferring onto a diagram seemed to cause some confusion, showing that more familiarity with the ways of representing the relationships in that way would be required for them to be used and interpreted confidently. As discussed in Section 2.3, the Concrete-Pictorial-Abstract approach heuristic for teaching mathematics, is likened to Bruner's enactive-iconic-symbolic modes of learning (Hoong *et al*., 2015). The schematic allows for representation of a multiplicative relationship in a visual (pictorial/iconic) way which would allow for ideas around multiplication and division to be developed simultaneously and the symbols for multiplication and division could be introduced after the use of the schematic is secure.

The tasks required learners to explore, establish and apply multiplicative relationships in a practical way and allowed for multiple opportunities to reinforce the relationships, whether through 'multiplication' or 'division'. Tasks such as these support the exploration of multiplicative relationships, as noted as a description of learning within the Curriculum for Wales (see Table 1, p.8) and, with further use of the schematic, would provide an avenue for more formal introduction of multiplication and division symbols.

7.8 DISCUSSION AND CONCLUSION

In this thesis, I have focused on the exploration of the learning and teaching of multiplicative reasoning through measures, using design-based research, with the following sub-questions:

S1: What are teachers' and learners' prior experiences of teaching and learning multiplicative reasoning?

S2: What are learners' prior experiences of learning number and measures?

S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?

S4: What is the impact of learning multiplicative reasoning through measures on learners?

S5: What are teachers' and learners’ views on teaching/learning multiplicative reasoning through measures using the materials developed?

Though the study has been small-scale, involving one school, six teachers and twenty-one learners, I have been able to undertake tasks with learners in their school environment. Through design research, I have not been seeking proof of what works or any statistical generalisation, rather I have been exploring a teaching approach and seeking insight into how and why it might, or might not, work (Bakker, 2018). I offer this insight for scrutiny and discussion, in line with the theoretical perspective taken within this work.

As outlined in Section 1.1, Coles (2017, p.206) sees the ‘counting world’ as a ‘predominant narrative’ in mathematics education and this was very much the experience of the learners I worked with. The interviews (Appendices J and O) suggested learners had experienced multiplication and division as extensions of counting, with a focus on the operations and ideas of ‘grouping’ and ‘sharing’ through counting discrete objects and counting in steps other than one, as outlined in Section 2.6. Interviews also suggested measurement is typically taught as a skill to be accomplished and not usually considered as a context for the exploration and introduction of number relationships (Appendices J and O). Whilst counting in steps other than one is seen as important in the learning of the multiplicative relationship (e.g., Anghileri ,1989), the use of discrete objects for counting often means counting in ones is always possible, and often reinforced. Davydov (1992) rejects the idea of multiplication and division as an extension of counting, focusing on the design of measure based tasks which necessitate a change in approach from counting in single units to counting composite units, with a focus on seeking efficiency, and on the relationships involved. However, Davydov and Elkonin designed a whole curriculum that introduced number and additive relationships to learners through a measures based approach (e.g., Davydov, 1990) and therefore learners were familiar with using measures contexts for learning about number and relationships. This research offers insight into the introduction of a measures based approach for developing multiplicative reasoning, where learners and their teachers have come from a 'counting world' (Coles 2017, p.206). I have explored the approach and have shared my concluding themes.

Watson (2021, p.19) notes that 'mathematical tasks can define what it means to do mathematics, so tasks which afford different forms of activity shape different views of the subject'. Through implementing tasks and analysing learners' and teachers' responses to them, I have shown that measures based tasks can offer rich and enjoyable opportunities for learners to explore multiplicative relationships *and* learn about measures. As the tasks are rooted in practical and real contexts and focus on reasoning, as noted by the teacher (Appendix Q), they could also shape a view of mathematics being rooted in reasoning and real contexts.

Though tasks offer rich opportunities to develop multiplicative reasoning, I have found that the measure context, and type of continuous quantity being used, can impact on the learning and teaching, something seemingly not explicitly explored in Davydov's work (e.g., Davydov, 1990; Davydov, 1992).

As discussed in Chapter 4, design research offers the opportunity to learn more about learning and teaching, and this study is not purely about the evaluation of tasks. Through this study, I have identified the importance of establishing equality relationships in measures contexts, and I offer an analysis of the methods by which equality relationships can be perceived and established within measures contexts. The choice of method varies according to the measure context, and the use of them has implications for the teaching and learning of any measures based tasks. I suggest that using a many-into-one approach when using units quotitively to establish an equality relationship is a progressive step, and this should be considered more explicitly when planning measures tasks. The concept of a unit is central to a Davydov and Elkonin curriculum (e.g., Davydov, 1990; Davydov, 1992), less so in other curricula, where it is used to refer to fractions, 'unit fractions', or units of measure 'standard or non-standard units' (e.g., Curriculum for Wales, WG, 2021; Department for Education, 2021), yet from the start of schooling, learners can apply units in learning about number and relationships and this research offers further insight into this process.

As discussed in Section 2.3, curricula cannot be easily transposed into different settings. Shape-shifting (Venkat, Askew and Morrison, 2021, p.399) or 'deconstruction' (Mellone, Ramploud and Carotenuto, 2021, p.382) is needed; ideas need to be re-interpreted and adapted to account for cultural contexts and circumstances. In this research, Davydov's (1992) ideas around multiplicative reasoning involving a change in units, with a focus on using measures tasks have been adapted to account for the learning experiences of teachers and the learners.

I began this study with a desire to explore the learning of the multiplicative relationship, and I wanted to do this in a way that might support the learning and teaching of it. Design research has allowed me to explore specific tasks *and* pedagogic approaches, whilst also developing insight into learning *both* multiplicative reasoning and measures. I recognise that more can be done to develop the tasks and the pedagogical approaches they require; future development should involve a focus on structuring and sequencing of the tasks, and consideration of the ways they may be enacted by practitioners. As Coles and Sinclair (2022) suggest, integrating a pedagogical focus on relationships into a curriculum would require significant training and development. Further iteration of this research could focus on the development, enactment and sequencing of the tasks, with teachers and learners as participant researchers, thus supporting teacher professional development whilst also drawing on learners' and teachers' invaluable insight and experience into learning.

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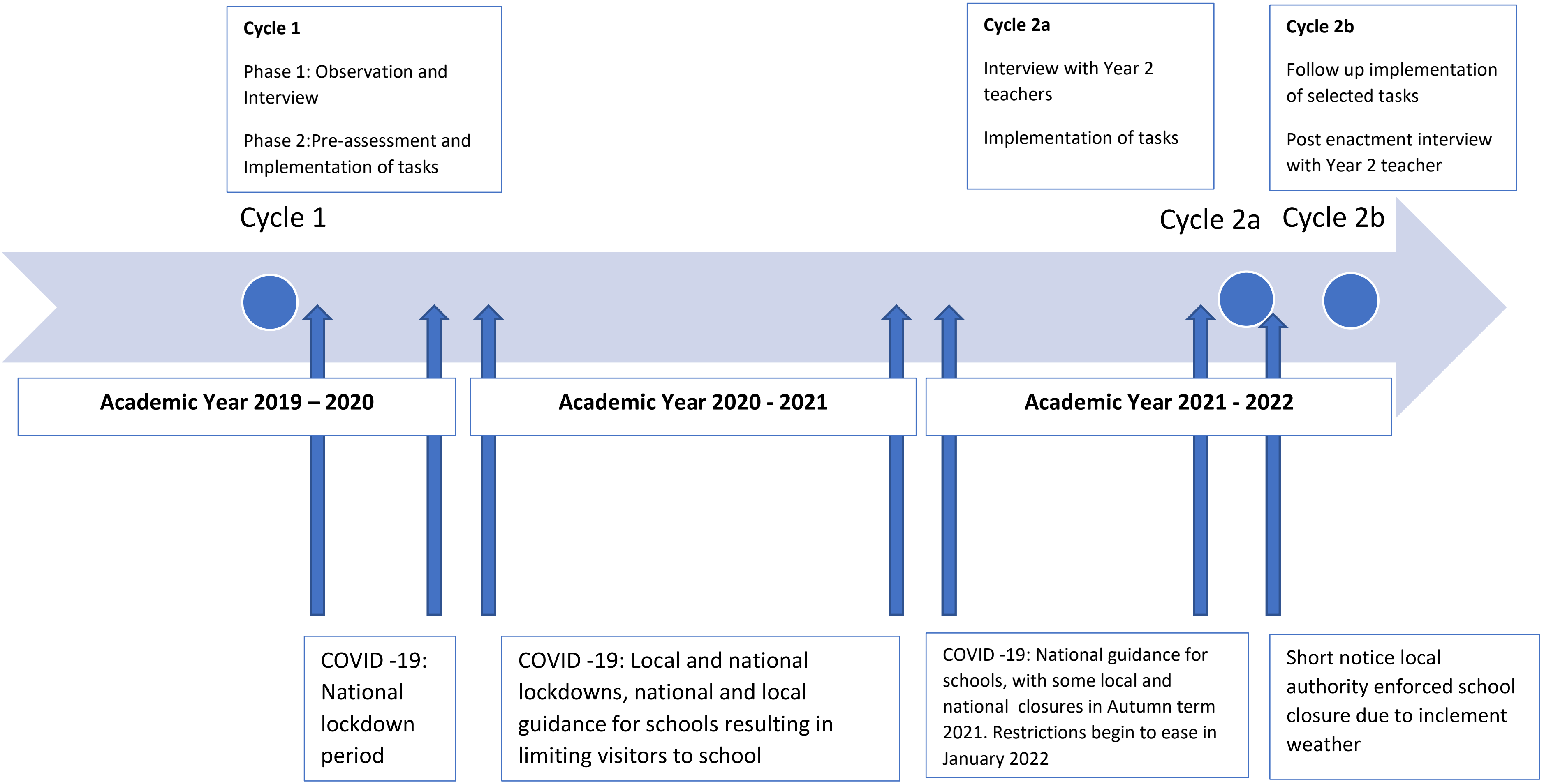
APPENDIX A: TIMELINE OF RESEARCH CYCLES



APPENDIX B: INFORMATION ON CYCLE 1, PHASE 1

Prifysgol Cymru Y Drindod Dewi Sant University of Wales Trinity Saint David | Yr Athrofa Institute of Education Addysg i Gymru Education for Wales

Information about research activity phase 1 (practitioners)

Title: A design-based research project to develop and evaluate materials for teaching multiplicative reasoning through measures.

Name and contact details of researcher: Rachel Wallis, rachel.wallis@uwtsd.ac.uk

Overview: The aim of this research project is to develop and evaluate teaching materials to support young children's learning of the multiplicative relationship, in particular, multiplication and division. The main research questions are:

S1: What are teachers' and learners' prior experiences of teaching and learning multiplicative reasoning?
S2: What are learners' prior experiences of learning number and measures?
S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?
S4: What is the impact of learning multiplicative reasoning through measures?
S5: What are teachers' and learners' views on teaching/learning multiplicative reasoning through measures using the materials developed?

Phase 1 (Exploration: focus on S1 and S2 above)
Through finding out about teachers' and learners' prior experiences, I aim to be able to begin develop materials (for phase 2) that could support teachers in using measures as a meaningful context for teaching the multiplicative relationship.
Through finding out about learners' prior experiences of learning number and measures, I aim to design learning tasks (for phase 2) that will build on prior experiences, using measures as a context and collaborative problem-solving as a strategy.

Research activity (phase 1)
Please note that the research activity is to develop my awareness of the experiences of learners and teachers, and to understand the provision they will be accustomed to. It will not involve making judgements.

1)Exploration of Foundation Phase setting and related provision:

- Observation of learning environment (not of individual lessons), taking into account continuous and enhanced provision and resources available.
- Ascertaining arrangements for mathematics learning across Foundation Phase (including planning/resources etc.)
- Analysis of learner work (where possible) to ascertain typical experiences

2)Focus group interview with Foundation Phase practitioners

- To explore, with practitioners, how mathematics (and in particular numbers and measures) is typically taught.

- It will involve some factual/information type questions (e.g. about general planning and provision)
- It will involve some questions that may explore opinions.

Data collected
Data collected will be in the form of notes (in a record sheet and researcher diary) and audio recording (for the focus group interview). It will be used to inform the next stage (development of tasks).
Any data collected would be available to view on request. No data collected will refer to individual learner or practitioner names and the data would be kept confidential.
Data will be stored by the researcher in a secure password protected folder on the university cloud storage system. It would be destroyed on completion of the PhD.
Once collected, the data would be collated and analysed to inform task design for phase 2 (implementation). Analysis of data (or aspects of it) may be referred to within the thesis dissertation which would be available for reading on request. No locations or individuals would be named within the thesis.

Consent
If you are willing to participate in the research activity (phase 1), please sign below.
You would be free to withdraw involvement and consent at any time, and no related data would then be used.

Name (please print)

____

Role (please print)

____

I consent to take part in phase 1 of the proposed research activity, in line with what is outlined above.
(signed)

____

Date:___________________________

APPENDIX C: CYCLE 1, PHASE 1 OBSERAVTION SCHEDULE

Aspect: Planning *Use examples where possible
Scope: School long-term, mid-term and short-term planning for number and measures teaching and learning.

Areas for consideration… **(including some prompt questions, not an exhaustive list)**	**Notes (include information about source of information where possible)**
Long-term planning What informs the long-term planning (e.g. blocking of mathematics topics and connections between)? Who is responsible? How often is it reviewed?	
Mid-term planning What informs the mid-term planning? Are certain resources (e.g. teacher support books) used to support planning	
Short-term planning What informs the short-term planning? Are certain resources (e.g. teacher support books) used to support planning?	
General notes	

Aspect: Learner work *Photograph anonymised samples where possible
Scope: Learner work (Year 1 and Year 2).

Areas for consideration… **(including some prompt questions, not an exhaustive list)**	**Notes (include information about source of information where possible)**
Experiences with number Note experiences such as: Learner recording (free recording/supported/to what extent) Use of notation Use of vocabulary Record of task given? Reasoning aspects? Collaboration?	

Experiences with measures Note experiences such as: Learner recording (free recording/supported/to what extent) Concept of unit Use of notation Use of vocabulary Use of measuring resources/resources to measure Record of task given? Reasoning? Evidence of collaboration?	
General notes	

Aspect: Organisation and enactment of mathematics/numeracy learning in classroom *Photographs taken of relevant spaces

Scope: Foundation Phase settings and the organisation of learning experiences
Foundation Phase setting:____

Areas for consideration… **(including some prompt questions, not an exhaustive list)**	**Notes (include information about source of information where possible)**
How is the mathematics taught? Consider: • whole class teaching • focus group • who teaches • groupings and group working? • support (e.g. use of teaching assistant) • timing (across week/daily timings) • collaborative work and problem solving approaches • feedback mechanisms and focus of feedback	
What resources/manipulatives are available? Consider availability and location of specific resources to support teaching and learning of number and measures, e.g. • unifix/multilink • blocks • Numicon • Dienes/base 10 • measuring equipment • support displays • technology	

Continuous provision Note continuous provision and location and arrangements of this. For example: • sand/water • role play • construction • small world play • table top activity • creative area • washing line • writing/graphics • technology	
Enhanced provision Note any mathematical enhancement of provision. Consider how this is communicated (e.g. orally/in writing/both/with talk buttons). Consider aspects such as vocabulary development.	
Outdoor provision Note outdoor provision and relevant arrangements (e.g. free/controlled access). Note resources used and how these may be managed.	

Observation of specific mathematical interactions
Location:
Year:
Context:
How did interaction come about (e.g. spontaneous/as part of planned learned etc./location/participants)

Notes:
What happened? Use of resources/mathematical language etc.

Thoughts:
Why has this incident been chosen? What might it suggest?

APPENDIX D: EXAMPLE OF REFLECTIVE NOTES: REFLECTIVE DIARY ENTRY

Day 1 Tasks

The learners were keen to use liquids! They remembered these.

The starter tasks worked well – interesting that Learner 2 and Learner 8 seemed to think that the larger number would go with the bigger cup.

Modelling seemed to work well on this task. Interestingly Learner 4 wrote x against the arrows showing understanding of multiplication sign. It made me wonder whether I should use it along the arrows.

The wool task worked fairly well – some (Learner 6) predicted the double relationship. The wool itself was not a good thing to use because it was quite springy. I actually ended up changing the task a little by not revealing the number of the smaller straw – this was a reaction to the learners predicting the relationship. This task showed me that the learners had some experience of the multiplicative relationship (e.g. using terms doubles) which would be expected. Learner 6 was able to state clearly that the small objects would result in a bigger number because we had to use it more times.

The main task worked fairly well, although I did feel the need to get the learners more involved at this point. The starter tasks led up well to this task, but I did make the mistake of putting the bigger cups out earlier. Interestingly, Learner 2 said we could use a jug that was marked, showing experience with standard measures. This did cause me to wonder whether I should build on the understanding of standard measures rather than relying on non-standard measures. My questions could be considered leading (need to check the recording) and I felt that I was doing a lot of the talking. This is something I need to consider further.

We worked out together that there were 7 big cups in the jug and they poured these out in turn which worked well – it struck me though that there could be a lack of accuracy in terms of the measures as some were fuller than others so we were always using approximates. Interestingly the learners did use fractional terms (e.g. half) for cups that were not fuller than others. Using marked containers should help with this and this will definitely be considered in the future. However, one thing to note here is the approximate nature of a measures task like this, which could be considered contrary to what the learners will already be working on (I.e. the developing accuracy and standard measures).

Trying to establish how many of the smaller cup were in the larger cup was problematic because the learners were using the cups already measured out – it struck me at this point that having sufficient little cups would have been beneficial from a visual sense. I need to think about how I use the resources to support the students in a visual and practical way. One pair of learners (led by Learner 2) tried to submerge the little cup into the large cup. This could be because I had said 'how many times it fits in' or it could be because there was attempt to fill the cup without pouring. Another learner, Learner 3, attempted to guess how many times by moving his finger up the cup (seemingly approximating the amount of liquid).

We eventually established that there were 10 little cups worth in the larger cup, but different pairs had resulted in different numbers. This was due to spillage. In this way the use of continuous quantities could be considered problematic for the reasons discussed above. There was a constant balance between trying to have the learners active in the task whilst also trying to ensure we agreed on the resulting numbers – this does perhaps highlight the 'tension' between accuracy and the reinforcement of a concept using these tasks. I was, of course, less interested in establishing the result (70 little cups) and more interested in establishing the notion of an 'intermediate unit' as a 'quicker way' but, nevertheless, it felt important to establish an actual number.

The use of the arrow diagram was fairly useful – I built this up gradually and modelled. We ran out of time at the end to complete the last task but I did see that Learner 3 and Learner 4 completed an arrow diagram. Learners 2, 6 and 8 seemed able to verbally state the relationship and Learner 7 seemed less sure.

Further reflection has caused me to wonder whether I should incorporate some standard units into the tasks, still focusing on the multiplicative relationship and the use of an intermediate unit but building on the notion of accuracy and the use of an intermediate unit. Such tasks might include making a given measurement using Cuisenaire (with restrictions in what is available) or weighing equivalents to 1kg by counting in multiples of g.

Of course, another perspective is that the learners are being challenged to consider solutions (because I am deliberately restricting what is available) which is causing them to think in other ways.

Plan for now:

-Try to involve learners more in sharing their thinking – allow time to think and discuss and ask for thoughts more dialogically

-Consider using marked objects to support notion of accuracy

-Reinforce relationships between units –and the idea that we can work things out more efficiently with a larger group

APPENDIX E: EXAMPLES OF REFLECTIVE NOTES: ANNOTATIONS AND MEMOS IN NVIVO

Example of annotations made when transcribing audio data from a task (Cycle 1, Phase 1, Day 1)

Annotations

Item	Content
1	This is an interesting suggestion because it could be that the learner is suggesting that a relationship could be established between the big cup and the little cup. What is noteworthy here is that I did not take the time to find out.
2	This suggests a difficulty with task set up - however accurately I tried to measure out, there is a lack of accuracy with liquids.
3	A difficulty with task set up - using wool meant that it had some elasticity and therefore needed to be pulled out straight. Thinking about the suitability of the materials is important.
4	This is where I had set up a relationship but suddenly doubted the relationship - to what extent is this manufacturing?
5	Here the learner seems to agree it is half and is establishing that two of them would make the whole
6	This sugggests Learner 7 was having difficulty iterating the units
7	This appears to be a moment of realisation for Learner 7.
8	Learner 7 is a very interesting case - learner needed support with physically measuring and so this may suggest she has learnt about iteration

Example of a memo made on an audio file.

First reflections 1d (before transcription and coding)
29/04/2020 13:34 Task 1d builds on what was done with Task 1c but with a different container. Task effficacy comes to mind here as the use of words cups could be confusing - we keep having to differentiate between big and little. I wonder whether I should have restricted the use of the small cup completely because learners were using this when they didn't need to. Learners were clearly not (from the audio evidence) working in multiples of 10. Of particular note is audio towards the latter part of the session where two learners (Learner 5 deinitely) were counting in 1s. Coding for working in multiples is something to consider, as well as working in 1s.
Another point to note here is that the context of rabbits is quickly dropped (possibly influenced by me, because I don't reinforce it) but that actually the narrative context does seem irrelevant - the learners don't seem to pick this narrative context up a lot in what they say either.

After coding 1d
This is clearly a rushed episode - there is some awareness of relationships between quantity and unit and perhaps some awareness of a change in unit but it doesn't appear as rich as 1abc. I think I will need to go back and have another look at this.

APPENDIX F: ETHICAL APPROVAL

APPLICATION FOR ETHICAL APPROVAL

In order for research to result in benefit and minimise risk of harm, it must be conducted ethically. A researcher may not be covered by the University's insurance if ethical approval has not been obtained prior to commencement.

The University follows the OECD Frascati manual definition of **research activity**: "creative work undertaken on a systematic basis in order to increase the stock of knowledge, including knowledge of man, culture and society, and the use of this stock of knowledge to devise new applications". As such this covers activities undertaken by members of staff, postgraduate research students, and both taught postgraduate and undergraduate students working on dissertations/projects.

The individual undertaking the research activity is known as the "principal researcher".

Ethical approval is not required for routine audits, performance reviews, quality assurance studies, testing within normal educational requirements, and literary or artistic criticism.

Please read the notes for guidance before completing ALL sections of the form.

This form must be completed and approved prior to undertaking any research activity. Please see Checklist for details of process for different categories of application.

SECTION A: About You (Principal Researcher)

Full Name:	Rachel Malca Wallis				
Tick all boxes which apply:					
Member of staff:	☒	Student:	☒	Honorary research fellow:	☐

Faculty/School/Centre:	Yr Athrofa		
Campus:	Swansea		
E-mail address:	rachel.wallis@uwtsd.ac.uk		
Contact Telephone Number:	01792 282039/07733 072081		
For students:			
Student Number:	140785	Undergraduate	☐
Programme of Study:	PhD	Taught Postgraduate	☐
Director of Studies/Supervisor:	Jane Waters/Anne Watson/Jan Barnes	Research	☒

SECTION B: Approval for Research Activity

Has the research activity received approval in principle?	YES	☒	NO	☐

(please check the Guidance Notes as to the appropriate approval process for different levels of research by different categories of individual)				
				Date
If Yes, please indicate source of approval (and date where known):	Research Degrees Committee		☒	**PG1 approval 2015**
	Faculty Research Committee		☐	
	Other (write in)		☐	

Approval in principle **must** be obtained from the relevant source prior to seeking ethical approval.

SECTION C: External Ethical Guidance Materials

Please list the core ethical guidance documents that have been referred to during the completion of this form (including any discipline-specific codes of research ethics, and also any specific ethical guidance relating to the proposed methodology). Please tick to confirm that your research proposal adheres to these codes and guidelines.	
British Educational Research Association [BERA] (2018) Ethical Guidelines for Educational Research, fourth edition, London. https://www.bera.ac.uk/researchers-resources/publications/ethicalguidelines-for-educational-research-2018	☒

SECTION D: External Collaborative Research Activity

Does the research activity involve collaborators outside of the University?	YES	☐	NO	☒
If Yes, please provide the name of the external organisation and name and contact details for the main contact person:				
Institution				
Contact person name				
Contact person e-mail address				

Where research activity is carried out in collaboration with an external organisation

Does this organisation have its own ethics approval system?	YES	☐	NO	☐
If Yes, please attach a copy of any final approval (or interim approval) from the organisation				

SECTION E: Details of Research Activity

Indicative title:	A design based research project to develop and evaluate materials for teaching multiplicative reasoning through measures		
Proposed start date:	June 2019	Proposed end date:	June 2020
Purpose of research activity (including aims and objectives) Outline the purpose, aims and objectives of the research activity, including key research questions. Show briefly how existing research has informed the proposed activity and explain what the research activity will add and how it addresses an area of importance. (Maximum 300 words)			

Multiplicative reasoning is a term used to refer to the understanding, applying and reasoning with multiplicative relationships and involves not only understanding and applying concepts such as multiplication, division, fractions and ratio but also being able to make connections between such concepts. In the last decade, understanding and applying the multiplicative relationship has been suggested as a key indicator of progress and later attainment (e.g. Siemon *et al*. 2008; Sieglar *et al*. 2012; Nunes *et al*. 2012). Hence, ensuring the multiplicative relationship is understood when it is first introduced (typically ages 7-9 in primary school) is vital.

Vergnaud (1979, p.264) comments that 'the concept of number would not exist if man had not met problems of measurement'. Davydov (1991) with a colleague Elkonin, designed a programme which aimed to developed concepts in number, additive reasoning and multiplicative reasoning through reasoning tasks involving measures. In this programme continuous quantities (such as length, area, mass, volume and capacity) were used as contexts for tasks in which children could explore and generalise mathematical concepts. Although research has been undertaken on teaching through measures (e.g. Schmittau and Morris, 2004; Dougherty, 2003), the programmes have used this approach from the start of compulsory education. For example, the U.S. 'Measure Up' programme (Dougherty, 2003; Venenciano, 2017) involves a curriculum based on measurement (informed by Davydov and Elkonin) for ages 5 through to 17/18 and is taught within a research school.

This research will aim to develop teaching materials to support the learning of the multiplicative relationship in Year 2, using measures as a context. The materials developed will need to reflect a context where measures may not have been a predominant feature of learning number concepts up to that point.
Sub-questions will be:
S1*: What are teachers' and learners' prior experiences of teaching and learning multiplicative reasoning?*
S2*: What are learners' prior experiences of learning number and measures?*
S3*: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?*
S4*: What is the impact of learning multiplicative reasoning through measures?*
S5*: What are teachers' and learners' views on teaching/learning multiplicative reasoning through measures using the materials developed?*

Proposed methods
Provide a brief summary of all the methods that **may** be used in the research activity, making it clear what specific techniques may be used. If methods other than those listed in this section are deemed appropriate later, additional ethical approval for those methods will be needed. (Maximum 600 words)

This research will take place in an educational setting (primary school) and will involve the development and evaluation of teaching materials to support the learning and teaching of the multiplicative relationship for children aged 7 –9 years. A Design Based Research (DBR) approach will be used. DBR (also known as 'research design') is an approach that focuses on the design and evaluation of an intervention, with a main aim being the production of outcomes which can be applied in educational settings (Anderson and Shattock, 2012). In the context of this research, the intervention will be the use of measures tasks as a context for the introduction of the multiplicative relationship. DBR typically uses mixed methods to gather data, and also involves collaboration between practitioners and researchers (Anderson and Shattock, 2012). One of the benefits of DBR is that it not only results in the production of materials that can be used in educational settings, but also that the process of development of the materials supports the professional learning of the teachers using and evaluating them (Swan, 2007).

A research map (Appendix 1) is attached. This outlines the phases of school based research activity and each research action. Each suggested research action is detailed below.

Table of research activity

Research activity and related sub-question	Notes	Participants
R1 Focus group interview with Foundation Phase practitioners. S1 and S2.	Focus group interview with semi-structured questions to explore experiences of teaching the multiplicative relationship and the teaching of number and measures. It will be audio recorded.	Practitioners (teachers) within the Foundation Phase setting (approximately 6).
R2 Audit of learning environment and provision. S1 and S2.	Observation of learning environment (auditing continuous/enhanced provision available within setting). Audit of secondary sources (school planning/ schemes of work/work books).	No data collection from individual participants. Year 1 and Year 2 learning environments and planning/work will be audited.
R3 Ongoing day-to-day discussion with practitioners related to suitability of tasks recorded via researcher diary. S3.	Once tasks and lessons have been devised (using data collected from R1 and R2) they will be shared with practitioners teaching Year 2. The researcher will keep an ongoing reflective diary to record day-to-day conversations about the tasks and their suitability.	Year 2 practitioners (teachers and teaching assistants)
R4 Pre-assessment with focus learners. S4	Pre-assessment in the style of a semi-structured interview, using adaptive assessment tasks and concrete resources (not a written test). This will be video/audio recorded.	Focus learners (approximately 6) from 2019-2020 Year 2 cohort.
R5 Teach tasks. Observation of focus learners. Reflection by researcher. S3.	Lessons will be taught by the researcher over a one week period within the normal school timetable and in line with normal routines. Practitioners (teachers and teaching assistants) will be asked to observe focus learners. Tasks will be video/audio recorded (depending on school/parental/learner consent). The researcher will also record reflective comments in a diary.	Lessons will be taught to the 2019-2020 Year 2 classes in line with normal practice (up to 60 learners). Focus learners will be observed (approximately 6).
R6 Focus group interview with practitioners S5	Focus group interview with semi-structured interviews will take place at the end of the one week block. This will explore practitioners' views of the tasks.	Year 2 (2019-2020) practitioners (teachers and teaching assistants), up to 6.
R7 Post-assessment with focus learners S4 and S5	Post-assessment in the style of a semi-structured interview, using adaptive assessment tasks and concrete resources (not a written test). This interview will also seek to explore learner views of the tasks undertaken. This will be video/audio recorded.	Focus learners (approximately 6) from 2019-2020 Year 2 cohort.

R8 Semi-structured interview with focus learners S4.	Follow up phase (in a subsequent term): Semi-structured interviews with focus learners to explore the images/models/representations used by the learners for the multiplicative relationship. This will be video/audio recorded.	Focus learners (approximately 6) from 2019-2020 Year 2 cohort
R9 Semi-structured interview with practitioners. S4 and S5.	Follow up phase (in a subsequent term): Semi-structured interviews with the practitioners to explore their views on whether the lessons have had any impact on the learners and/or their own practice. This will be audio recorded.	Year 2 practitioners (teachers and possibly teaching assistants), up to 6.

(this box should expand as you type)

Location of research activity
Identify all locations where research activity will take place.

The research will take place in a primary school which is in the Yr Athrofa Professional Learning Partnership (APLP). Currently it is proposed that the research activity will take place in SCHOOL NAMED AT REQUEST OF ETHICS PANEL, although this could be subject to change. This school is a lead school in APLP and is also identified as a research school, meaning it already has an established research relationship with Yr Athrofa. The head teacher and practitioners have indicated initial interest in being involved and an email can be provided on request to verify this. Such interest is not considered consent and should this situation change another school within the APLP would be approached for voluntary informed consent in line with the procedures outlined within this form.

(this box should expand as you type)

Research activity outside of the UK
If research activity will take place overseas, you are responsible for ensuring that local ethical considerations are complied with and that the relevant permissions are sought. Specify any local guidelines (e.g. from local professional associations/learned societies/universities) that exist and whether these involve any ethical stipulations beyond those usual in the UK (provide details of any licenses or permissions required). Also specify whether there are any specific ethical issues raised by the local context in which the research activity is taking place, for example, particular cultural sensitivities or vulnerabilities of participants.

N/A

(this box should expand as you type)

SECTION F: Scope of Research Activity

Will the research activity include:	**YES**	**NO**
Use of a questionnaire or similar research instrument?	☐	☒
Use of interviews?	☒	☐
Use of diaries?	☒	☐
Participant observation with their knowledge?	☒	☐

Participant observation without their knowledge?	☐	☒
Use of video or audio recording?	☒	☐
Access to personal or confidential information without the participants' specific consent?	☐	☒
Administration of any questions, test stimuli, presentation that may be experienced as physically, mentally or emotionally harmful / offensive?	☐	☒
Performance of any acts which may cause embarrassment or affect self-esteem?	☐	☒
Investigation of participants involved in illegal activities?	☐	☒
Use of procedures that involve deception?	☐	☒
Administration of any substance, agent or placebo?	☐	☒
Working with live vertebrate animals?	☐	☒
Other primary data collection methods, please explain in this box Focus group interviews	☒	☐

If NO to every question, then the research activity is (ethically) low risk and **may** be exempt from **some** of the following sections (please refer to Guidance Notes).

If YES to any question, then no research activity should be undertaken until full ethical approval has been obtained.

SECTION G: Intended Participants

Who are the intended participants:	**YES**	**NO**
Students or staff at the University?	☐	☒
Adults (over the age of 18 and competent to give consent)?	☒	☐
Vulnerable adults?	☐	☒
Children under 18?	☒	☐
Prisoners?	☐	☒
Young offenders?	☐	☒
Those who could be considered to have a particularly dependent relationship with the investigator or a gatekeeper?	☐	☒
People engaged in illegal activities?	☐	☒
Others (please identify):	☐	☒

Participant numbers and source Provide an estimate of the expected number of participants. How will you identify participants and how will they be recruited?

The research will take place within an educational setting (focus primary school). Participants in the research activity will be practitioners (teachers and possibly teaching assistants) and learners within the school. Voluntary informed consent of the headteacher and the practitioners involved will initially be sought. Informed consent from parents and informed assent of learners taking part in data collection will also be sought. The number of practitioners involved will be approximately 6 and they will be identified as potential practitioners because they will be teaching in the relevant year groups. Please note that initial interest in taking part in the study has already been indicated by a research school (see Section E) and the related practitioners (email verification can be provided on request).

The number of learners involved in lessons could be up to 60 (two classes), although the learners involved in data collection (observation and semi-structured interviews) is likely to be approximately 6. Focus learners (those within the data collection sample) will be identified by selecting learners for whom parental consent has been gained; learners who may broadly be considered 'representative' will be considered in consultation with practitioners and these learners will be approached for assent. They would be asked (in a child friendly manner) whether they would be willing to take part in the study and, due to their age, this would be repeated each time with a right to withdraw observed. Body language would also be considered.

The main research (involving the 2019-2020 Year 2 cohort and practitioners) would be considered normal classroom practice as it will involve teaching and learning of curriculum related content. Details of each research activity suggested in the research map (Appendix 1) and the table of research activity (Section E, Proposed methods) outlines intended participants and number.

this box should expand as you type)

Information for participants:	**YES**	**NO**	**N/A**
Will you describe the main research procedures to participants in advance, so that they are informed about what to expect?	☒	☐	☐
Will you tell participants that their participation is voluntary?	☒	☐	☐
Will you obtain written consent for participation?	☒	☐	☐
Will you explain to participants that refusal to participate in the research will not affect their treatment or education (if relevant)?	☒	☐	☐
If the research is observational, will you ask participants for their consent to being observed?	☒	☐	☐

Will you tell participants that they may withdraw from the research at any time and for any reason?	☒	☐	☐
With questionnaires, will you give participants the option of omitting questions they do not want to answer?	☒	☐	☐
Will you tell participants that their data will be treated with full confidentiality and that, if published, it will not be identifiable as theirs?	☒	☐	☐
Will you debrief participants at the end of their participation, in a way appropriate to the type of research undertaken?	☒	☐	☐
If NO to any of above questions, please give an explanation			
(this box should expand as you type)			

Information for participants:	YES	NO	N/A
Will participants be paid?	☐	☒	☐
Is specialist electrical or other equipment to be used with participants?	☐	☒	☐
Are there any financial or other interests to the investigator or University arising from this study?	☐	☒	☐
Will the research activity involve deliberately misleading participants in any way, or the partial or full concealment of the specific study aims?	☐	☒	☐
If YES to any question, please provide full details			
(this box should expand as you type)			

SECTION H: Anticipated Risks

OUTLINE ANY ANTICIPATED RISKS THAT MAY ADVERSELY AFFECT ANY OF THE PARTICIPANTS, THE RESEARCHERS AND/OR THE UNIVERSITY, AND THE STEPS THAT WILL BE TAKEN TO ADDRESS THEM. If you have completed a full risk assessment (for example as required by a laboratory, or external research collaborator) you may append that to this form.
FULL RISK ASSESSMENT COMPLETED AND APPENDED? Yes ☐ No ☒
Risks to participants For example: emotional distress, financial disclosure, physical harm, transfer of personal data, sensitive organisational information
This research will take place within a partnership school, identified as a research school. This means that the school often supports the Yr Athrofa in classroom/school based

research. There is a risk that practitioners may feel obliged to participate because of this existing relationship. This risk will be mitigated by explaining to any potential practitioner participants that they are not obliged to take part and that, by choosing not to take part, this would not affect the status of the school as a research school. Practitioners would be asked for voluntary informed consent, with information provided to practitioners in advance.

There is a risk that practitioners may feel their practice is being judged, particularly during phase 1, which involves exploring school provision. I will mitigate this risk by explaining that the research aims to explore current provision rather than judge existing practice. I will reinforce that their views as practitioners will be a valuable contribution to the research.

There is a risk that practitioners who are involved in the research may feel obliged to invest more time than they would like to in the project, adding to their workload. This risk will be mitigated by negotiating times for interviews that suit the practitioners and their classes; as I researcher I will aim to be as flexible as possible and will take into account practitioners' other commitments when arranging interviews. I will keep an ongoing reflective diary to note down day-to-day discussions during the phase 2 research to avoid practitioners feeling they are being constantly interviewed. As a teacher educator and university tutor to students on Professional Teaching Experience, I am familiar with the need to be flexible within the school setting and the need to be mindful of practitioner workload.

There will be potential risks in working with young children, particularly that they may feel obliged to take part in something they may not want to be part of. As a previous primary teacher, now working in Initial Teacher Education, I have experience of working with children of this age and have planned the project to ensure I spend time in the classroom prior to the main trial, getting to know the learners so that I will not appear unfamiliar to them. The tasks planned will be in line with normal classroom activity and within the settings familiar to the learners, with class teachers and teaching assistants present. Learner participants involved in data collection will be asked for voluntary informed assent to take part in the research and this will be conducted in an age appropriate manner. Parental/guardian consent will also be sought.

The well-being of the participants will be a guiding feature of data collection design and implementation and the participants will be assured that they can cease involvement in data collection at any point with no adverse reaction and no questions asked. In the assessment tasks and learner interviews, an 'opt out' card (or similar) will be used. The learners will be informed that if they do not want to continue they can point to the card and will be able to stop with no questions asked. The assessment tasks and interviews will take place in the school environment with which the pupil is familiar. The assessments will not take the form of written tests. Assessment tasks will involve the use of mathematical resources that the learners will be familiar with. Questions asked as part of the assessment will be designed to be adaptive i.e. pupils will be scaffolded (through the use of planned support prompts) to succeed in demonstrating understanding.

As there will be a time lapse between interviews with learners, voluntary informed assent will be sought each time from the learners involved, with the option of withdrawing at any time. The interviews will take place in a setting familiar to the learners, in either an open plan area or in a room with an open door. I will be particularly mindful of body language when working with the learners and, if at any time a learner appears to be indicating that he/she does not want to continue with data collection, then I would stop.

(this box should expand as you type)

If research activity may include sensitive, embarrassing or upsetting topics (e.g. sexual activity, drug use) or issues likely to disclose information requiring further action (e.g. criminal activity), give details of the procedures to deal with these issues, including any support/advice (e.g. helpline numbers) to be offered to participants. Note that where applicable, consent procedures should make it clear that if something potentially or actually illegal is discovered in the course of a project, it may need to be disclosed to the proper authorities
The research activity does not include any sensitive, embarrassing or upsetting topics. If a learner discloses sensitive information then I would follow the school and Yr Athrofa safeguarding policy. (this box should expand as you type)
Risks to investigator For example: personal safety, physical harm, emotional distress, risk of accusation of harm/impropriety, conflict of interest
There is a risk that, as a researcher, I would be accused of harm or impropriety. I will mitigate this risk by trying to ensure I do not impose on the time or good will of any participants and by ensuring that when I work with the learners, this is in an open, accessible and visible space within the school, with practitioners present if they (or the learners) wish. All research activity will take place within the school. Teaching and learning activity I will undertake within the school is in line with typical school practice and so this should minimise the risk of any accusation of harm. I will, however, ensure that the interests and well-being of the participants is paramount so that there should not be a conflict of interest between me (as a researcher) and the school. I have planned ongoing practitioner feedback on tasks into the research process (taking into account learners' reactions), thus trying to ensure that the tasks are building on learners' prior experiences and taking account of practitioner views. This should minimise the risk of any conflict of interests because I will be seeking to develop tasks alongside the practitioners and with their feedback. (this box should expand as you type)
University/institutional risks For example: adverse publicity, financial loss, data protection
There is a risk that any breakdown in relationship would adversely affect the University. As a research school, there is an already established research relationship and I would endeavour to promote and enrich this relationship through professional behaviour rather than put this relationship at risk. (this box should expand as you type)
Adverse outcomes List measures put in place to limit any adverse effects or outcomes of research activity where appropriate. Include any emergency protocols.
An adverse outcome of research could be that the teaching or learning has little impact on the learners' understanding of the multiplicative relationship. The tasks are planned to take place over a week (within the regular daily mathematics) and each task will be reviewed and evaluated on a daily basis (including evaluative comments from the learners and the practitioners). I will 'ring fence' time each day to evaluate and further develop any tasks for subsequent days to maximise the learning potential of any tasks produced. There is a risk that any of the planned research activity may be affected by unexpected circumstances (e.g. emergency school closure/fire drills/inclement weather/learner or practitioner absence). In any instance participant well-being will be given priority and I will seek to be flexible in by possibly rearranging scheduled times (with agreement of relevant

participants). In addition by working with a number of participants (around 6 focus learners and most Year 2 practitioners) I would hope that there would be sufficient data even if there is some absence.
(this box should expand as you type)

Disclosure and Barring Service			
If the research activity involves children or vulnerable adults, a Disclosure and Barring Service (DBS) certificate must be obtained before any contact with such participants.	YES	NO	N/A
Has a DBS certificate been obtained?	☒	☐	☐

SECTION I: Feedback, Consent and Confidentiality

Feedback
What feedback will be provided to participants, how will this be done and when?

Participants will be thanked for their involvement and practitioner participants and the school will be offered the research project when complete.

(this box should expand as you type)

Informed consent
Describe the arrangements to inform potential participants, before providing consent, of what is involved in participating. Describe the arrangements for participants to provide full consent before data collection begins. If gaining consent in this way is inappropriate, explain how consent will be obtained and recorded.

All participants will be asked for voluntary informed assent/consent as follows:

Practitioner participants: An information sheet outlining the aims of the research, giving detail about the nature of involvement, the right to withdraw and data collection/storage and asking for consent will be given.

Parents of learner participants: An information letter outlining the aims of the research, giving detail about the nature of involvement, the right to withdraw and data collection/storage and asking for consent will be given to all parents. This will be in line with school policy and in negotiation with the school. Any learners for whom consent is not given would not be included in any of the data collection.

Learner participants: A child friendly information sheet will be given to the focus learners. Voluntary informed assent will be sought and learners will be given an opt out card to use at any time. Assent will be sought at each interview to account for time lapse.

(this box should expand as you type)

Confidentiality / Anonymity
Set out how anonymity of participants and confidentiality will be ensured in any outputs. If anonymity is not being offered, explain why this is the case.

Learners and practitioners will not be anonymous because I will be working with them within the school environment and aim to get to know them so that I can use their names as they

would be normally used within the school setting. Confidentiality will be ensured through all outputs; learners and participants will be given unique identifiers and the school or any individual associated with it would not be named. (this box should expand as you type)

SECTION J: Data Protection and Storage

In completing this section refer to the University's Research Data Management Policy and the extensive resources on the University's Research Data Management web pages (http://uwtsd.ac.uk/library/research-data-management/).

	YES	NO
Does the research activity involve personal data (as defined by the Data Protection Act)? ***"personal data"*** *means data which relate to a living individual who can be identified—* *(a) from those data, or* *(b) from those data and other information which is in the possession of, or is likely to come into the possession of, the data controller, and includes any expression of opinion about the individual and any indication of the intentions of the data controller or any other person in respect of the individual.*	☒	☐
If YES, provide a description of the data and explain why this data needs to be collected:		
Assessments, tasks and interviews will be video recorded (if possible) or audio recorded. This will support analysis of data. This audio/visual data will be kept securely in a password protected digital folder (cloud storage) and will only be used for the purpose of this research. This audio/visual data will not be shared, although will be available to the school and practitioners with accompanying transcriptions to check agreement for accuracy of reporting. (this box should expand as you type)		
Does it involve sensitive personal data (as defined by the Data Protection Act)? ***"Sensitive personal data"*** *means personal data consisting of information as to –* *(a) the racial or ethnic origin of the data subject,* *(b) his political opinions,* *(c) his religious beliefs or other beliefs of a similar nature,* *(d) whether he is a member of a trade union (within the meaning of the Trade Union and Labour Relations (Consolidation) Act 1992),* *(e) his physical or mental health or condition,* *(f) his sexual life,* *(g) the commission or alleged commission by him of any offence, or* *(h) any proceedings for any offence committed or alleged to have been committed by him, the disposal of such proceedings or the sentence of any court in such proceedings.*	☐	☒
If YES, provide a description of the data and explain why this data needs to be collected:		
(this box should expand as you type)		

Will the research activity involve storing personal data on one of the following:	YES	NO
Manual files (i.e. in paper form)?	☐	☒
University computers?	☒	☐
Private company computers?	☐	☒
Home or other personal computers?	☐	☒
Laptop computers/ CDs/ Portable disk-drives/ memory sticks?	☒	☐
"Cloud" storage or websites?	☒	☐
Other – specify:	☐	☒

For all stored data, explain the measures in place to ensure data confidentiality, including details of password protection, encryption and anonymisation:
All data will be kept in password protected cloud storage on the University Office 365 system which will not be shared. Audio/visual data will be transcribed and would be shown to practitioners to check accuracy of reporting. All participants will be given a unique identifier to ensure confidentiality and this list will be kept securely in the password protected folder. (this box should expand as you type)

Will the research activity involve any of the following activities:	YES	NO
Electronic transfer of data in any form?	☒	☐
Sharing of data with others at the University?	☐	☒
Sharing of data with other organisations?	☐	☒
Export of data outside the European Union or importing of data from outside the UK?	☐	☒
Use of personal addresses, postcodes, faxes, emails or telephone numbers?	☐	☒
Publication of data that might allow identification of individuals?	☐	☒
Use of data management system?	☐	☒
Data archiving?	☐	☒

If YES to any question, please provide full details, explaining how this will be conducted in accordance with the Data Protection Act (and/or any international equivalent):
Audio-visual data will be transferred from the recording devices to a password protected cloud storage system and will then be deleted from the recording devices. (this box should expand as you type)

List all who will have access to the data generated by the research activity:
I will have main access to the data generated and this will be shown (but not shared), as appropriate, to practitioners to check for accuracy of reporting. It will also be shown (but not shared) with research supervisors.

(this box should expand as you type)
List who will have control of, and act as custodian(s) for, data generated by the research activity:
Rachel Wallis (this box should expand as you type)
Give details of data storage arrangements, including where data will be stored, how long for, and in what form. Will data be archived – if so how and if not why not.
All data will be stored in password protected cloud storage using the University Office 365 system. It will be kept for the duration of the project (until completion of PhD) and would normally be destroyed afterwards. A possible outcome is that audio/visual data may be considered potentially useful for professional learning purposes. If this occurs, further permission would be sought from participants involved for the use of the material for professional development purposes. The audio-visual data would be deleted on completion of the project if the above outcome does not occur or should further permission not be granted. (this box should expand as you type)

SECTION K: Declaration

The information which I have provided is correct and complete to the best of my knowledge. I have attempted to identify any risks and issues related to the research activity and acknowledge my obligations and the rights of the participants. In submitting this application I hereby confirm that I undertake to ensure that the above named research activity will meet the University's Research Ethics and Integrity Code of Practice		
Signature of applicant:	Rachel Wallis.	**Date:** **11/01/2019**

For students:

Director of Studies/Supervisor:	
Signature:	
Date:	

For staff:

Head of School/Assistant Dean:	
Signature:	
Date:	

Checklist: Please complete the checklist below to ensure that you have completed the form according to the guidelines and attached any required documentation:

☒	I have read the guidance notes supplied before completing the form.
☒	I have completed **ALL RELEVANT** sections of the form in full.
☒	I confirm that the research activity has received approval in principle
☐	I have attached a copy of final/interim approval from external organisation (where appropriate)
☐	I have attached a full risk assessment (and have NOT completed Section H of this form) (where appropriate)
☒	I understand that it is my responsibility to ensure that the above named research activity will meet the University's Research Ethics and Integrity Code of Practice.
☒	I understand that before commencing data collection all documents aimed at respondents (including information sheets, consent forms, questionnaires, interview schedules etc.) must be confirmed by the DoS/Supervisor, module tutor or Head of School.

RESEARCH STUDENTS AND STAFF ONLY
All communications relating to this application during its processing must be in writing and emailed to pgresearch@uwtsd.ac.uk , with the title 'Ethical Approval' followed by your name.
You will be informed of the outcome of your claim by email; therefore it is important that you check your University and personal email accounts regularly**.**

STUDENTS ON UNDERGRADUATE OR TAUGHT MASTERS PROGRAMMES should submit this form (and receive the outcome) via systems explained to you by the supervisor/module leader.

This form is available electronically from the Academic Office web pages:
http://www.uwtsd.ac.uk/academic-office/

APPENDIX G: EXAMPLE OF PRACTITIONER INFORMATION AND CONSENT

Prifysgol Cymru Y Drindod Dewi Sant University of Wales Trinity Saint David | Yr Athrofa Institute of Education Addysg i Gymru Education for Wales

Information about research activity phase 2 (practitioners)

Title: A design-based research project to develop and evaluate materials for teaching multiplicative reasoning through measures.

Name and contact details of researcher: Rachel Wallis, rachel.wallis@uwtsd.ac.uk

Overview: The aim of this research project is to develop and evaluate teaching materials to support young children's learning of the multiplicative relationship, in particular, multiplication and division. The main research questions are:

S1: What are teachers' and learners' prior experiences of teaching and learning multiplicative reasoning?
S2: What are learners' prior experiences of learning number and measures?
S3: How can tasks using measures be developed to introduce and consolidate multiplicative reasoning, taking into account learners' and teachers' prior experiences?
S4: What is the impact of learning multiplicative reasoning through measures?
S5: What are teachers' and learners' views on teaching/learning multiplicative reasoning through measures using the materials developed?

Phase 2 (Implementation: focus on S3, S4 and S5)
Phase 2 will involve the trial and evaluation of materials for teaching the multiplicative relationship through measures.

Research activity (phase 2)

- A (possible) pre-lesson to establish working practices and for learners to get to know me
- Pre-assessment with individual focus learners (video/audio recorded)
- A series of lessons involving the observation of focus learners (recorded audio and observation notes)
- Post lesson interviews (practitioners - audio recorded)
- Post lesson interview and assessment with individual focus learners (video recorded)
- Ongoing reflective notes

Data collected
Data will be stored by the researcher in a secure password protected folder on the university cloud storage system. It would be destroyed on completion of the PhD.
Once collected, the data would be collated and analysed so that it could inform further development of materials. Analysis of data (or aspects of it) may be referred to within the thesis dissertation which would be available for reading on request. No locations or individuals would be named within the thesis.

Consent

If you are willing to participate in the research activity (phase 2), please sign below.
You would be free to withdraw involvement and consent at any time, and no related data would then be used.

Name (please print)

____

Role (please print)

____

I consent to take part in phase 2 of the proposed research activity, in line with what is outlined above.
(signed)

____

Date:____________________________

APPENDIX H: EXAMPLE OF LETTER FOR PARENTAL CONSENT

Prifysgol Cymru Y Drindod Dewi Sant University of Wales Trinity Saint David | Yr Athrofa Institute of Education Addysg i Gymru Education for Wales

Dear Parent/Guardian,

My name is Rachel Wallis. I am a primary teacher working in teacher education at the University of Wales Trinity Saint David in Swansea. I have recently started research into children's understanding of number and measures and I will be undertaking some of this research at your child's school with your child's class. Through children's participation in the research, I hope to develop teaching materials that can be used in school to support mathematics teaching.

I would like to ask for your consent for your child to participate in my research project. This project will involve lessons looking at how number and measures can be linked. With your consent, your child may be asked to participate in some short individual tasks (which would involve verbal questions and the use of familiar mathematics equipment) and your child may also be observed during particular lessons in order to consider your child's response to the tasks in the lesson. After the lesson, your child may be asked to give his/her views on the lesson taught.

The activities, lessons and interviews would be audio recorded and all will take place in the school with school staff nearby. They will be designed to be typical lessons and activities. If at any time your child does not appear comfortable, then it would be assumed he/she does not want to take part and his/her participation in the study would cease. Also, your child would be given a card which he/she could use if he/she wants to indicate he/she would like to stop taking part. You would be able to access recordings or the research reports on request. The recordings will be destroyed following the completion of the project and during data collection they will only be seen by those involved in the project.

Please note that all research activity would adhere to educational research guidelines and strict ethical procedures would apply. Your child's participation would be voluntary and identities of the school and the learners involved in the project would not be revealed in any research reports.

If you would consent to your child taking part, please complete the form below. If you would like to discover more before giving consent, please contact me using the details below.

Thank you in advance for your time. If any questions do arise at any point, feel free to contact me at your convenience.

NAME & ADDRESS OF RESEARCHER
Rachel Wallis rachel.wallis@uwtsd.ac.uk
University of Wales Trinity Saint David
Yr Athrofa (Institute of Education)
Dylan Thomas Centre
Swansea
SA1 1RR

Mathematics Research project reply

Name of parent:____

Name of child:____

Please place a tick/cross in the box as appropriate.

☐

I confirm that I have read and understand the information sheet date for the above study.

☐

I understand that my child's participation is voluntary and that my child is free to withdraw at any time, without giving any reason.

☐

I agree for my child to take part in the above study.

Signed:____

Date:____

APPENDIX I: TEXT READ TO LEARNERS

My name is Mrs Wallis. I am teacher who is interested in maths learning. I would like to find out more about how to help children learn maths. I am making some lessons and activities that I would like to try out and I would like to find out what you think of these, and whether they help you learn.

I would like to find out about your maths by doing some maths activities with you and your class. Sometimes I may need to record or write notes about what we do so that I can learn from what you say and do. The work we do will always be in school with your teachers and other adults that you know nearby.

I would be very pleased if you would like to take part but if you do not want to take part in the work I am doing, you can say no and that would be fine – this would mean you may still be in your lessons but you would not be asked about the activities.

If you want to stop being a part of this work, you can tell me or another adult and you would not have to take part in the discussions about it anymore. You could also hold up this card (show red card) if you want to stop taking part, or this card (show amber card) if you are worried and want to talk about it. This card (green card) shows you are happy to take part.

APPENDIX J: INTERVIEW WITH PRACTITIONERS (CYCLE 1, PHASE 1)

Key:

I – interviewer (researcher)

T1 – teacher 1

T2 – teacher 2

T3 – teacher 3

T4 – teacher 4

(teachers numbered according to order in which they introduced themselves around table)

N.B. Italicised words are words that *seem* to have been emphasised by the speaker in that they seem to stand out as being intentionally emphasised. This is, of course, my interpretation of what was emphasised.

… used when a statement may not appear to have been completed or there is a gap in speech

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I: OK, so thank you for the interview. So, can we go round and if you just say your names so that I can recognise the voices when we're on?

T1: Says name

T2: Says name

T3: Says name

T4: Says name

I: Thank you, and again thank you for coming to the interview. So the first question, I've got a few questions that are really just about general learning and teaching of maths. So are there particular principles you follow in the Foundation Phase for teaching and learning of maths and, if so, what are these principles?

T2: I hope that everybody uses CVA concrete, visual, abstract but I hope that wouldn't just be Foundation Phase but right the way through for any concept taught.

T1: I agree.

T4: Yep

T1: We do. They go, start off with the practical activities. We have moved from it just being nursery using practical to expecting them to be recording in Year 2. It's dependent on the child.

T2: And I'm very mindful, I went on a course and it was, you do not just offer your Numicon and Dienes to your bottom group, which happen to sit at this table with me for particular activities. You offer it to anybody if anybody wants any practical so that they are not made to feel…and always giving everybody the opportunity to use it at the beginning so they've got the visual image.

I: So when you say that, in terms of what the children are used to, they're used to, sort of having access to the resources by choice in a way?

T2: Yes

I: Is that something you plan for? That they've got, they can have sort of free access to those resources?

T2: Yes. They know where they are, that's the maths table (sound of agreement from others).

I: OK. Are there particular ways you manage and plan, because when I came last week I noticed so you all do maths at the same time? And then you work with a focus group?

T4: So we, in Year 2 the children are, actually no in Year 1, some of the Year 1's as well, the children go into different classes for…

T2: Ability grouping

T4: Yes, based on ability. I've lost my track of what I'm going to say, it's because it's being recorded.

(laughter in background, and T3 continues a response)

T3: In reception we, you know, we teach an introduction and then you'll have a little focus group, maybe sometimes one child, sometimes two, sometimes more with maybe the other TAs working again on another focus activity but related to the same topic and then, you know, some of our enhanced will relate as well back to what we've done or done the previous week as well, so obviously it's not as sit down as maybe Year 2 but it, that's kind of, the way we work, isn't it.

T2: Year 2 is more class based. They will be taught and then the work is differentiated and I go around them all, but I focus on my less able unless I need to focus on other groups depending on what they're doing.

I: Thank you. So I've got some prompts here and one of them is 'are there any ways you believe young children learn maths effectively', so you've mentioned CVA, you've mentioned enhanced provision…

T4: I think putting it into a context as well. For example, if we're getting it into our daily routine, for example if we're going into assembly, and of course they line up in twos, so right, let's count, what do we need to count in, so we count in twos. And putting it into a real life context, so at milk time, some of us have milk shops so the children have to come and pay for the milk, just to get them used to using those things all the time, but recognising where they will use it in their lives as well.

T2: Making maths real. It is…(sounds of agreement)…in every aspect of our lives (further sounds of agreement).

I: Because some of the tasks I am hoping to develop will be about things like, how many small jugs will fit the large jug, and if you've got an intermediate jug, so it's trying to make, find a context that would be relevant I suppose for that isn't it…

T2: One of the Year 2 questions, and it's because of the LNF. It's not because of the LNF, but that is why we have the major push is…Today we were doing measurement, we're doing measurement this week, but the children need to learn if I'm measuring water, it means I need a measuring jug and I measure in litres and millilitres, if I'm measuring time I need a clock or a stopwatch and they get them so muddled up because the language is so so similar. Centimetre, millimetre, millilitre and it's so the drumming drumming drumming and that continual…Mrs G*, she's killer G* and so they know if you're weighing Killer G* always weighs to just get that K G because the language is so similar for them, it's very very difficult, but then they've got to have the practical to know that Mrs G* always weighs whether we're cooking and doing real things or measuring dinosaurs or plastic animals or what not.

I: Thank you. And then are there any things that if you're doing maths that you particularly focus on so praising and rewarding, in terms of their…when they're doing maths.

T2: Effort. How do you think you solve a problem? Have a go. That's come up today, how can we record it, have a go, it doesn't matter if you're wrong.

T1: Language. The vocabulary they use, if they use the right terminology they get praised.

I: Right, thank you. That sort of links on to my next question, which is, you mentioned effort and language and my next question was how you feel the learners respond or may respond to challenges and collaborative challenges in maths, in terms of working together. If they're stuck on something and how they respond to that.

T3: I think they like working, like whole class, especially with reception, they come in and you can set up like something's happened and they've all got to work together and they quite enjoy talking to one another, you know, especially if you're like mixing the abilities so they're not, you know, if they're struggling to think of ideas then you've got somebody who maybe is a bit stronger and can think of ways to solve a problem, you know regardless of, even if it's a maths or you know any topic really, but yeah they're quite, I think they enjoy it mixing the abilities and discussing it that way.

T2: It's odd that I do tend to focus, if it's a number problem, they're in their sets but if it's when we do time and measure and everything they're put into mixed ability, until I know that right who can go this far with the clock so they may become sectioned to push that…I hadn't really thought about it.

I: So that's something, Because, because I'm using, I'm planning to use measures as a context I'll have to think about whether they are in mixed ability groups or ability groups because it's not, the number comes out of the context if you see what I mean (sounds of agreement) and it's the collaborative challenge of them thinking together and to introduce the concept of multiplication and division so that's something I think I'll have to think about when I've thought about the tasks…

T2: If they know quarter past, half past, quarter to they can't keep on doing it so they've got to go on to all the past times and then…

I: Yes absolutely. Thank you. So onto my next question. In terms of multiplicative reasoning, so teaching multiplication and division, what would say are the key milestones or experiences for learners in coming to understand the ideas of multiplication and division in the Foundation Phase?

T2: Understanding. If you're doing the two times tables it means that five twos is you've got five groups of two, five mountains of two…And the practical, Numicon is the best thing (sounds of agreement) and coins (sounds of agreement), applying it to money, five two pences.

T1: And we give them items that would relate to that group, so if we're doing pairs, if they were counting in twos, we'd give them pairs of socks, if we were counting in fives we'd give them gloves to have the five fingers, that kind of…

I: So they've got that visual (sounds of agreement) going back to what you were saying about the concrete visual (sounds of agreement), they've got those sort of images of what two groups of five means, or two groups of two and so on (sounds of agreement). OK, thank you. Would you say there are key resources, I think you've mentioned some of these, and images that learners typically use in understanding multiplication and division in the Foundation Phase. You've mentioned Numicon, coins, pairs of socks, are there any other sort of images and resources that the children are used to using?

T4: It kind of depends on the topic really. Earlier on in the year when we were doing animals and we were counting in twos, did we do legs or something like that?

T3: Yeh (sounds of agreement), we do like (sounds of agreement)…

T4: I think it depends on…But they are all, from nursery, they're very familiar with Numicon (sounds of agreement)…

T3: You know, songs as well, there's lots of songs, like counting in twos, counting in tens, you know, like with reception, it's an easy way for them, without realising, they're singing along but counting in twos and tens as well, so things like that…

I: And that goes back to the key milestones then doesn't it (sound of agreement) because, you know, the counting in twos and tens is that understanding of that repeatedness of it (sound of agreement). And are there particular words and phrases, because you mentioned language T1, are there particular words and phrases that you'd particularly want the children to be using and that you'd reinforce as teachers…

T2: It depends on the age. I mean mine are doing the commutative law at the moment, so they know to reverse their times, you know if you know a fact then you've got another fact, but it's all our…

T1: Sharing, you always say for division, sharing…

I: And lots of you mentioned earlier…

T4: Sets of…

I: Sets of

T2: Sets of, groups of, piles of, I just, why I say it in so many different ways, I've never really…

I: I suppose that goes back to what you were saying earlier T4, if you're making it relevant to the children (sound of agreement) I suppose if you say it in lots of different ways it helps them to understand that…

T2: And memory, something to trigger the brain.

I: And so, I think I've mentioned this, what words and phrases, we've mentioned that, and symbols would you expect the learner to use for multiplication and division.

T1: We do it with our hands, yeah, cross your arms over (sounds of agreement).

I: Yes, thank you. Are there any other things related to multiplication and division, I think, that we haven't covered?

T4: No, I don't think so.

I: OK, so onto number, sorry measures really. What would you say are the key milestones and experiences for learners in coming to understand measures in the Foundation Phase? So T2, you were talking earlier about the use of the particular units and the language related to that…Are there any other aspects that you think are really key for them when they're learning measures?

T3: When they're younger it's more like the simple language of taller, shorter, longer, (sounds of agreement) bigger, heavier, like making sure they understand, like…

T1: full, empty…

T3: Yeah, making sure they understand full, empty, half full and making sure they know, you know, which relates to which and obviously as they get older developing and showing them, yeah, like centimetres and things in regards to length then, and yeah, just making sure they have an understanding of what those things mean and relate to the different, you know, when they're maybe doing capacity, they're not saying things like you know it's tall, they're saying it's full, so things like that…

T4: With measure as well, you know, we'd use things like Duplo, you know, to measure length initially and giving them the choice as well, so you know saying we need superhero capes, what do you want to use to measure, and if the cubes are smaller, well let's see what the difference is, and just getting them to use lots of non-standard units first of all…

I: I notice you've got, these happen to be [on table was a box of measuring jugs] (laughter from all)

T2: That was today, yes…

I: And you mentioned capacity, so I was going to ask about…I know you've got, you're lucky in the sense that you've got lots of outdoor provision and that's set up for the use of that…

T2: That was just free play, yesterday they did have somebody out there with them who, they were measuring sticks they'd found at the beach. So centimetres and they were doing…I didn't bring all those down (sound of laughter).

I: (laughs) I know, I know

T2: That was somebody else, yes…

I: Thank you. Are there any other points related to that question?

T2: I think they've just got to have an understanding. We start off with what do you think is a minute, what can we do in a minute, how many times can you write your name in a minute, can you stand up. When you think, because they've got no no real understanding of time, it's nearly my birthday, but it's not, it's in seven months time, you know it's so hard, so bringing it back to real life.

You know we tell the time constantly, you know, I will say you've got ten more minutes to do this and now we're moving on, so bringing it back to their real understanding…

I: To real context again, yes, and so in terms of resources and manipulatives, you've mentioned that you have different resources and things out for the children, and is that something they're quite used to, being able to access those resources, you mentioned measuring sticks for example, so they're used to being able to just go and get the resources and measure those sorts of things…

T2: Yeah…

T4: We do, particularly with the younger ones, well with me, they do feel like they've still got to go to a teacher to ask, you know I don't think they've developed that independence yet to think I need the Numicon to help me with this or, but I think we encourage it (sounds of agreement).

T2: They do get there, because if (sound of agreement), if they're learning a new table and they want to do in their independent time, they'll go and get that to learn and they'll get out a ruler for…

T4: It's just encouraging them (sound of agreement) to not have, to not need the constant reassurance from a teacher, but I think that comes with age as well (sound of agreement).

I: And you mentioned '* challenges' (sound of agreement) last week to me, T4, so that's something you do and that relates to enhanced provision tasks that they have to do across the week (sound of agreement) and that's the sort of thing, you may have measure tasks related to those?

T4: Yeah, and when the children come in in the morning, they come into an activity so they're in, I think, yeah (sound of agreement)…We do different things in different classes, but in mine they come in and they find their face and there's numeracy activities set out, but they're all things they've covered previously so…And they do them independently then and I think that because I've got a mix of Reception and Year 1 in my class, saying that they come to the teacher for reassurance, they are used to now, if I've got a couple of quite able Year 1's, so we'll say right on this table [name] is going to be the teacher, so if you've got any questions you can go to him, so we've got and just getting them used to that. And so the * challenges, that's the enhanced provision, so there's a challenge for each, for lots of different areas and it's worked quite well…

T3: It's lovely yeah, and you can relate it like to whatever you've done the week before, so if we've been doing capacity then that challenge might be relating to capacity, whereas obviously it's an independent task that they've then got to do, but using like what they've learned the week before, or something that you've done that week, or maybe it's something that we've just covered in the year and you just want to go over. But it's just a nice way for them to get a bit of independence. And they are, in the beginning, they will come up how do I do this, how do I do this, but like you said, when you've got some more able children, you can kind of give them some responsibility then, can you check, can you help so and so to do that activity and they'll help them then and like you said about the working collaboratively, I suppose that's an approach to it as well.

I: Thank you. And again related to what we've been saying…You've, I think we've covered resources. There's no other resources that you can think of, that?

T1: Counting stick.

I: Counting stick.

T2: Yeah, and the IR resources I was telling you last week, so interactive teaching resources really…

I: And words and phrases that you would reinforce when you're teaching measures, or particular words you would want the children to use, you've mentioned units, are there any other words and phrases you'd particularly…

T2: Taller, not bigger (sound of agreement) and a ruler is to measure length and height. They get stuck into one group because if you've permanently been measuring something on a table and they're going around, they forget it can also be upright…

T1: We're still hammering the teen and ty, ty, ty and we do it, they just get confused, seventy, eighty, ninety, twenty (sounds of agreement)

I: It is confusing isn't it, it doesn't make sense the English language.

T2: And time is crazy. The language of time is absolutely bonkers. You know, half an hour is thirty minutes, well why is it…

T1: Especially if they're English second language (sound of agreement), it's really confusing.

PLEASE NOTE THAT THE INTERVIEW DOES CONTINUE BUT WITH A FOCUS ON ORGANISATIONAL ASPECTS SUCH AS TIMINGS ETC.

APPENDIX K: INTERVIEW QUESTIONS FOR LEARNERS

Thank you for working with me over the last week. I am hoping to ask you some questions to help me see what you have thought about the activities we have been working on. You can give an honest answer, I won't mind if you didn't like any of the activities.

Put out the pictures of the different activities.

The activities I have been making are to try to help children like you with maths. Have the activities helped you with any maths?

If so, what maths might they have helped with?
What have you learned about measuring?
What have you learned about multiplication?

Did any of the activities make you think/make you think really hard?

What did this make you think about? Can you tell me more?

Did you find any of activities confusing?
Could you tell me more about why it was confusing?

Which activity do you think helped you learn maths most?
Why?

Which activity did you find the most fun?
Why?

I am looking to see how the activities I have made could be made better for other children like you. Can you think of ways they could be made better?

APPENDIX L: LEARNER RESPONSES TO CYCLE 1 TASKS (OVERVIEW)

Black text – from what Learner said in relation to question
Blue text – inference from interview

Task	Helped me learn the most maths	Made me think the most/was hard	Most enjoyable	Notes about improvement	Other notes
Straws and string	Learner 4 – learnt how to measure	Learner 2 – alludes to double/half relationship Learner 3 – because not using standard measure Learner 5 (suggested in what is said) Learner 6 Learner 7 Learner 8 (though comment actually about material)		Learner 3 notes different sized straws Learner 8 – wool stretchy	Most learners relate comments to pre-assessment task rather than task intended Learner 5 'we didn't have enough straws' and later 'the teacher should give them the amount of straws that they think that they need'
Jugs and little cups for rabbits	Learner 1 Learner 2 Learner 7		Learner 2 Learner 8	Language implication from Learner 2 comments	Learner 6 comments that liquids were easy to work with Learner 8 – liked pouring water
How many pancakes? Flour and cupfuls.			Learner 1 Learner 3 Learner 7		Learner 7 makes a comment to suggest that seeing all the cups helped
Cuisenaire – how many lengths?	Learner 1 Learner 2 Learner 3 Learner 5				

Task	Helped me learn the most maths	Made me think the most/was hard	Most enjoyable	Notes about improvement	Other notes
	'It helped me to um count in centimetres' *'How long (20cm is) in two centimetres, how long in five centimetres and how long in four centimetres'* Learner 6 Learner 7 Learner 8				
Medicine for dog – how many spoonfuls?			Learner 4 Learner 7	Learner 2 notes use of line Learner 3 notes size of bottle Learner 5 'I think we need to use the bottle first and then pour it into the spoon'	Learner 6 comments that liquids were easy to work with
Weight – sugar cubes and weighing sugar	Learner 1 Learner 5 Learner 6	Learner 2 'it was helping me think as well because, because I didn't know you had to put, you could put the cubes into there first' Learner 7	Learner 5		Learner 5 suggests that equality is easier to see

Task	Helped me learn the most maths	Made me think the most/was hard	Most enjoyable	Notes about improvement	Other notes
		'Well it was kind of hard...because we had to measure things and see and I tried to guess which one was heaviest' Learner 6 I had to think hard of the one that we did yesterday. Learner 4 'we had to measure in fours'			

APPENDIX M: INITIAL NOTES FROM AUDIO

First listen through reflections
Tasks 1a, 1b and 1c
29/04/2020 11:35 **Up to 11.51 Capacity and units** is the Task 1a, which involved establishing which sized cup would need most scoops to make an amount of liquid.
Of note here is that one learner (Learner 2) mentions using both the big cup and the little cup together at the start.
This was really a way of establishing learners' understanding of size of unit, but also of note is the automatic link some learners make with capacity and millilitres. One learner (Learner 7) in particular says 'miliilitres' quite frequently.
29/04/2020 11:50 **Up to 25.28 Length and units** is the Task 1b, which involved establishing the length of a piece of wool with two different sized straws. The size of the straws were such that the small straw was half the size of the large straw, thus allowing the consideration of a multiplicative relationship of halves and doubles. It is noteworth that one learner (Learner 5) notes the relationship at the start and another learner (Learner 2) reinforces this quite strongly at the end. Learner 7 seems to display instances of thinking aloud, when her thinking seems to change.
29/04/2020 12:46 **25.28 - End Capacity Cups and little cups for rabbits** is Task 1c which involved finding out how many little cups filled the bottle of water (medicine for rabbit) using an intermediate cup. Comments about measuring jugs were suggested, although one learner commented that it was being made harder. A perplexing comment was made when one learner (Learner 3) commented on a pattern 'up, down, up, down' - this was a muttering which was responded to by Learner 7. I am not sure what was being commented on (possibly the pattern in pouring out) but it caused me to wonder about learners' mutterings - there is a link to literature here. This audio also caused me to consdier how (or whether) I should code issues related to task efficacy because there were points at which learners and me were spilling water and some accuracy was lost. There was definite evidence in the audio of multiplicative thinking from learners which showed that, once we had established 7 little cups were worth the same as a big cup, they were making guesses that were multiples of 7 (e.g. 14, 21, 28, 42). Another muttering picked up on audio was 'I have two numbers'. I led a bit too much at the end of the audio because it was approaching play time!

Task 1d
29/04/2020 13:34 Task 1d builds on what was done with Task 1c but with a different container. Task efficacy comes to mind here as the use of words cups could be confusing - we keep having to differentiate between big and little. I wonder whether I should have restricted the use of the small cup completely because learners were using this when they didn't need to. Learners were clearly not (from the audio evidence) working in multiples of 10. Of particular note is audio towards the latter part of the session where two learners (Learner 5 definitely) were counting in 1s. Coding for working in multiples is something to consider, as well as working in 1s.
Another point to note here is that the context of rabbits is quickly dropped (possibly influenced by me, because I don't reinforce it) but that actually the narrative context does seem irrelevant - the learners don't seem to pick this narrative context up a lot in what they say either.
Tasks 2a and 2b
29/04/2020 14:21 00.00-20.28 Capacity cups The learners needed reinforcement on what 1 cup represented (10 little cups) but it is noteworthy that the multiplicative relationship we had looked at with halves and doubles reflected in first predictions rather than multiples of 10 (despite fact that we were using same cups as previous day). This might be because of the order in which I

approached this particular task (starting with how many big cups), or could be because the half/double relationship is one they are more familiar with. Another reflection here is that the use of straws and the half/double relationship is one which is easier to visually accept (i.e. that two little straws = 1 big straw) than a capacity relationship which they may explore physically but is harder to 'see' - even with cups laid out as I had done on this day, the equality is not as obvious. Task efficacy is again a consideration as bits were spilled and, although I had marked cups for use on this day, this still created more of a challenge to identify pre-planned relationships. The arrow diagrams and asking learners to represent what they did in a calculation showed that some learners (Learner 5) needed more support in understanding this relationship as she is heard to say '3 times 30'. Learners could be heard to be using multiples of 10 (although sometimes with prompting from me).
29/04/2020 15:06 20.28- 45.00 How many pancakes? This task had a relationship of 1 cup of flour making 6 pancakes. The audio in which learners discuss how they could find out how many pancakes could be made from a bag of flour was fascinating to listen to, but also quite frustrating because I clearly could have given them more time. One pair of learners (Learners 5&6) discuss this, with one seemingly contemplating and with one learner making a prediction of a multiple of 6 (18). Another pair of learners (Learners 3&4) discuss splitting 'cutting' the bag in half, or a quarter and that being 6 pancakes and then saying 'we can just count in sixes then'. This seems to suggest that the learner is estimating how many cups worth are there and the partner responds that it is not possible to cut. When I bring the group together, one suggestion (I think Learner 7) says '14'. I ask the learners whether using the cups could help, and there seems to be collective assent that it would. Task efficacy comes into play when we discuss spilling things. Some conversations can be overheard as the learners discuss the feel of the flour, and one can be heard to say 'I love how satisfying it is'. Learners can be heard to be discussing how many cupfuls they have and the need to measure to the line and the need for it be flat at the top of the cup. As they are working one learner (Learner 3 I believe) says 'Once we have done this, we can count in sixes, anyone know their six times tables'? with one learner replying 'No'. As learners begin to complete the task, they start to predict how many cupfuls '4 or 4 and a half'. I encourage the learners to consider whether they have similar amounts in each cup (a benefit of having enough cups) and learners can also be heard to estimate 'I don't think we have enough for another cup' and 'those 2 have the same amount but this doesn't have the same amount'. Learner 7 can be heard to say 'This is fun'. Learners then start to compare with each other 'How many cups have you guys got' and 'That's over the line'. Again, the feel of flour is commented on 'Flour is so fun to play with' (Learner 7) and (Learner 3 or 4) 'It makes a giant cloud'. As we draw together how many cups, establishing each group has about 4, and then say 'Mmm, so how many pancakes can we make', Learner 5 can be heard saying '6, 12' and then whispering '18' Learner 3 says '24, I think 24 pancakes'. When I do not verify this and ask others, Learner 5 says 'Maybe about 30'. This learner had been observed (and can be heard) counting in sixes using fingers. This learner was in teh group of learners who had been discussing previously that they had '4 and a half cups'. Therefore the suggestion of 'about 30' could be because the learner believed the bag had more than 4 cupfuls, but nevertheless her suggestion is the next multiple of six. It is unfortunate that I did not explore this idea with the learner. After establishing that 6 pancakes taken 4 times could be written as 6 x 4 and the creation of an arrow diagram, I ask the learners to consider how many pancakes if they had 5 cupfuls of flour. The learners can be heard to say '30' to each other and agree with each other. Learner 5 says '24 and 6 more so that is 30'. This is brought together by counting on from 24 to 30. On reflection, asking them how many with 10 cupfuls or another number would have reinforced the multiplcative thinking further. 39.00 onwards is clearing up!
Task 3a
30/04/2020 13:59 Length with Cuisenaire Unfortunately the very start of this session did not record so it starts with a learner saying 'I was saying that if you said millilitres, that would be funny because that is used with liquids'. This is noteworthy because the learners do seem to automatically (and

routinely) associate a type of measure with a unit. This links back to the practitioner interview where a teacher discusses the importance of this in the year group and the need to 'drum' in the types of measure. This session also has audio of me talking to students who were present in the school observing. On presenting learners with a 1cm rod and a 2cm rod, the learners are asked to predict what length the 5cm rod will be. Various guesses are heard '3cm, 4cm and 5cm'. A learner is then asked to establish that the rod is 5cm using 5 x 1cm rods. On presenting the learners with the 10cm rod, a learner (Learner 2) says 'I know what that ones going to be' and is asked to explain why. The response from Learner 2 is that 'it looks like 10 blocks' and other learners can be heard to agree. This is where I am glad that I had restricted the amount of 1cm cubes that were visible to the learners because I say 'Can we find out another way, because look, I've run of cubes now, I don't think I have got enough 1cm cubes to check'. Learner 2 says 'You can use them, these' and I respond by 'saying Oh, these, the 2cm rods, so how could we use those'? and Learner 2 says 'You can count in twos' (another learner is audible joining in with the word two). The learners are asked how many of the 2cm blocks are worth the same, and Learner 7 is heard to say 'I think it is seven'. This is noteworthy because throughout the audio so far, Learner 7 is a learner who has given incorrect guesses and seems one of the least secure in multiplicative thinking. There is some discussion around 2cm taken 5 times (whilst sticking up the rods as a display), and a learner (Learner 5) is heard say 'You could have used' and then again 'Miss, you could have used 8 of the little blocks and one of the 2cm blocks'. This shows the learner was able to partition 10cm in another way (albeit in an additive way). This comment is used as a way of reinforcing the next task involving using the same colour block each time to think about 'multiplication' (a learner is heard to say 'What's multiplication') and a learner (Learner 5, I think) seems to mutter '2cm, 10 times' (although the example discussed had been 2cm, 5 times) and is then reinforced collectively. Whilst laying out the blocks, Learner 2 says 'Miss, Miss, I know how we could measure this again now' (10cm rod) 'We could use two of these (5cm rod)'. The learners are challenged to make a 20cm line using orange rods, which they quickly do and are when asked to say how they did it. Learner 7 can be heard to say '10 and 10 is 20' and other learners can be heard (as I am reinforcing 10cm two times) 'two times 10' and 'ten times 2'. Learners quickly make another line 20cm long using another colour and Learner 2 is heard to say beforehand 'I think it'll be easy'. It is noted as part of the discussion that one learner (Learner 5) has used 2cm rods whilst the other learners have used 5cm rods, but learners are noted to indicate consensus on the number of 5cm rods and when asked how many Learner 5 will have, Learner 6 quickly says 'ten, ten twos'. Creating another line with the other colour rod was managed quickly, with Learner 2 saying 'It's easy'. The lines created are described by Learner 2 as 'a burger with a crust and the bun for the burger' and another learner (Learner 7) is heard to say 'I've made a burger'. Learners are then asked to write multiplication calculations (with the example of 10cm taken two times, 10cm x 2 = 20cm) provided. I reflect on this activity being quite teacher directed and functional - much more in line perhaps with the learners' conventional experiences of multiplication (although I had been told by class teacher that they had not used Cuisenaire before). The implication of having student observers (which had been unexpected) should be noted - as I felt a pressure to be seen to model tasks successfully but also I can reflect on feeling more confident in the use of this resource - it is much less messy and more controllable than liquid or flour! The learners can be heard speaking out in whispered voices the calculations as they write them (e.g. two times 10 is 20). Learners can be heard to be working like this for a few minutes before they are challenged to, in pairs, make a line 40cm long using different coloured rods each time. Learner 6 can be heard to be counting in fives and another learner counting in twos. Learners 6 and 5 can be heard to be counting in twos up to 40. There is establishment that there are 8 5cm rods '5cm taken 8 times'. On asking the learners how many 2cm rods made up 40cm there is audible counting (in ones) as they count how many they have. It is noted then (by me) that Learner 7 has made a line 30cm long (How many 10cm rods). Meanwhile Learner 2 seems to be responding my my previous question saying '20, It's 20'. Learner 7 is asked to comment on how many of each rod and

Learner 7 then acknowledges how many of each colour rod there are. Up to 22.00, after that it is clearing up. I can be noted as saying 'After we clear up, we are going to do 'Liquids'' (to which some of learners can be heard to join in that word'. This task was meant as a warm up, to reinforce multiplication through length and to reinforce number relationships that would be used within next task - longer than anticipated

Task 3b

1/05/2020 09:06 Capacity - medicine for the dog, how many spoonfuls (using millitres)

Task starts with a reinforcement that we are going to look at liquids at which point Learner 7 says 'millilitres'. This is another example of the association of a unit with a type of measure. I set up a narrative explaining that my dog has to take medicine (a spoonful each day). Some discussion takes place where learners try to establish whether what they are seeing is actually medicine. These discussions are noteworthy because I am reflecting on the value of the context - it could be argued to give some sort of reason for trying to find something out but the learners also seem to quite happily accept that they are not really seeing and using medicine. When taken back to the question of millilitres and what they know about millilitres, Learner 5 says 'It is less than a centimetre' which I do not explore. Learner 2 can be heard to say 'that's a full cup that is' - it is not clear whether this learner is responding to my question. As I establish that a spoonful is worth 10ml, Learner 2 asks 'Are we going to use other bottles'? This implies perhaps a growing familiarity with the sorts of tasks they are going to be undertaking. As I set up the idea that I want to know how many spoonfuls I have in the bottle, I do lead the learners because I say 'we've been doing liquids for a few days now and it could awkward to count spoons, could we do something else' and two learners are heard to say 'get a cup'. It is a pity I led this too much - I was conscious of time, having taken longer in first task than anticipated. I introduce a bottle, saying 'We're not going to use a cup today, we're going to use this bottle'. Learner 2 then says 'Miss, I know why there is a black line there' and I say 'Why is there a black line' to which Learner 2 responds 'So you know how full it's gonna be'. Learner 2 had not been present on Day 2 when I had decided to use visual markers to support task efficacy but clearly seemed to understood the value and purpose of the use of the marker. I reinforce this response 'That's right, because we were finding we were making different measures' and in the background a learner can be heard to say (I think) 'We dropped some medicine' - perhaps suggesting the idea of spilling liquids and the effect this may have had been recognised although this is difficult to tell. At this point I model using a funnel and little bottle (again too teacher led) to find out how many spoonfuls are in the bottle. Learner 2 says 'Miss, is it actual medicine' and another learner says 'No'. Two other learners (Learner 7 and Learner 3) can be heard to have a conversation (initiated by Learner 7) about the colour of the liquid and the heart. I model that there are 5 10ml spoonfuls in the little bottle. Learner 2 can be heard to say 'Miss it's over the line' and I respond 'It's just a little bit over the line, I must have not measured very accurately, but I measured this morning and I know that that (must have been pointing to the little bottle) is worth 5 of these (spoon)'. On reflection this shows that I was very teacher led here because I had planned for the learners to work with these facts, but also by introducing the units (which had been a deliberate choice in terms of progressing the tasks) I was bringing in another need for accuracy both in the actual measure but also in terms of ensuring I was reinforcing understanding of what 10ml or 50ml 'looked like'. I can be heard to say 'If I know that there are 10 millilitres in the spoon, how many millilitres will be in the bottle?'. Learner 2 can be heard to say '50' as I am starting this question and then '50' more loudly at the end of the question. I say 'Learner 2 thinks 50, do you agree' to which Learner 7 says 'half'. 'Half of what, Learner 7?' The bottle - Learner 2 can be heard to say 'No'. I say 'You think that that's half of the bottle (big bottle)'? Learner 2 says '30' and I say 'If I know that there are 5 of these and each one of these is 10millitres' at which point Learner 3 says '50' In the meantime another conversation can be heard 'Wait that's not a half, that's a quarter'. I believe that the learners were, at this point, estimating what fraction of the whole bottle the little bottle might have been but I can only hypothesise because I did not explore Learner 7's suggestion (and had not heard the conversation about half and quarter). This incident, however, might suggest some multiplicative thinking from Learner 7. I can then be heard to reinforce this relationship using the arrow diagram, but as I complete the arrow diagram Learner 8 (typically quiet) says '50' when I am reinforcing 10ml taken 5 times. Learner 2 had clearly been confident about this relationship so this episode when I am trying to reinforce the relationship probably seemed unnecessary for him. When I say 'Everybody happy with that' Learner 3 can be heard to yes 'Yes' and there is a collective sound of agreement. As a model the counting of how many 50ml bottles there are in the larger bottles there is a conversation about whether the liquid is squash or food colouring and also a reinforcement for Learner 2 who can heard to say '100' as the second bottle is filled 'Well done, Learner 2 is counting in 50's'. On the third bottle Learner 2 says '150, it's 150' meanwhile I say 'How many bottles'? And then 'This is 3 bottles' (Learner 2 again says 150) and I say, which is 150'. On the fourth bottle Learner 2 says 'Miss, it's a little bit full' and I say 'Yes it is, but approximately, it was meant to be four bottles'. Learner 2 and another Learner say '200' and I say '200ml, you're absolutely right, yuou were very good at counting in 50ml, how many bottles'. Learner 5 can be heard to say as I say this '250, and a half'. It is possible that Learner 5 was referring to a possible total and was counting the extra bit. It is noteworthy that the earlier conversation about half and quarter might have correctly reflected the prediction about the relationship of little bottle to big bottle. At this point, I ask the learners how many spoonfuls in the big bottle. How many day's worth of medicine? ' A learner can be heard to say '200' and then I appear to scaffold this question heavily (without actually giving time for the learners to discuss and think) using the arrow diagram and 4 bottles worth 5 spoonfuls. Nevertheless, the learners assert that there are '20 spoonfuls' in the bottle. I then introduce their task which is to find out how many spoonfuls are in another bottle. Ironically, as I hear the spoons, I can be heard to say 'Are you going to use the spoons?' and then reinforce 'it would take too long, we can use the bottles, how many spoonfuls in the bottle?' and learners can be heard to respond '5'. I also reinforce the fact that they only have one small bottle each for this task. This highlights another task efficacy issue - the use of the bottle as a repeated measure of which there is only one. Arguably it is far easier if they can fill each bottle (which they had done with the cups of flour the previous day) and 'see' how many there are, rather than keeping count whilst pouring excess liquid into a bowl. There is then a lot of chatter as resources and pairs are organised. Learners can be heard saying as I distribute the bottles 'There's our spoons'. I check whether the learners understand what they are finding out and some can be heard to say 'How many bottles' meanwhile I can be heard to say 'How many spoonfuls will fill that bottle, but you're not counting the spoonfuls, you're counting the little bottles'. I then say 'How many spoonfuls in the bottle' and Learner 3 quickly responds '5'. The learners are then heard be telling each other to stop as they take it in turns to pour and watch. Learner 5 can be heard to say 'That's definitely 50' I can then be heard to reinforce the relationship to a group of learners of spoonfuls to the bottle and then have a conversation with another group of students who have come to observe. The groups of learners can be heard telling each other where to measure to. Learner 2 can eb heard to say 'It's 5' I then can be heard trying to establish how many little bottles are in the big bottle and a learner says 'We have 250'. It is noteworthy that I appear to assume the learners are talking about millitres here because I say 'So you have 250ml'. During this task, Learner 2 refers several times to making jelly. This is possibly because of the colour of the liquid. I say 'I like that you have told me how many millitres, but you haven't told me how many spoonfuls'. Another group of learners can be heard talking (I believe to the students) about 50ml and 100ml. Whilst we are waiting for a pair to finish a group of learners (which includes Learner 5) to say 'We had 5 bottles'. There is then some discussion about spilling and accuracy (task efficacy). As I try to gather together thoughts, Learner 5 says 'we had 5 bottles but we spilled a bit'. Learner 7 (whom we had been waiting for with Learner 3) says '150' with a questioning tone and Learner 3 says '150ml'. I say 'You had 150ml, how many bottles?'. I don't know why I did not respond to 150ml because it would have been 250ml but I believe that it is because this pair of learners had been talking to a student teacher who may not have understood the task and appeared to have reinforced a relationship around 3 bottles being 150ml (hence their delay in completing the task'. I ask 'How many bottles did you count' Learner 3 appears unsure as a response of 'I think 5' and Laerner 7 '4'. I say 'You seem undure, did you lose count? to which they appear to agree' and I then establish with the other two groups that they had counted 5 bottles. This episode could highlight an intervention but also possibly the task efficacy as they only had one bottle to count with. It also highlights the extra difficulty with this task because there are several things being counted and discussed - spoonfuls, bottlefuls and miliitres. I then reinforce the relationship between bottle and spoonful and that each time they count a bottle, that is 5 spoonfuls and ask 'So there were 5 little bottles in the big bottle. how many spoonfuls in the bottle' and there is quiet for at least 10 seconds and then say 'how many spoonfuls in the bottle' to which Laerner 4 says '250' and I say 'You're telling me how many millilitres aren't you' and the learner says 'yes'. There is then an spisode where I scaffold by taking the collection of five little bottles and putting them out to model that there were 5 bottles equivalent to the large bottle and reinforced again that there were 5 spoonfuls in one bottle. It is noteable that I go back to the narrative here 'I wanted to know how many days I could feed my dog because she needs one spoonful a day'. It appears here that the narrative (I believed) gave some reason to me wanting to know this information. I asked again 'How many spoonfuls'. The fact that there were 5 in each bottle and 5 bottles did possibly make this task harder. Having, say, 6 little bottles might have differentiated what we were counting. I can then be heard to model with each bottle - 'If that's 5 spoons worth, that's 5 and then Learner (3 or 4) says '10' (I reinforce that would be 10), two learners then say '20' and I say '15' and then '20' to which a few learners can be heard to join in and Learner (3 or 4) says '25' before we reach it. As I am saying 'so we had 10 ml taken 25 times' and as I am saying this Learner 5 says '5 times 5'. I then ask the learners to record what they learned and what they liked. It is worth noting that the ending was quite rushed because other classes were moving for play. This task was a step up in the sense that I was asking learners to use measure unit relationships as well as a more basic multiplicative relationship that had been introduced previously, and was also affected by the availability of unit of count.

Tasks 4a and Task 4b

01/05/2020 12:43 00.00- 22.32 Length with Cuisenaire I start by thanking learners and reinforcing that we are looking at using measures to help us understand multiplication, mentioning that today we are going to use length and weighing. I remind the learners about a task previous day looking at length, using rods to help and whether they remember this and there is sound of assent. I ask 'How long did we say the orange rod was' and a male learner (Leraner 3 or 4 I believe) can be heard to say '10' quickly. I say '10 what?' and Learner 3 or 4 says '10cm'. I say 'How did we know it was 10cm' (at which point a learner says 10cm as I say it) and Learner 5 says 'we measured it with the little rods'. At the introduction of the 2cm rods, the learners eem to quickly agree that this is 2cm and, that 10cm is 2cm, 5 times (some learners join in with 5) similarly with the 5cm rod when a learner (I believe Learner 2) says '5 times 2'. At this point I say 'We're going to be using this rod today' and a learner (I believe Learner 6) says 4. Another learner must also have said this (Learner 3, though not audible on audio) because I then say 'You said that very quickly Learner 3' and other learners can be heard to say '4' although one says '3' and I say 'What do other people think?'. Learner 2 says '3 or 4' and Learner 7 says 'I agree with Learner 2, 3 or 4'. On asking how we could check, Learner 5 suggests we can see how many little blocks it is worth. I then ask 'Could we find out in another way, could we use other rods' but don't seem to give them a chance to respond as I say 'it's also the same as 2 of these isn't it'. I then set the task of making 20cm like yesterday but also using the 4cm rods. At this point I ask the learners to predict whether they will need more or less of the 4cm rods than the 2cm rods and to discuss this. Learner 6 can be heard to say 'less, because it is longer' and Learner

5 can be heard to say 'in the 3 times table it goes up and is one less' then appears to correct herself saying' it is one more, 21'. It seems to be that Learner 5 believes the rod to be a 3cm rod because when I ask whether there will more or less of the (4cm) rods, Learner 5 says ' It will be an odd number, 21 cm' She appears to have misunderstood my question. Meanwhile Learner 7 appears to change suggestions, being heard to alternate between more and less. I say 'Learner 6 disagrees, tell us what you think Learner 6' and Learner 6 says 'the pink ones are smaller ones, and they only have two and that one has four'. I can be heard to reinforce this saying 'so you are saying because they are smaller we need more', what do you think Learner 1? Learner 1 responds saying 'less' and when asked to explain thinking says 'Because when you make 10, it's 4 and 8 and then it's not making 10'. Learner 1 appears to have recognised that you can't 'fit' complete fours into a 10cm rod. I can then be heard to say 'So what you're saying Learner 1 (using the rods) 'so if we make 10cm with 4cm, we have one, two and a bit of another'. In the audio, Learner 3 (or 4) can be heard to say '3' and then someone can be heard to say 'two and a half'. I then say 'so we've got one, two, and a half of it' at which point a learner (I think the same laerner' says again 'two and a half'. At this point I suggest that they make 20cm using the different rods, including the 4cm rod. The learners can then be heard to lay out the rods with Learner 5 saying 'easy peasy'. As I reinforce that most learners have laid out 2 10cm rods to help them, Learner 2 says 'Its five fours'. There is then some audio chatter with other learners (from a different class) in background whilst some learners can be heard counting in steps and words like 'times' can be heard. A learner (I believe Learner 6) can be heard saying 2cm times 10, and again 5cm times 4 equals 20cm 'as the calculations are written out. This appears to be mirroring what had been modelled earlier. I then talk to Learner 4 and point out that the 2cm, 4cm and 10cm have been used and ask how many 5cm? I repeat the question, and say 'fantastic' a few seconds later though the learner can not be heard on audio to give a response. I can then be heard to challenge learners to use 40cm again and think about the different combinations. The learners are asked to predict how many they might need - I could have put more emphasis on this but I didn't. Learner 5 discusses a sandwich 'I'm making a sandwich' - different to a burger from day before and at that point I ask whether the rods have been used before and Learner 5 says 'Yes we used them yesterday with you'. I prompt Learner 7 to use the 4cm rods and Learner 2 says 'this will take hours'. I say 'It won't take hours' and Learner 2 says 'It won't take hours, Learner 2, it'll take a couple of minutes'. There is some chatter about running out of rods and not having enough room and I (conscious of time) say 'don't worry about pink, the 2cm, I want to know particularly how many 4cm rods are needed to make 20cm'. There is come counting in steps of two audible and a learner says 'I can't seem to do this'. Learner 7 says she is going to do 2cm (having done 4cm) and is asked 'can you predict how many 2cm rods' and, as I proceed to say for each 2cm Learner 5 interjects to say '20, you need 20 twos' and when asked 'Why do you need 20 twos', Learner 5 says 'because it is half of the number'. Annoyingly I react too quickly by saying 'Ah, because it is half of the number so you need double the amount' instead of exploring the idea further - this Learner could have been saying you needed 20 2's because she related it to 20 divided by 2 and made this conection or she might have been building on the point I had begun with Learner 7 that a 2cm rod was half of a 4cm rod. Either way, I did not establish her idea around half and what number she she was referring to, although the learner is showing multiplicative thinking. I appear conscious of time as I discuss stopping and finding out how many 4ck rods there are (leading into next task). Learner 7 (on my saying I am going to stop them says 'Are we doing liquids?' and I say 'I wanted you to find out how many 4cm rods make up 40cm and I think you have found that out haven't you?'. Learner 2 appears to say 'No' and then says (seems to himself) '32' as though the learner is counting in fours. At this point I can be heard reinforcing the relationship (whilst writing it up) that 40cm = 4cm x 10 and 20cm = 4cm x 5. I ask 'Do you notice anything about the numbers there?' It is quiet for quite a few seconds and I say 'What is the same and what is different about these two number sentences'? It is quiet so I say 'What is the same?' and Learner 6 says 'They both have a 4', I reinforce 'They both have 4cm don't they'. Learner 6 says 'and they both have a times' and I say 'they both have a multiplication

sign, the times sign, don't they' yes'. Learner 6 murmurs agreement and I say 'What's different' and then (I believe) Learner 5 says 'one has 20 and the other has 40' and then Learner 6 'one has a 5 and one has a 10'. I remember this moment because I had been hoping for them to recognise the half and double relationship but also hadn't wanted to push it too much and was conscious of moving on to the main task. I then move to a number line and say 'we are going to count in 20's on this number line, are you good at counting in 20's?' We count (with them joining in up to 140) with me marking jumps of 20 on the number line. I say 'How many 4's did we say are in 20, how many 4's?' and Learner 6 says '10'. I say again 'How many 4's are in 20' Here is 20cm, how many 4cm did we say are in 20cm? Learner 6 says '5' and I say '5, so there are five 4's in 20, that's 4 taken 5 times'. I can see the next question I ask is really confusing because I ask 'How many 5's did we say are in 40?. Learner 5 says '8' and another learner says '10'. I'm so cross listening to this back because I had changed the unit (mistakenly) and I say '10', thinking that I had referred to 4's. Interestingly at this point learners can be heard counting - it is possible they were checking - this moment was confusing because I had changed the unit being discussed. I bring this to an end by writing on the number line 4 x 5 under the 20 jump, to try to reinforce that for each 20 we had four five times. I had added in these Cuisenaire episodes to support length understanding whilst reinforcing relationships that would relate to what would be used in the next tasks; i recognise that they promote multiplicative thinking but not necessarily problem solving in the way the Davydov tasks are set up, but because of the quantities I was setting up and limitations in materials I was using them to support the understanding of particular relationships. They did this (and there is evidence of this). It is noteworthy that as I introduce the next activity, one learner (Learner & I believe) says 'liquids' and when I say 'we're not doing liquids today', I hear a 'awh' (although not too pronounced!). I mention 'weighing' and hear a 'mmm!'.
Cleaning up unti 26.09
05/05/2020 12:03 We start at 26.09 with Learner 2 saying 'Is it liquids!?' and me saying 'It's not liquids, it's solids' and another learner (Learner 6?) saying 'What are solids'. I respond by saying 'We'll see in a moment, liquids are things that move around but solids don't move around as much, solids are harder'. Another learner (Learner 3 or 4) discusses a weird solid that when you hit it can become soft - mentioning 'ubleck?' and 'corn starch'. I then try to re-focus the learners by bringing out the pan balance, saying 'we're going to use these today, have you used these before?' to which there is a 'no' response from one learner (7?). Some then seem to say 'yes, no, yeh, yeh' as though they are remembering. I then say 'these are called pan-balances'. As I am starting to say 'When I use a pan balance I have to make sure' another learner (6?) says 'you see which thing is heaviest', so I then say' we can use these to see which things are heavy or light, but we can also use these to see what weighs the same, because if they weigh the same, they will balance'. I then model putting some Cuisenaire blocks into the pan and a learner (6?) says 'that's definitely the heaviest'. I say 'Yes, there's nothing in there (referring to the other pan), but if I wanted it to balance...' and Learner (3 or 4) says 'You'd have to put two tens in there'. I repeat 'I'd have to put two tens in there, they have to be the same' and then 'and then we have to wait'. At this point Learner 6 says 'They're both balancing'. I say 'when they weigh the same' and start to whisper 'the arrow should point down to the the green triangle , to that point there, and we know then that they are both equal, they are both worth the same'. It is worth noting that I said (when modelling the Cuisenaire) that the things in the pans should be the same (when I meant they should weight the same). I seem to hear Learner 2 saying 'I knew it was going to be same' but am not sure. I then introduce the idea that they were really good at telling me that we use centimetres to measure length and we use millilitres to help us to measure liquids (Learner 7 can be heard saying 'millilitres' in a seemingly drawn out way) and ask 'What do we use to help us measure weight, mass of something, we sometimes call it mass?'. On reflection, it is annoying that I didn't use the word 'unit' here. A learner (not sure which) can be heard to say 'kilograms' and I say 'kilograms, and grams, we use grams and kilograms to measure weight and today we're going to be talking about grams and kiligrams, and grams in particular'.

Learner 7 (I believe) says 'kila', and then again 'kila'. On reflection, it could be that Learner 7 is repeating the prefix here (as Learner 7 can be heard to say 'milli' in a drawn out way earlier. On the other hand, it could be that Learner 7 is emphasising the prefix because the word gram is possibly less familiar. I say 'We do use kilograms, kilograms are for big heavy things, and we use grams for things that are a bit lighter'. On reflection it would have been useful to have a 'benchmark' weight of 1kg available for the learners to feel at this point. I say 'We're going to be using grams today'. In the background I seem to hear two learners talking 'and kilograms ‘says one and what the other says can't be heard. I then introduce the sugar cubes. One learner says 'Ow, that's sugar' and I then ask them to pass around. One learner says 'Can I have a marshmallow'. Learner 7 says 'This smells so nice'. I ask them to feel the sugar cube and ask them whether it is very heavy or very light and they asll seem to agree that it is light. I start to say 'It is very light, if I told you' and a learner (7?) says 'Is it a real sugar cube?'. I seem to ignore this and say 'If I told you a suga cube weights about 4g, 4g, that's quite light isn't it, 4g?' (lowering voice as I finish sentence). I then say 'We're going to weigh out sugar as well' (as I must introduce actual sugar, there are gasps'. I say 'We're going to try and find' and a Learner (3 or 4) says 'which one is heaviset'. I say ' we're going to try and find the weight of the sugar, using the weight of the sugar cube to help us to know how heavy the bag of sugar is'. I then say 'But it's not easy to count in fours, and 4g isn't very heavy. I then ask 'Could we make it easier for us to weigh'. I say 'Could we do anything to help us, talk to the person next to you, can you think of a way we can make it easier, because we're going to need lots of these. We are going to use them, but could we make it easier'. There is audible sounds of thinking such as 'mmm' and 'we could use another one' and then a learner (Learner 4) says 'we could put 5 in there and another 5 in there' I ask for ideas and Learner 7 says 'we could put them in there' and I say 'Yes we are going to put them in there, how can we make it easier to count'. Learner 4 is asked about the idea given. I say 'What would that weigh, if we put 5 lots of 4g in there and 5 lots of 4g in there?'. On reflection this is ambiguous question because I haven't clarified the total or each pan. *I*nterestingly I can hear in the background Learner 5 and 6 saying 10, 100. I repeat the question 'What would that be worth', What would 5 lots of 4g be, 4g taken 5 times'. Learner 4 says '20'. I (annoyingly) jump on this and say '20g' and then say '4cm taken 5 times was 20cm wasn't it, we can check can't we, we can count in fours'. I then model and say 'We can say 4, 12' and then correct myself '4' and a learner (Learner 5) says '4, 8, 12'. And as I drop them, I say '8, 12' and Learner 5 says '16' ahead of me. We do this up to 20 and I say 'How many 4g cubes there?'. Learner 4 says '20'. I say 'How many 4g cubes' and Learner 4 says 'ugh, 5' and there seems to be sound of agreement. I say '5 cubes, but you're right, 20g' and 'to make it equal on this side, we'd have to put 1, 2, 3, 4, 5, 5 4g cubes on this side' . I then reinforce that I like the thinking that 20g was a good number to use and that we could use bags of 20g. On reflection, I probably rushed this view forward. I then say 'We could use bags of 20g to help us to weigh, because each time we put a bag of 5 lots of 4g, we know we have ...?' Learner 4 says '20g'. I say 'Learner 2, do you agree?' and Learner (6 or 7) says '4 add 4 add 4 add 4 add 4 ... is 20' (it's not clear whether 4 is repeated 5 times'. I say 'yes, we know that 4 taken 5 times is 20'. I then reinforce that we can make up bags of 20g and check understanding by asking Learner 7. I then ask, if I want to make a bag of real sugar and I want to weigh that, because I am going to do some cooking' (notice the start of a narrative), and I need to make a bag of sugar that is worth 40g, it's worth 40g and I can put on this pan, how can I work out if it is 40g'. Learner 2 says 'you could use these' (presumably Learner 2 is pointing to the bags of sugar cubes'. I say 'I could use these' and then i say 'I could have one bag of 5, because I know that that's 20'. At this point Learner 5 says '2 bags'. I say 'I could have another bag of 5, worth 20g, because I know that that will be 20g and 20g, and Learner 4 says '40g'. There is audible counting in ones as I put spoonfuls of sugar into the bag and whisper 'tell me when it is balanced'. They call stop and I say 'OK, so how much sugar is in this bag'. Learner 7 seems to say '9 metres'. The 9 may be explained by the spoonfuls? Another learer (Learner 4) says '40g'. I say '40g, how do we know it is 40g' and someone says 'because they balance'. Learner 2 says 'the sugar cubes'. I say 'It balances with the bags of sugar cubes, so 20 and

20 is 40 and that is 40 too'. On reflection it is clear that a few learners are more confident in this task than others. I then introduce the arrow diagram. There is then some discussion around stick tack and the arrow diagram as I try to reinforce that they have 4g cubes, 5 times making 20g and that for 40g they needed 10 cubes. I can see that there were a lot of variables in this task. I then explain the task, asking them with bags of 5 sugar cubes. On reflection having had the sugar cubes made up in advance would have been beneficial and I was also moving quite quickly. There is then some clarification of task and partners. I can then hear talking to student teachers as the learners get started. Interestingly, I don't hear me clarifying what they are weighing out. There is a task efficacy issue with the soil trays and pan balances. I then reinforce (as I distribute resources) that each time they make a bag of 5 sugar cubes they will be counting in 20g. There is discussion as the learners count out in 5's (shame I didn't get them to count in grams), five cubes in each bag. There is some reminding as they have counted out some bags. I then distribute a sheet with some questions - there were a lot of resources in this lesson and there was a lot going on. I then ask the learners to make a bag of sugar that weights 80g, using the bags of sugar cubes to help them. Interestingly, one learner says 'this will be impossible', meanwhile Learner 2 can be heard to say 'we need 4 bags' and then 'we need another bag'. There is quite a lot of movement and chatter as the learners settle to the task and I ask them to chat about their approach. Learners can be heard discussing it 'how are we going to do it?, how are we going to balance it?' (Learner 3) and 'we need 2 more bags' (Learner 4). There is also discussion about 'What are we going to do, we haven't got enough sugar cubes'. I then can be heard asking 'you have got four bags, how much have you got there, can we use this (pointing to number line' to help us, so each bag is worth 20'. In the background I can hear someone say '80'. There is lots of chatter about 20's and how many bags. Unfortunately, the latter part of the task has a student teacher intervening with a pair of learners - I feel I need to discard this for ethical reasons as the student teacher was not part of the research. It is clear from listening to other discussions in the background that some groups had worked out that they needed 4 bags but the issue was in the number of variables. This task could work for an older age group but was just a step too far for the younger learners. There is some gathering together and also discussion about sugar 'I love it' (Learner 7).

APPENDIX N: EXAMPLE OF CODES

Awareness of quantity in relation to unit

<Files\\Cycle 1\\Day1\\Day1Task1d> - § 4 references coded [10.43% Coverage]

Reference 1 - 3.16% Coverage

It has to be just enough. You put it all the way in there

Reference 2 - 3.06% Coverage

Miss can I have more water to pour?

Reference 3 - 4.21% Coverage

Girls can I borrow one of your cups?

Reference 4 - 4.21% Coverage

I don't need all of these

<Files\\Cycle 1\\Day1\\Day1Tasks1abc> - § 17 references coded [38.51% Coverage]

Reference 1 - 2.30% Coverage

and you could that cup

Reference 2 - 2.30% Coverage

Lots of times

Reference 3 - 1.24% Coverage

So for two times it's going to be this one, because it's bigger, because you can put more amount of liquid in it

Reference 4 - 1.24% Coverage

I know, and that one's going to be twelve because that's smaller

Reference 5 - 1.24% Coverage

In this one, you can have a less amount in it because it's only little

Reference 6 - 1.24% Coverage

You can use this one two times and that one twelve times because this one is way bigger

Reference 7 - 1.24% Coverage

And then this one you'll only have two because you can get bigger scoops

Reference 8 - 11.11% Coverage

Miss we're going to have more being scooping up with this one because it's a bigger, it can fit a bigger amount in it

Reference 9 - 11.11% Coverage

You have to pour more times

Reference 10 - 11.11% Coverage

Well I think that the big one might be two cupfuls because you can fit more in there

Reference 11 - 11.11% Coverage

And the smaller one you can't fit as much liquid in as the big one

Reference 12 - 2.35% Coverage

If you used, if you used the big straw, it'll be more quicker, you will rush

Reference 13 - 5.20% Coverage

Learner 7: But how, how do we need to use it ten? Oh yes, because it's the little one.

Reference 14 - 5.37% Coverage

Well with the smaller straw, you'd think that there would be less amount because the big straw is bigger than it but because it's little, there'll be more, like the number, ther'll be a bigger number than with the bigger one

Reference 15 - 5.37% Coverage

There's more, there's more of them

Reference 16 - 2.04% Coverage

It's going to take forever

Reference 17 - 8.89% Coverage

Miss mine was ten but we did spill a bit

<Files\\Cycle 1\\Day2\\Day2Tasks2ab> - § 3 references coded [18.71% Coverage]

Reference 1 - 6.23% Coverage

A lot!

Reference 2 - 12.48% Coverage

Yes...we need three more scoops maybe. OK. That's way more than that one. They are two of the same amount but this one, there's not that much flour...Yes all of them are the same

Reference 3 - 12.48% Coverage

I don't think we might have enough for another cup

<Files\\Cycle 1\\Day4\\Task4ab> - § 10 references coded [70.29% Coverage]

Reference 1 - 7.18% Coverage

Learner 6: I think it's less because it's bigger than two

Reference 2 - 7.18% Coverage

Learner 1: Because when you make ten it's four, eight and then it's not making ten

Reference 3 - 9.84% Coverage

Learner 7: I think I have enough tens

Reference 4 - 4.70% Coverage

Learner 7: Forty. We need more fours

Reference 5 - 4.70% Coverage

Learner 5: We're also running out of pink

Reference 6 - 1.65% Coverage

RW: Can you predict how many twos you are need Learner 7?
Learner 7: Ah, a thousand?

Reference 7 - 17.71% Coverage

Learner 5: Two bags

Reference 8 - 29.36% Coverage

Learner 7: I've already got two bags

Reference 9 - 29.36% Coverage

Learner 2: We need four bags
Learner ?: So we need some more

Reference 10 - 29.36% Coverage

Learner 3: We've got sixty there, we need two more bags.

APPENDIX O: INTERVIEW WITH PRACTITIONERS (CYCLE 2)

Questions (shown to practitioners)

1. When I visited 2 years ago, I spoke with members from Foundation Phase and undertook a learning walk to find out about your approaches to teaching mathematics. You spoke about the importance of concrete, visual and abstract approaches and linking mathematics to real life experiences. Are there any differences in the way you approach teaching mathematics since that time?
2. The Covid pandemic has affected the way learners have been able to access learning provision. What factors do you feel I should consider when planning mathematics activities for Year 2?
3. How do you feel the learners respond (or may respond) to collaborative challenges in mathematics?
4. What experiences might the Year 2 learners have had to support their understanding of the multiplicative relationship?
5. What experiences might the Year 2 learners have had to support their understanding of measures?

I – interviewer (researcher) T1 – teacher 1 T2 – teacher 2 (teachers numbered according to order of initial responses)

I: Thank you for agreeing to answer some questions and so you can see that first question. When I came two years ago and I spoke with people within the Foundation Phase and did a learning walk, you spoke about concrete visual and abstract approaches and linking maths to real life experiences, are there any differences in the way that you approach teaching maths since that time that I might need to take into account?

T1: I'd say reasoning. We're very much still concrete visual abstract, very much real life wherever possible I mean you still need to teach multiplication before you can apply it, but I'd say everything is more or less the same. The *reasoning* I think we're far more aware of trying to get a reasoning problem into *any* situation, be it maths or whatever really. And that's our core purposes and everything.

T2: And letting the children decide if they want to use certain resources, others might want to draw something, so you know offering an array of different resources and techniques and allowing them to choose then.

T1: So yes not just for the less able given all the concrete apparatus…

T2: Mmm (sound of agreement)

T1: …For everybody I think with far more mindful of that now as a as a school.

T2: Yes.

I: That's great thank you is there anything else in relation to that question. That seems to be lots of things for me to think about.

T1: We have since…September, no, during lockdown we purchased White Rose Maths, which we’re loving. I'm loving the flashback fours so that's bringing to memory everyday five questions one on time and all the concrete not letting it fall away, keeping building, keeping chipping away at the language of maths. So it’s something that we have changed and I think it has changed our teaching.

I: And in terms of language of maths when I'm thinking about… this sort of relates to those last two questions, you talk before about the language that would be expecting their children to use for multiplication, has any of that changed because of what you're talking about there in terms of language of multiplication or the language of measures?

T1: I think it it's a different scheme so it has brought different things in. It's just displayed in a different way, obviously it's 2D but you go away and do the 3D, the real life tasks, but it's we’re constantly talking about the greater than and less than sign and it’s equal and it's bringing things up all the time that does develop the language…and the mastery. We've been doing estimating today so they have practical tasks to do, and we get back together and it was going over that language constantly. And as a school we are very aware of the acquisition of language, writing the new curriculum and (another practitioner’s name) is: ‘How can we get the language drilled into our children?’. And a lot of them are starting at a very basic level.

I:Anything else?... So you could see that second question in relation to Covid. I know that, you know obviously it will have affected the children's experiences, so in terms of that what factors do you think I'll need to consider when I'm planning those maths activities for Year 2?

T2: We’re sort of in bubbles. We've been lucky you've got my class and T2’s class bubbled together, but we're not allowed to mix with other people, so sort of things like space will have to be aware beforehand because we've got our own allocated space in the hall, so sharing and things like that. So if we know beforehand that can all be sorted. We will have our own resources and that's fine. Before we’d have to wipe it all down and I do think that it's all in our bubbles and we’re OK at the moment. So I think that's probably the only...

T1: And I just think they are *not* where they should be, they are *not*.

T2: No, they have…(seems to be agreeing)

T1: …certainly not at the level…I mean you've got your obvious highflyers and they've been able to push, and we’ve gone onto our five times table now, but even last years, able as they were, and they could, they had strategies for their tables. But learning them, and the Year 3 teacher has said they *don't know them*. They’ve got the strategies but I don't think they had the daily drilling so I do think that we’re still covering the basic skills…

T2: The basics skills, mmm (agreeing)

T1:The basic skills, yes, which I didn't feel would happen because I thought we had that term back and we would have caught up but I *certainly* don't think so, in maths and reading.

T2: It's taken a much longer time to get them back into routine…

T1: Yes, yes

T2: let alone academically

T1: And being independent… in *every* aspect of school.

I: Yes because that was one of the things I was going to ask about is…you know you mentioned language development earlier, but also things like working together. Some of the tasks will be paired tasks…

T2: Sharing

I: And sharing

T2: So I've had quite a bit of problems in my class where sharing has been, so we've had to talk a lot more than I *ever* have done before and actually had to physically show them how we share, how we take turns and yeah I've not had that before. I think they are slowly getting there.

I: Yes

T2: We've been back, touch wood, for a while now…

T1: Nearly half a year…

T2: So getting back into routine

I: Yes and also you mentioned language earlier so in terms of language within multiplication or measures then do you think they would understand certain terms or would that be something that's a new concept to them. It doesn't matter either way but it's just being aware.

T1: I'd say they wouldn't be where they should be, where they would have been. We wouldn't have done so much on capacity…we weren't allowed to cook…

T2: Yes, we weren't allowed certain things for a long time…

T1: So weighing…

T2: Until recently

T1: Yes, equipment…

I: So, handling of equipment, things like, some of the tasks have pan balances

T1: Yes

I: And that links to what you were saying about equipment. Last time I bought in my own equipment from uni I'm happy to, because it's just easier to have it

T1: Yes it is easier

T2: Yes

T: Rather than search around, so is that OK

T1 and T2: Yes

I: …and make I can make sure it's wiped down and everything.

T2: Yes that's totally fine. That's fine.

I: Yes

T2: I think we try and much use as much language as we can. Say for times table it's groups of and we try, but as you said they're not where they should be and where they would have known that a long time ago, we seem to be doing that now, but try to use as much…

T1: inverse operations and things… The commutative law, they love learning the commutative law and these lot I'm finding they haven't, even my more able couldn't…we were doing arrays as we were doing the two times table and they couldn't see it…as much as they would have in other years. I think there has been two terms out, the younger the children more of an impact Covid has had and I think maths has definitely taken a hit.

I: Yes. So is there anything else in relation to that?

T2: I don't think so

I: And then you can see the next question. I suppose this links to what we were just saying. In the tasks I'm trying to set up, you were saying earlier about reasoning, tasks do start with a reasoning sort of idea so: *How* can we solve this problem? What are your ideas as to how we can solve this problem? How do you feel the learners will respond to that sort of, a collaborative sort of problem solving challenge?

T1: I don't know. I don't feel mine are offering. I keep saying just tell me anything… be creative in your thinking because they've got to be creative, and I'm… it's always the same group of children but I am not having as much…

T2: I think there are quite a few children, and whether this is Covid related or not I don't know, they are frightened of having a go.

T1: Yeah

T2: They feel it has to be right, and we're forever saying 'This is why we're here, I'm learning constantly and it doesn't matter if it's right, wrong, we're here to talk about it' and a lot are still frightened. And, as I said, whether it is Covid and they've been at home and things have been done for them I don't know, but we try that a lot in school 'Let's have a go', well 'Let's try it your way, let's have a go your way' and get them to say. I have got some children that will I'm quite confident and will come out with ideas and they don't mind if it's right or wrong, but some will hold back and I think you get that in any year group anyway.

T1: We did estimating this morning so we had the bags of everything all out, and they had to go along fill in their estimates, and it took them a while they wanted to count. And I said 'Don't count, we're not allowed to *touch* the bags, we're just guessing'. We'd been through the language of '*roughly*' and…I'm thinking all those words now that the children came out with this morning! And they weren't prepared to guess. They wanted to be right and it took a little while… it doesn't matter it can be near, it doesn't matter if you are over or under and they eventually were quite happy then.

I: So that's it, yes and that does link to some of the tasks that I'll be doing because it isn't about the answer in the end.

T1 and T2: No

I: It's about the this about the process

T2: Exactly

I: And they are thinking about how many of this or in this, and that means I can find out how many are in this, and that relationship, but actually finding out how many rabbits we feed

T2: No

I: You know it's not ultimately the main purpose of the activity

I: So in relation to my other questions there, so I think we've discussed most of this already, but what experience might learners have had to support their understanding of the multiplicative relationship? So ideas around multiplication and division…you mentioned earlier arrays.

T1: Yeah we've been… well lots of work on White Rose maths as the *theory*, and the grouping, they've thing had tasks, with a circuit of tasks, where they had to go along rolling dice making twos. Lots and lots of the physical, and today when they were doing the estimating, when it actually came back to counting, I had half a class so that was lovely, should we count in two, fives and tens and so they couldn't see what I meant first of all and then they couldn't see why I am not counting everything in ones, well there was no need and that that was *lovely* to see like a light switching on. So applying everything back, with two groups of 10 so that's 10 and 10 makes 20.

I: And that links very much to what the tasks are about. Because they are measures they can't count in ones…

T1 and T2: Mmm (sounds of agreement)

I: So it's trying to encourage them to count in, in the other units.

T1: Yes

I: So they are being restricted from counting in ones because they're not going to use the little cup.

T1: No

T2: Yes, yes

I: They've got to use that middle cap which is how worth however many. So, or you know, when they are weighing they've got to use an intermediate one, so it is trying to encourage that counting in steps rather than…

T2: It's just getting them to that point where they are having to do it. They'll be reluctant at first I think and then once they do, as you say, they'll have that lightbulb moment.

I: But as I say that their ability to actually count in those units that, in a sense, doesn't matter

T2: No

I: It's more the idea that that's what they need to do

T1: Yes

I: Rather than the outcome, so if they, if they counted in twos fives and tens, even if they're not very, you know they don't know it by rote, that doesn't matter to their success in the task

T2: Yes

I: It's more that it's encouraging them to think in that way

T1: They came, when we started doing the twos, I mean I know they've done it in Year 1, but it's two, four, six, eight, ten and they chant together (makes noise to indicate some mumble rather than count)

T2: (laughs)

T1: And they haven't had the experiences or had forgotten then about…So it's being practical getting that concept solid before we can move on and understand.

T2: I mean we've lined up for dinner with little let's see how many pairs of children, getting in twos and try and do it that way, and actually after we've been doing the two times table and to suddenly go to that they were a bit of a loss…

T1: And reasoning problems after doing, you know while you're doing the two times table, and you throw in a reasoning problem it's well 'Woah', it's like you're doing another language, 'what you doing now, you were doing times table a minute ago, not realising the connection at all.

I: Yes

T2: And that's when you know they haven't really got that

I: Yes

T1: Solid understanding

I: Absolutely

T1: Dripfeed

I: Yes. So in relation to that last question there, their understanding of measures, and I think we mentioned this earlier, that sort of experience of hands on measuring and the use of things like pan balances. I… one of the things I sort of thought from my previous phase was trying to make more use of standard measures again not the main aim being the use that their understanding of the standard measures but in terms of the equipment that I'm using. I was sourcing the bottles from, you know, places like Home Bargains and things…

T2: Yes

I: And emptying them out now and then sort of establishing the relationships but making more use of bottles that you can buy that that are empty that already have that sort of you know, a 10ml bottle and 20 ml bottle and 50 ml bottle or whatever, so it's sort of their experience of handling weighing equipment and so on. Is there anything there?

T1: They are just going to need a lot of practical. I mean…rulers we've been doing standardised, you know Christmas time we were measuring practically to do our craft, so it's using maths across the curriculum…

T2: I mean I do need to do more to measure and bits but it has been hard lately

I: Yes

T2: Staff being off as well and not being able to cross in bubbles so that has had an impact.

I: Yes, yes.

T2: We've had to be more adaptable.

T1: We do need to be putting more of the capacity

T2: Yes

T1: It's too cold to be outside in water at the moment, they'll freeze, and we've now got non-carpeted floors

I: Ah

T1: So we can start putting things out…

I: But you, as I say it's not, it's more for me to take into account so that when I'm introducing the equipment I can assume, you know I can sort of go from a level that takes into account the fact that they may not have used it for a while.

T2: Yes, start at the basics, back to basics.

T1: And unless they have done the weighing at home, making pancakes and fairy cakes and what not…

T2: But also they don't cook at home so they've not been exposed to that language, of that we need a certain amount of food. Very few I find have cooked.

T1: And they haven't had the experience in school of us doing regular baking

T2: Because we're not allowed to cook, so of course they're not shown scales so I mean I would just take into account everything is very basic.

I: Yes

T1: Definitely

I: So that's that's all my questions I think is there anything else that we would you want to tell me or anything you want to ask really?

T1: No just interested to hear what they come up with because I think that helps us then...

T2: Yes

I: Yes

T1: and our understanding of their learning so

I: Well as I say the last time, the things that things that I learned was that particularly, they said they enjoyed the tasks because there were always comments about, particularly in the first few days, about you know enjoying working with the materials and so on. The did show, in what they were saying, they did show that they were using the multiplicative relationship and that the tasks were developing certain ideas about the multiplicative relationship. At the end of the week I showed them pictures of the tasks that we done and asked them which they felt that they learned the most from, and that was *really* fascinating because they showed, they said things like it was about that perception of what equality was. They didn't use up those words obviously, but the things that they said made me think about how they see that relationship, so when you're thinking about liquids seeing equality with a little cap and a big cup it's actually quite hard isn't it because they're different containers...

T1: Mmm (agreement sound)

I: ... and even if you have ten of those little cap is equal,

T1: yes

I: It doesn't look the same

T2: Mmm (agreement)

I: Whereas when you're using a pan balance...

T1: Yes

I: Equality is much more obvious and the same with Cuisinaire which we use so for centimetres but we didn't have the centimetre one, and equality is much easier to *see*

T1: That’s a nice one, yes

I: So it's that sort of aspect that's got me thinking about that sort of thing and making sure that establishing the initial relationship where one of that object is equal to however many of that, which they can't use, establishing that is really important and that's the sort of thing that, that it’s the things that they said showed me that

T1: Yes

I: So it’s building that into the tasks that I'm doing now

T1: So do you adapt the tasks

I: Yes

T1: And as you can see this is not working

I:...and adapting the way in which I introduced the tasks

T1: Yes

I: As well so when I'm thinking about the reasoning tasks that I introduce to them, I have to think about the way in which they get that quality relationship.

T1: Mmm (sound of agreement)

I: To allowed them to work on the on using that intermediate unit

T1: Yeah yeah

I: Yes, it’s not so much the tasks it is the way I introduce them, so taking into account all of these things

T1 and T2: Yes

I: and what I've learned is more about the way that I use them

T1: Yeah

APPENDIX P: LEARNER RESPONSES TO CYCLE 2 TASKS (OVERVIEW)

Task	Helped me learn the most maths	Made me think the most/was hard	Most enjoyable	Notes about improvement	Other notes
C2a1a Making same quantity of liquid	Learner 6 'everything'	Learner 1 'because when me and Learner 8 were partnered up, we didn't know what to do and that was the most maths' Learner 7: I was going to say the liquids because at the start when I had my partner, I didn't know what to do but then my partner helped	Learner 2 'I liked the liquids because it was satisfying' Learner 7 'I agree' Learner 2 'satisfying when you poured it out' Learner 1: I liked the liquids…because I liked pouring in cups		
C2a1b Using straws to measure	Learner 6 'everything'				When asked: Is it always good to use a small unit to measure? Learner 1: With straws no because it is too, because it is super annoying
C2a1c How many flapjacks?	Learner 6 'everything'				Learner 2 discusses porridge but might also be alluding to this task which involves oats
C2a2a How many rabbits?	Learner 6 'everything'		Learner 2 'I liked the liquids because it was satisfying' Learner 7 'I agree' Learner 2 'satisfying when you poured it out'		

			Learner 1: 'I liked the liquids…because I liked pouring in cups'		
C2a2b How much porridge?	Learner 2: I think the porridge…because we had to remember how many…how much there was and for every cup we had to measure up to the line and we needed to remember loads of stuff Learner 6 'everything'			Learner 12: 'I think this could be better for other people…because they could have a little bit longer straws if they are a little bit littler...'	
C2a3a Making lengths without using single centimetres	Learner 6 'everything'		Learner 6: I liked the straws, because you were helping	Learner 12: 'I think this could be better for other people…because they could have a little bit longer straws if they are a little bit littler...'	When asked: Is it always good to use a small unit to measure? Learner 1: With straws no because it is too because it is super annoying
C2a3b How much medicine?	Learner 6 'everything'		Learner 2 'I liked the liquids because it was satisfying' Learner 7 'I agree' Learner 2 'satisfying when you poured it out' Learner 1: 'I liked the liquids…because I liked pouring in cups'		
C2a4a Exploring relationships between different masses	Learner 6 'everything'		Learner 12: 'I think weighting' (was most fun)		
C2a4b How many portions of pasta?	Learner 6 'everything'				
C2b1a Using straws to measure (Similar to C2a1b	Learner 9 this helped 'with sizes, length'	Learner 9 Learner 11: 'Um this one was hard because it was super hard to get those things to			Learner 12: Notes it as confusing 'It's hard to go like one, two, three, four' Learner 11 points to this as confusing

		make into a straight line			Learner 10: 'For the straws you need to measure and it's going to take longer'
C2b1b Making lengths without using single centimetres. (Similar to 2a3a)	Learner 9 this helped 'with sizes, length'	Learner 9 Learner 11: 'Um this one was hard because it was super hard to get those things to make into a straight line'			Learner 12: Notes it as confusing 'It's hard to go like one, two, three, four' Learner 11 points to this as confusing Learner 10: 'For the straws you need to measure and it's going to take longer'
C2b2a Exploring relationships between different masses (Similar to 2a4a)	Learner 9 this helped with 'strength…I mean weight… And how much one gramme is really like, two, uh, twenty grammes is like..' Learner 10 suggests these helped 'because we know how hard it is to weigh'		Learner 12: 'I think weighting' (was most fun)		Learner 12: So these, I think these are useless. RW: They are useless? Learner 10: No they are not useless! Learners: No! RW: How interesting Learner 11: I'm with Learner 12. Learner 12: You can't just put a bed in there and then a something else in there to weigh a bed. You need straws to length a bed. RW: So you think length is really important to understand. Learner 10: No it isn't Learner 12: It's useless Learner 10: No it isn't RW: Learner 13 and Learner 10 disagree. Tell them what you think. Learner 13: These are actually really good because when you are like trying to measure how

					much sugar you need and how much like... Learner 10: Yes Learner 13:...things for a cake you need you need to measure it
C2b2b How many portions of pasta?	Learner 9 this helped with 'strength…I mean weight… And how much one gramme is really like, two, uh, twenty grammes is like..' Learner 10 suggests these helped 'because we know how hard it is to weigh'				Learner 11 suggests counting each one Learner 13: 'These are actually really good because when you are like trying to measure how much sugar you need and how much like...things for a cake you need you need to measure it'

APPENDIX Q: FINAL INTERVIEW WITH PRACTITIONER

Information: This interview was conducted a few weeks after the end of Cycle 2b. The teacher was provided with:

An overview of tasks used in both cycles (similar to tables provided in Chapters 5 and 6)

Interview questions (below)

And was then provided with an overview of tasks with implementation notes

Questions for practitioner/s:

The table provided gives an overview of the tasks I used with learners. I am keen to hear your views on the tasks as your thoughts will be valuable in helping me develop the tasks further.

What might you anticipate being possible benefits for learners in using these tasks?

What might you anticipate being possible limitations for learners in using these tasks?

In what ways are these tasks similar to tasks you may have used before, and in what way are these tasks different to tasks which you have used before?

Would you use any or some of these tasks to develop learners' understanding of the multiplicative relationship? Why/why not?

These tasks have been designed for use with Year 2 learners. Would you use such tasks in different age ranges?

If you were going to use tasks such as these, at what point in the learning of the multiplicative relationship might you use them (e.g., introduction of concept, consolidation, practice of skills)?

If you were going to use tasks such as these when teaching the multiplicative relationship, how would you structure their use (e.g. as a unit of consecutive tasks across a week, interspersed with other tasks across a series of weeks)?

Here is an overview of learner responses. What are your thoughts on the responses?

If you were going to use tasks such as these as part of your teaching of the multiplicative relationship, what would you require or find helpful in terms of teacher guidance and materials?

Interview:

RW: You're an experienced practitioner so it's really getting your...

P1: Old (laughs!)

RW: ...opinions. I'll just check it is recording, yes, it is recording. It's getting your opinions really on what I have done and trying to sort of, not square the circle...

P1: Yes (sound of agreement)

RW: ...because it's no way near complete but it's sort of at a point where I've got some data and I'd like I just like to get your opinion on what I've been doing and how you might use it and how I might further develop it. So, what I'm going to do if that's okay I've got this. *(Refers to practitioner* *information sheet)*. This - you have seen this before, so that's just the information about the research that I'm undertaking. The focus is to try and understand how children are learning, to see if I can develop tasks that help to develop an understanding of the multiplicative relationship through measures. So, what I've got here is this... *(Refers to overview of tasks document)*. Now this is where you are going to hate me because I've got lots and lots of information about the tasks, so perhaps if I just give you a moment...You probably won't be able to read through it all.

P1: Right yes

RW: *(Referring to columns on table)*. These are the tasks that I did in Cycle One, when I came in a couple of years ago, and in Cycle Two. You can keep that afterwards.

P1: Right, ooh lovely.

RW: It's got information about the tasks that I did and why I did them.

Allowing some time to read.

P1: *(as reading)* Were the straws half – two yellow and one red?

RW: Yes

P1: It really is using and applying isn't it.

Allowing some time to read.

P1: Oh that's tricky, (with weights?)

Allowing further time to read

P1: Thank you

RW: So, I've just got some questions...

P1: I love the activities

RW: Thank you! These are the questions *(Referring to questions document)* I was going to ask you, so I'll give you a moment to read through them and we can then discuss them as we go through.

P1: Right

Allowing some time to read

P1: Right

RW: So as I say, it's really..., I'll go through the questions

P1: Yes

RW: But I am sure lots of things will come up as we discuss it...

P1: Mmm huh

RW:...and then we can talk about the results, I've got some information on results as well

P1: Ah yes, so it says on that question doesn't it *(Referring to question document)*

RW: Yes...So what I was going to ask you first of all is if you were using task like this what would you think possible benefits would be?

P1: I would be able to see who understands the multiplicative rule and just about capacity and all the different mathematical concepts because I think a lot of them are taught in isolation. Alright, we are doing this, and it needs to be underpinning everything. You know the maths isn't just about, um, numbers, we have got to be using it and applying it, and real life, and all of these are real life, very good real life problems, actually.

RW: What was interesting is when I was working with the group when I came in a couple of weeks ago, when I was asking about the tasks, they were saying... they didn't use the word authentic

P1: Ah

RW: But they were saying this relates to things you would do...

P1: Ah! Yes.

RW:...and it was still quite interesting that they were saying that because I know that there's been a focus on that in school

P1: Ah, reasoning

RW: Yes

P1: Getting a good reasoning question, it's so hard and some of them are so random

RW: Yes...so in terms of limitations, for learners or for teachers I suppose, what would you think might be the limitations in using tasks like this, or any particular tasks that you see there?

P1: Umm, teachers setting up. I mean you need the right equipment, you've got to have it. To explain it and to give them the hands on they'd have to have their own sets of equipment in small groups so you need time, you need space and you need all the right equipment. Um, some children would really struggle, you know so you'd need a lot more of the basic work, so it would have to be brought down a level. But in term of the actual problems, they are lovely problems but they would have to be adapted, differentiated, but yes, time, space, money.

RW: Yes, laughs (as though to agree).

P1: But if the children haven't, don't know, their multiplication, you know, they've got to be at a certain level to understand it anyway. That's the main thing.

RW: And that's what I was going to ask you about really, because I designed the tasks so that in a way...particularly for this round, after our initial conversation I focused on relationships, when I came in two years ago, I focused on relationships where there was, perhaps, they were thinking about six, for example, the oats one, I did pancakes, and every cup made six pancakes but this time after coming in to speak to you I kept the relationships a little bit simpler, so it was thinking about those aspects of it...

P1: So if they didn't know them...

RW:...but in a way it didn't matter if they didn't *know* their multiplication tables, but they had to be able to think about counting in units of a number...

P1: Yes! Yes.

RW: ...so it's thinking about those relationships...

P1: Yes. They are supposed to know their two, five and tens tables in Year 2, and be able to apply them.

RW: Yes. Yes.

P1: But others are on their three, four, six.

RW: Yes

P1: But it's the children who still are using Numicon, daily. They still can't do three twos...

RW: Yes.

P1:...never mind I've got the coins out, or the Numicon, or we've got two dots on each finger, it's using and applying, so that would be the main problem.

RW: And I did find, I anticipated you might say the set-up, because even for me with a small group, making sure you've got the resources, and making sure you are thinking about the relationships you are going to be using...

P1: Yes...

RW:...so making sure you're resourcing the cups and the containers that are going to give you those relationships that are authentic.

P1: Yes, mmm, and changing the numbers, did you change the numbers? Well, you only worked with the more able, well sort of, didn't you. I know one little boy came in with you, because I just didn't get the forms back, even though I sent double what you asked.

RW: Yes. I focused on twos, fives and tens in the last round, because based on what you had said, what you and Practitioner 2 had said to me, so this time I focused on two, fives and tens. Last time, because of the group I'd had, and after our conversation, I did focus on different relationships.

P1: Yes, yeah.

RW: So yes, resourcing those containers and making those relationships...

P1: Mmm, yes.

RW:...in an authentic way. Even things like, um, you know, with the oats, that a cup does actually make that many pancakes, because you don't want to give them the...

P1: Yes, yes...

RW:...you want the relationship to be slightly real, because if it made...

P1: It's...yes, well...

RW:...it's that sort of aspect, do you know what I mean? It's that sort of aspect of it.

P1: I hate saying, especially for a low-level child, this is four pence, when it clearly isn't four pence but you cannot work in real numbers until they have got the concept.

RW: Yes.

P1: I have struggled more with that this year than other years, I don't know if it's because maths has got to be meaningful and reasoning, it's got to be understandable

RW: Yes, OK so in terms of tasks that you've used before, in what ways would you say these tasks are a similar to tasks you have used before, and in what ways may they be different, if they are different that is.

P1: I just like the way you've applied it to real life, maybe I have not made them enough of a reasoning problem. I've just said how many cups do you think this will hold and then we'll do it practically, were we right, who was nearest, write down everyone's in that groups trial, and try and refine so they are very different in that they are so applicable. The straws, *everybody* can do it hands on, um...

RW:...Although interestingly, what I found with the straws is that practical practically they roll around...

P1: Ah!

RW:... so I had to flatten them (laughs)

P1: Ah that's a shame because I was thinking 'Oh I could use these' when you haven't got enough Numicon...

RW: Yes, yeah...

P1:...for the children doing the two times tables because I normally borrow them from other classes

RW: You can, but my advice would be, flatten them...

P1: Flatten them

RW:...because otherwise they roll around and some of the children were finding that...

P1: Right yes...

RW: ...and some of the children were getting frustrated because they were trying to make their relationships...

P1: Yes, yeah...I can see that.

RW:...and they were finding that they rolling around so my solution was to flatten them (laughs)

P1: Right, OK. Um...(*looks through tasks*) medicine. We had someone in to talk about medicine and who can give you medicine so you've got to be a little mindful of that, that you don't just feed yourself medicine

RW: Yes, yes absolutely. That was for my dog that medicine.

P1: Yes.

RW: But again it's making it, um, and they *know ...*

P1: Oh yes...

RW:...even at Year 2 they know it's not medicine...

P1: Yes

RW: They know it's water with food colouring in

P1: Yes

RW: Bit they seemed to like to know, or to think about it as something else.

P1: Its, yes. It's for a purpose. I bring dragon's tears in, and they know they are not dragon's tears!

RW: Yes, and I found this with the first task actually...

P1: Yes

RW:...which was a sort of pre-assessment task, where they had to make the same volume of liquid, of yellow liquid as red liquid, without having the same container...

P1: Oh yes...

RW:...so it's forcing them to think about a unit. I found that would have been easier to explain, I think, if I had given it a context and I didn't and it was actually quite difficult to explain.

P1: Ooh, I can't even think of a context...that's difficult

RW: It's that aspect of it I think, but, yes...

P1: Yes if you have got a context for the *real* problem that you are trying to solve, and you need a proper answer

RW: Yes

P1: I think if it's a random one

RW: Yes...so in terms of any of those tasks, have you used any tasks like this before and in what context have you used them?

P1: Mmm...I don't think I have. The measuring one, we do Christmas, always a Christmas activity that I tend to go back to, and they do lots of measuring and make Christmas trees and I know the Year 3s they have measured theirs, but maybe not putting it into a reasoning problem. I think we talk about it but we haven't done it specifically, so when you cook, though we haven't done it in Covid times, so roll on next year, um, that's when it would come up naturally

RW: Yes, yeah. Cooking is a great context actually isn't it.

P1: Ah yes. There's a result at the end that they want, but I don't think I have, and *I can really see that these will make it more meaningful.* For my more able, maybe my middles. My lowers...I... don't know. They might still be on the on the full, half-full, empty and knowing the different units of measure for...I struggle in Year 2, we are constantly, constantly going back to how heavy something is, it's kilograms. Even yesterday I talked to the pupil voice group, so first of all I got them so they are spread over the Key Stage 2 children, and what we're doing in maths. So, I said 'Right, what are we learning in maths?' first of all. They named lots of concepts, but they were very numerical. They weren't thinking of other things. And then I said 'Why do we need maths?' and they were talking in pairs and they came back. They didn't realise a lot and then they kept going back to money. Money, because that is the one...

RW: Yes

P1:...It's not abstract, I mean they probably see their parents dealing with it, like, you can't have those new trainers, there's a budget. But it was really interesting, but when I was saying 'What other units of measure?', even the Year 6s were struggling to get their heads around it, and then once they realised what I was talking about...I think we've got to make it more mindful; we don't leave it until the week you're doing capacity, it's got to be brought through the curriculum...

RW: Yes, yes...

P1:...and *doing* it

RW: And that's why I think, with tasks like that, the focus isn't about, it's not so much about the measuring

P1: No...

RW: It's about the concept of multiplication, even if they struggle with the concept of multiplication, it's to support them in understanding that if they are using a different unit and if that unit is biggerand it's actually *quicker* to count...

P1: Yes

RW: in another unit, so it can help with that I think.

P1: They've just got to understand what we are doing, we're not just pouring water

RW: Yes, yes, absolutely. So would you use any or some of these tasks to support your learners' understanding of the multiplicative relationship?

P1: Definitely. I love the straws one, because I think it's so hands on and you can give it to everyone and they can have it individually or in pairs. The medicine one is an obvious one, because it's something, it's *anything* that applies to them. Next year as well I want to give the children more of the weights to use themselves. We use my electronic scales and we measure the plastic animals and we look for those things, but I do like the idea you know...Ideally the ten grammes of pasta...we've got the ten gramme weights.

RW: Yes, yes.

P1: Um, definitely and letting them play first.

RW: Yes.

P1: So having that hands on knowledge of what ten grammes...

RW: Yes

P1:...actually is

RW: Yes exactly, I found that actually. They wanted to spend quite a lot of time playing with the scales.

P1 Yes they've got to play with them first.

RW: Yes

P1: Yes. One of my questions to the staff, when I do, um, my maths to the fifty, you know, to every member of staff, is I brought in a melon which I'd weighed that morning, a watermelon, and said, 'How heavy is this?' and then there was a prize. But it took a long time!

RW: (laughs): I can imagine!

P1: And some of them were *so far out*, because they didn't actually know and I said 'I actually buy my vegetables from the veg man who comes around, and I'll just ask for some carrots and I don't know how many carrots'

RW: Yes, yes.

P1: People are, they're not...you know everything is weighed up in bags ready.

RW: Absolutely.

P1: I mean they do it for you

RW: Yes, yes.

P1: And that came through one of the old, um, SATs papers. It was a Year 3 question, with a kilogramme of carrots and a kilogramme of potatoes but I want to buy one and a half, how much would it be, with prices on.

RW: Yes, yes.

P1: But unless they've got that understanding of what a kilogramme is...

RW: Yes

P1:...anyway...it's very, very abstract for them...they need more concrete.

RW: Yes. I totally agree. What I found with that one is...*(referring to pasta task)*...I was hoping to use things like paperclips to be able to say, you know, this is a gramme but we're not going to count the paper clips, I want you to find out how many paperclips there are without counting, give them loads of paper clips and find out how many, so they could *weigh.*

P1: Mmm.

RW: But...finding something that is worth a gramme, was really hard, or finding *anything* that has a sort of nice...

P1: Round

RW: Round number weight...

P1: Yes

RW:...was really tricky.

P1: But that's...we will definitely next year. I am going to put things out. I've got a tuff tray and we're going to have lots of, just problems...

RW: Yes, yes.

P1: ...on that. I think they've got to go through the playing...

RW: Yes. Yes..

P1:...part of it first. Bags of pasta weighed out is quite nice.

RW: Yes, yes.

P1: That's done for them? (difficult to pick up this bit)

RW: Yes, and again I experimented because the first time I did this, the first time I came in, I used oats and I've used flour, but of course they're messy

P1: Mmm, yes.

RW: And it's getting the balance between the children enjoying it... they loved using the flour,

P1: Yes

RW: They said things like, you know, 'this is so satisfying, I love playing...'

P1: Mmm

RW: But, as a teacher you're thinking, I've got all this mess, because they will drop things and...

P1: Rice is another one.

RW: Yes.

P1: It's more easy to sweep up, especially on our floors now.

RW: Yes, and lentils. Yes absolutely, yes. So as I've said these tasks have been designed for using with Year 2 learners, would you use those tasks, or tasks like this, with different age ranges? And why?

P1: Yes. Definitely, the older children, I think. Um, having been on a maths course, you do that one question and then the next time you do it, you give it to them, you just change the problem, the numbers and everything are very, very similar. So I would definitely use it in Key Stage 2, lower down, no. They're not ready, obviously, they...unless you've got a very able child, or if you simplify it but then you're simplifying it and you're not getting the multiplicative, you're just doing the capacity. So, definitely I would like to pass some of these on (laughs) to some of my colleagues to use, as our reasoning problems...

RW: Well I've...

P1:..to start things off.

RW: And I found that things like the, um, the medicine one, for example, in that one, I used, I starting using towards the end of the, the latter tasks, I started using the, the standard units but because I was focusing on their understanding of counting in a different unit, I didn't take that as far as I could have, but you could use the same problem...

P1: Yes, yes.

RW:...but bring out more in terms of the relationships because there's so many relationships there...

P1: Yes.

RW:... within, within, the standard units as well.

P1: Oh, Year 3 and Year 4, definitely.

RW: Yes.

P1: And I would start off at that level...

RW: Yes.

P1:...and see where you can go with it.

RW: Yes. So, if you were going to use tasks like these, at what points in the learning of the multiplicative relationship would you use them?

P1: Mmm...

RW: So, would you use them if you're introducing a concept, or if you're consolidating, or if you want to practise certain skills? How would you use them?

P1: Mmm...depending on which one it was. I mean they'd have to have *an understanding* of the units of measure first of all, so they'd have to know how to use a ruler to measure, which is a problem in itself, you know, start at the zero, count your Micky Mouse, and those basic things, and the capacity...It would nice maybe to give it to them, randomly, at the beginning, develop your work, and then see if they can apply.

RW: Yes.

P1: But if they have *no concept* of times tables but they still get something out of it, they just don't get maybe what *you* had planned out of it, but there is going to be all the language there and anyway they will learn all the different things leading up to it. ....Um, but then it's also nice if you know they understand the multiplicative rule...I don't know.

RW: A suppose...

P1: At any point!

RW: Yes, I suppose that, as you say, you could use it, it depends on the purpose of the task...

P1: Yes...

RW:...You could use it as a sort of assessing and planting a seed

P1: Yeah

RW:...Or you could use it...

P1: As a stimulus

RW:..to bring out...

P1: The assessment at the end...

RW: Yes. Yes.

P1:...of it all....Yes, definitely as a stimulus, with that purposeful problem. And let them play, and then they would come out of it and then they would...Did they all *want* an answer?

RW: *(Pauses)*. That's a tricky one actually because they all got to an answer, but whether they wanted it, I'm not sure. I sort of wanted them to get to...

P1: Mmm

RW:...to the answer. And that's...

P1: They were satisfied without an answer...

RW: But they were satisfied I think without an answer...

P1: Ah.

RW: Yes, if I show you now, *(hands out tasks with some learner responses),* and again you can keep this, this is, so that bit's the same, this is, those are the sort of responses that I've got, there. So yes, they did, in each one, we did mainly get to an answer, but I think that's what I struggled with is...

P1: Yes

RW:...how much should I have wanted for them to resolve it with a specific answer ' this is how many oats, or this is how many flapjacks you can make', or was I satisfied with them understanding the process that they were actually counting in units of five.

P1: Yes

RW: Most of time they did get to an answer and I think that's partly because I felt that was necessary but that's the bit I...not struggled with, but you know when...

P1: As an adult...

RW: ...when you listen back you're thinking 'am I...?'

P1: Yes, yeah

RW: '...am I pushing that bit too much?'

P1: Mmm

RW:...of them getting how many fives there are, or …

P1: Yeah

RW: That was the thing I struggled with I think

P1: They can be the same if they touch, it's comparative language

RW: Yes, yeah

Quiet as practitioner reads through responses

P1: It would have been, if I could have hand-picked the children, even though I had picked them for the letters going out for you...

RW: But in a way, it's nice that I had a range of children...

P1: A range yes...

RW:...because it gives me an insight into...

P1: Mmm, mmm

RW:...how different children may respond to the tasks

P1: Yes, when we do estimation, of anything, somebody brings in a bag of sweets, they've been on holiday, and we'll estimate, always, how many are in there. And they're random estimates, and then as we are counting you can see...'Oh no I want to change my mind now!'. You know, 'shall we count in twos, fives or tens?' is always the standard thing, so they are by Year 2, understanding more...

RW: Yes, yeah

P1:...about estimates and refining and rounding...

RW: Yes, yes. And actually, they made..., what was interesting in both cycles, sometimes they were making estimates that showed that they were thinking about the unit that we were thinking about...

P1: Ah, oh...

RW:...so if it was...

P1: Ah that's good

RW:...with the oats task, the estimate was a multiple of five in some cases and with the pancakes task which was a couple of years ago, which was the same as the oats task...

P1: Yes

RW:...but just with flour, um they gave estimates that were multiples of six

P1: Oh, mmm

RW:...so it showed that they were actually thinking about the units...

P1: Oh, right OK.

RW:...that were being considered really

Quiet as practitioner looks again at responses

P1: It's lovely to read what they've said

Quiet as practitioner looks again at responses

RW: And, for example with the rabbits one, when, when the child says 'it's going to take forever' , that's the purpose of them seeing that that little cup...

P1: It's not suitable

RW:...it's not suitable...

P1: Yes, yes

RW:...and that's why measures is quite a good one...

P1: Yes

RW:...you know you were saying earlier about learning to use a ruler, I had little centimetre straws, straws cut up into centimetres...

P1: Mmm...

RW:...and they didn't...the whole point was that I would then restrict it, but so they knew what a centimetre looked like...

P1: Yeah

RW:...and they could see that using lots of little centimetres, well that's not efficient...

P1: No

RW:...well we might as well use a ten centimetre...

P1: Yeah

RW:...or we could use a ruler (laughs)...so...it's that...

P1: You've still got those children who count in ones though, even though they know 'what is that ten?'...

RW: Yes

P1:…'come on think about it now'...

RW: Yes

P1:...they sometimes just need that little prompt...

RW: Yes

P1:...to remind them that you can actually do this, you know your numbers

RW: Yes, absolutely

P1: But if they're not solid, they've got to count...

RW: Yes

P1: ...and don't stop them

RW: Yes, yeah

P1: Yes, what is, I taught stripe how to whistle, but I can't hear him, I said I taught him, not he'd learnt it. What's that one? I showed all the staff, because I think it's so...we cannot just assume the children know...

RW: Yes, yes

P1:..because 'I've done that, they should know it' but no, you've done it but not necessarily every child has caught on

RW: Absolutely, yes, yeah.

P1: That is one of my favourites.

RW: (laughs)

P1: And I would say all teachers, including myself, are guilty of it.

RW: Oh yes, yeah

P1: Welsh, 'we've done this, we've done mynhwy, why can't they remember it?. .

RW: Yes, yes.

P1: One child...

RW: We do it with the students as well. They'll say, I don't know how...and we'll say 'well we've done that'

P1: Yes, mmm

RW: Yes, it's the same

P1: And they're adults

RW: Yes exactly...(laughs)

P1: (Referring to a learner comment) Oh I like the 'we counted in fives'

Practitioner continues to read

P1: Oh, I like...I'm going to have to check I have the 1 kilogramme and the five, ten and twenty kilogramme weights. My maths cupboard...

RW: Yes, that's the thing. I bought a little set of weights...

P1: This is brilliant.

RW:...the hexagon weights

P1: Yes, I can see, I saw the picture

RW: Yes, it as, as you say, trying to, sourcing things

P1: I'm putting in an order, the maths order's gone in, but...

RW: Laughs

P1:..they are...I hadn't thought of them for using multiplication before...

RW: Yes, yes

P1:...but it makes, it's using and applying and it's reasoning and there's...that is brilliant

RW: So, if you were going to use things like this as part of your teaching, what would you find helpful then? Because my next, my next sort of aspect is...analysing what I have got and thinking about how I would develop these into a set of materials that teachers could use

P1: Mmm

RW: So, what would you find helpful in terms of teacher guidance and materials?

P1: Provide materials!

RW: laughs

P1: Um, well having the questions there, knowing the expected...answer. I think as teachers, as an experienced teacher, you know what they're going to come out with

RW: Yes, yes

P1: But maybe...um, the other two levels so you can extend and how you can bring it down to something meaningful for the less able learner...what previous knowledge might they need, before they start the activity, but, um, the actual, no just having that problem there that you've thought of takes out the hard work

RW: Yes

P1: Teachers could then apply, I mean if you did that as well that would be better

RW: And I've got to admit that's where I found it hard, I was going to have to go to places like (SHOP) and get little bottles, empty them, and then work out relationships, so...

P1: Yes, see...

RW:...you're not going to have the time...

P1: No

RW:...to do that

P1: It is the materials, it is

RW: You want to be able to, you want a pack that says...

P1: Use

RW: ...use these

P1: Yes

RW: Or, things like straws that you can easily...

P1: Easily source

RW:..adapt

P1: Yes, things that you don't have to go looking for the correct container that you're going to fill up

RW: Yes

P1: That's not, that's not insurmountable, straws is an easy one, apart from the rolling

RW: Laughs

P1: I'll remember that, to plan

RW: Well I did use Cuisenaire

P1: Well as you said it and you were saying about using the ones

RW: Yes, yeah. The reason I didn't use Cuisenaire the last time is because, um, it can, well, as you know, it can be used in schools and part of it is the children being able to be flexible...

P1: Yes, yeah

RW:...in what you call those units, so not...

P1: Yes

RW:...trying to not get them to see that the smallest one is one centimetre all the time, if you want it to be something else...

P1: Yes, yes

RW:...there's that balance you have to consider, but on the other hand, it is a material that you could use and...

P1: Yes...

RW:...children can be quite flexible in their thinking so it's about thinking about those aspects

P1: Yes, children are more flexible than adults

RW: Yes

P1: You tell them something and they will take it on

RW: Yes, yeah

P1: Yes, so it's having the resources that are easy

RW: Yes and as you say thinking about prior learning...

P1: Mmm

RW:...what the expectations might be and how you might adapt it for different...

P1: Well for anybody

RW:...different groups

P1: Yes

RW: Yes, OK. Is there anything else you'd like to sort of comment on, or ask?

P1: No, am I allowed to ask names of children?

RW: You can, but to be honest I'd don't think I'd be able to tell you in here...

P1: Ah, yes

RW:...because I call them Learner 1 and Learner 2 and Learner 3...

P1: Yes I've done it with mine, I know and I had to keep going back

RW: Laughs...in my research and so I can't remember off the top of my head which learner is...

P1: Oh yes

RW:...which

P1: Because there was one much less able, a real struggle in maths

RW: In the last group I had?

P1: Yes

RW: Yes, he, he … he did find it harder

P1: Yes, I knew he would

RW: Yes

P1: But (learner) is very on the ball and knowledgeable about other things, so I wondered what (learner) might bring to the party

RW: Yes. But the task that he...that he...showed quite...that (learner) really engaged with, (learner) engaged in all of them, but the task (learner) engaged in the most was the weighing, and I think because they were, it was very visual, those relationships and (learner) worked with another learner and they were trying to get the scales to balance. (Learner) actually, (learner) worked well on that task

P1: Ah, see if I knew that I could have done his multiplications through...

RW: Laughs

P1: Weighing

RW: Yes

P1: That's one hundred percent what I am going to take on next year, and I am going to find out if I have got them. I think that's excellent. It's covering...it's ticking lots of boxes.

RW: I have said...if you want any...you are welcome to keep those...

P1: Thank you

RW:...and if you want any more details about any of the things I have used just contact me

P1: Thank you

RW: That's no problem at all

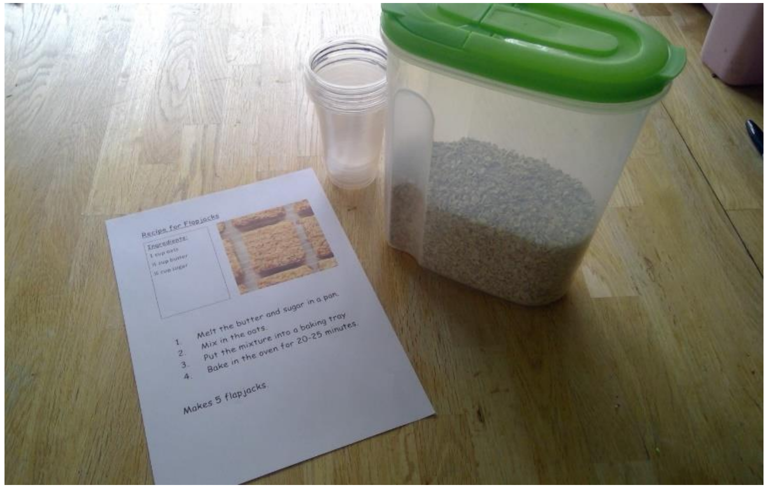
P1: Ideal for starting points...they really are, and I will share them upstairs

RW: Lovely, there we are, thank you, thank you so much.

APPENDIX R: PRACTITIONER OVERVIEW OF TASKS (1)

Task	Summary of expected learner activity	Purpose	Design notes
1a Making same quantity of liquid Questions: Here is some red liquid and here is some yellow liquid. Can you make the same amount of yellow liquid as red liquid in this container? How will you be sure that you have the same amount of liquid 	Suggesting ideas for how they can ensure the same amount of yellow liquid as red liquid. Showing awareness of a need to quantify/measure the amount of red liquid in order to reproduce the same amount of yellow liquid.	To assess learners' understanding of concept of unit.	Acting as an assessment of learners' understanding of unit. Restriction on pouring red liquid directly into container. Different shape container for red liquid to necessitate quantification.

Task	Summary of expected learner activity	Purpose	Design notes
1b Using straws to measure Questions: Here are some straws – red straws and yellow straws. Do you notice anything about the relationship between the straws? If you measure with the red straws and then also measure with the yellow straws, how will your answers be different? Can you measure these sticks with both the red straws and the yellow straws? Could you predict what the number of yellow straws would be if you knew the number of red straws? 	Discussing relationship between yellow and red straws. Showing awareness that the yellow straws will give a larger number than the red straws. Possibly being able to predict that the number of yellow straws will be double the number of red straws.	To assess learners' understanding of unit and the relationship between units and referent number in measure.	Acting as an assessment of learners' understanding of relationship between unit and referent number in a measure. Red straw measures 10cm and yellow straw measures 5cm. All sticks multiples of 10cm. Restrict number of red and yellow straws available to necessitate iteration and possible prediction of yellow straws. Ask one partner to use red straws and other to use yellow.

Task	**Summary of expected learner activity**	**Purpose**	**Design notes**
1c How many flapjacks? Questions: Here is recipe for flapjacks. One cup makes five flapjacks. How can I find out how many flapjacks I can make from this container of oats? 	Discussing how to find out how many flapjacks can be made from the bag. Showing awareness that each cup represents 5 flapjacks.	To assess learners' understanding of a composite unit.	Acting as an assessment of understanding of a composite unit. Have enough cups so that each cup can be filled for visual representation.

Task	Summary of expected learner activity	Purpose	Design notes
2a How many rabbits? Questions: This little cup is enough water for one rabbit for a day. I want to find out how many rabbits I can feed with this amount of water (in the jug)? How could I do this? Is there a quicker way? 	Discussing how to find out how many rabbits can be fed and whether there may be a quicker way of finding out. Recognising (with support) an equality relationship between a little cup and an intermediate unit. Using the composite unit to find out how many rabbits can be fed from a jug.	To introduce notion of intermediate unit	10 little cups = 1 big cup. This relationship will be established together, by counting the number of times the little cup needs to be filled and how many cups fill the intermediate unit. Learners are then asked to find out how many rabbits can be fed with a jug of water but they are not given access to the little cup.

Task	Summary of expected learner activity	Purpose	Design notes
2b How much porridge? Questions: This container contains enough oats for one person to have a portion of porridge. How many portions of porridge are in this bag? Is there a quicker way of finding out (intermediate unit cup available)? 	Discussing how to find out how many portions of porridge are in the bag, building on 2a. Using intermediate unit to work out how many portions of porridge there are.	To reinforce concept of intermediate unit	3 little containers = 1 cup. The relationship is established together as a group. Learners are then asked to find out how many portions of porridge are in a bag (12 portions).

Task	Summary of expected learner activity	Purpose	Design notes
3a Making lengths without using single centimetres. Questions: If this is 1cm, what might these lengths be (show straws)? How do you know? If I make a line 20cm long, how many 2cm will I need? How many 5cm will I need? What if you make a line 40cm long, or 60cm long? 	Discussing and establishing lengths of green, yellow and red straws. Exploring relationship between 20cm, 10cm, 5cm and 2cm. Making lines 20cm, 40cm and 60cm long and finding out how many 2cm, 5cm and 10cm straws are equal to these lengths.	Reinforcing the use of composite unit, restriction of counting in single unit. To make links with standard units of measure.	Making lengths as multiples of 10cm encourages consideration of multiplicative relationship as learners will need to establish how many red straws are needed. They are then asked to work out how many yellow and green straws are needed to make the same lengths. For the longer lengths (40cm and 60cm, insufficient numbers of greed and yellow straws are available so learners will need to predict).

Task	Summary of expected learner activity	Purpose	Design notes
3b How much medicine? Questions: My dog needs 10 millilitres of medicine each day. This spoon is worth 10 millilitres. I want to find out how many spoons worth of medicine is in this bottle. How could I do that? Is there a quicker way than counting spoons? 	Suggesting ideas for a quicker way of counting spoons. Recognising an intermediate unit (bottle) could help. Finding out how many spoons worth are in the bottle by using the intermediate unit.	To reinforce the notion of an intermediate unit. To make links with standard units of measure.	5 10ml spoon = 1 50ml bottle. There are 9 50ml bottles worth in the big bottle. Relationship between spoon and little bottle is established as a group. Learners are then asked to find how many spoons worth are in the big bottle, but the use of the spoon is restricted, necessitating counting the bottle as equal to 5 spoons.

Task	Summary of expected learner activity	Purpose	Design notes
4a Exploring relationships between different masses Question: How many 1g weights are the same as these weights? How many 5g weights are the same as these (10g, 20g). 	Exploring relationship between 5g, 10g and 20g masses. Recognising that it is easier to weigh in multiples of 5g, 10g or 20g.	To reinforce use of composite unit. To establish relationship between 1g, 5g, 10g and 20g.	1g will only be used to introduce what 1g feels like as a mass/weight. Once the relationship between 1g and other weights is established, its use will be restricted.

Task	Summary of expected learner activity	Purpose	Design notes
4b How many portions of pasta? Questions: 10g of pasta is needed for one portion of pasta soup. How many portions of pasta coup could be made from these bags? How could you find out? 	Suggesting ways to find out how many portions of pasta can be found. Recognising that the weight can be established through use of a composite unit.	To use composite units as a measure.	The use of 1g will be restricted. Learners are restricted to using one particular composite unit (5g, 10g or 20g each time).

APPENDIX S: PRACTITIONER OVERVIEW OF TASKS (2) WITH LEARNER RESPONSES

Task	**Summary of expected learner activity**	**Purpose**	**Design notes**	**Learner responses (overview)**
C2a.1a Making same quantity of liquid Questions: Here is some red liquid and here is some yellow liquid. Can you make the same amount of yellow liquid as red liquid in this container? How will you be sure that you have the same amount of liquid? 	Suggesting ideas for how they can ensure the same amount of yellow liquid as red liquid. Showing awareness of a need to quantify/measure the amount of red liquid in order to reproduce the same amount of yellow liquid.	To assess learners' understanding of concept of unit.	Acting as an assessment of learners' understanding of unit. Restriction on pouring red liquid directly into container. Different shape container for red liquid to necessitate quantification.	Learners seemed excited to be using liquids. e.g. *'I love this', 'Wow'* They seemed keen to call the liquids potions. Learners found the initial challenge difficult, e.g. '*I don't know what we are going to do'.* They needed lots of encouragement to share ideas. One learner suggested measuring the levels *'What we could do is put the two bottles next to each other and measure the sides.'* – which allowed discussion of the level

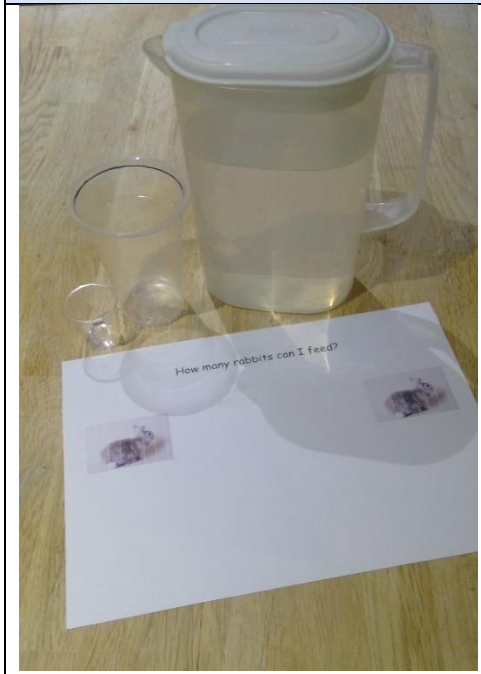
Task	**Summary of expected learner activity**	**Purpose**	**Design notes**	**Learner responses (overview)**
				being different in different sized containers (i.e. heights would be different). Although the small cups were visible, learners needed their attention drawn to them. Then the learners began to suggest ideas, showing awareness of cup as unit: *'Put the red liquid in the cups and and put the yellow liquid, in the cups and have the same amount in both'* *' I think I know, I think I know...On those cups there's lines…Maybe that's how far you need to go to put the yellow liquid in'* They then managed the task, e.g. '*Keep on track*

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
				with this, so every time I pour this we'll count' There was indication of understanding of the need for units being counted to be of equal size *'Miss I made it up to the line'*
C2a.1b Using straws to measure Questions: Here are some straws – red straws and yellow straws. Do you notice anything about the relationship between the straws? If you measure with the red straws and then also measure with the yellow straws, how will your answers be different? Can you measure these sticks with both the red straws and the yellow straws? Could you predict what the number of yellow straws would be if you knew the number of red straws? 	Discussing relationship between yellow and red straws. Showing awareness that the yellow straws will give a larger number than the red straws. Possibly being able to predict that the number of yellow straws will be double the	To assess learners' understanding of unit and the relationship between units and referent number in measure.	Acting as an assessment of learners' understanding of relationship between unit and referent number in a measure. Red straw measures 10cm and yellow straw measures 5cm. All sticks multiples of 10cm. Restrict number of red and yellow straws available to necessitate iteration and possible	Learners needed some initial encouragement to recognise the half-double relationship between the straws *'They can be the same size if they touch'.* When using the straws to measure, some learners began to recognise the relationship between the resulting measure.

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
	number of red straws.		prediction of yellow straws. Ask one partner to use red straws and other to use yellow.	*'There's a relationship!' Because...if you add four on, there's eight'.* Learners did not consistently use the half-double relationship to predict or check their resulting measures, though this is possibly due to the way the task was implemented.
C2a.1c How many flapjacks? Questions: Here is recipe for flapjacks. One cup makes five flapjacks. How can I find out how many flapjacks I can make from this container of oats? 	Discussing how to find out how many flapjacks can be made from the bag. Showing awareness that each cup represents 5 flapjacks.	To assess learners' understanding of a composite unit.	Acting as an assessment of understanding of a composite unit. Have enough cups so that each cup can be filled for visual representation.	Learners quickly accepted that one cup represented 5 flapjacks worth of oats. When asked to suggest how they could find out how many flapjacks could be made from the container, they initially made guesses, which all reflected an understanding of multiples of five. *'I think we can make twenty'*

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
				'I think about forty' *'I think like sixty or fifty'.* When prompted learners then started to suggest how they could find out: *'We could get lots of cups and then we could fill it up until we get all of the oats... Then we can count how many* *'You can count in fives'* Although learners generally managed the task well, recognising that each cup would represent five flapjacks, one pair believed they needed to use all the cups they were given, and tried to fill all those cups (unevenly).

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
C2a.2a How many rabbits? Questions: This little cup is enough water for one rabbit for a day. I want to find out how many rabbits I can feed with this amount of water (in the jug)? How could I do this? Is there a quicker way?	Discussing how to find out how many rabbits can be fed and whether there may be a quicker way of finding out. Recognising (with support) an	To introduce notion of intermediate unit	10 little cups = 1 big cup. This relationship will be established together, by counting the number of times the little cup needs to be filled and how	Learners initially suggested the little cup: *Keep on doing it… Pouring it. You have to have lots of little cups and fill them up to the top.*

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
	equality relationship between a little cup and an intermediate unit. Using the composite unit to find out how many rabbits can be fed from a jug.		many cups fill the intermediate unit. Learners are then asked to find out how many rabbits can be fed with a jug of water but they are not given access to the little cup.	*The things we did yesterday…Pour water into the little cups and count how many* When prompted with a question about how long this might take, a learner said: *… it's going to take forever* Learners started to suggest that the bigger cup could be used *I know! I know one. We could keep on filling the big cup until all the water has gone.* Learners needed some prompting to recognise that they would need to find out how many little cups filled the bigger cup but after

Task	**Summary of expected learner activity**	**Purpose**	**Design notes**	**Learner responses (overview)**
				discussion and the establishing of a relationship between the small cup and the intermediate cup through demonstration, they quickly accepted the relationship. When discussing the use of the intermediate cup there was recognition of its value as a composite unit. *Bring the water up to the black line.* *Learner 2: And then you'll know that it's worth the same* (as ten little cups)

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
C2a.2b How much porridge? Questions: This container contains enough oats for one person to have a portion of porridge. How many portions of porridge are in this bag? Is there a quicker way of finding out (intermediate unit cup available)? 	Discussing how to find out how many portions of porridge are in the bag, building on 2a. Using intermediate unit to work out how many portions of porridge there are.	To reinforce concept of intermediate unit	3 little containers = 1 cup. The relationship is established together as a group. Learners are then asked to find out how many portions of porridge are in a bag (12 portions).	They successfully suggested a quantity of little cups by using the intermediate unit and were able to cope with spills by adjusting to a sensible estimate (e.g. 39 rather than 40 when a cup wasn't completely full) Learners struggled more with this task than the similar rabbit task – partly because the same sized cup was used, which they had already associated with ten little cups, even though the little container was a different capacity. The establishing of the relationship between the little container and the intermediate cup was quicker and involved fewer learners.

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
				Some learners did recognise the relationship: *I did six people* *I can feed nine people* *I was counting in threes* The task was messy! Part of the difficulty was because of the design (same cup) and sequencing (same day in use of same cup).
C2a.3a Making lengths without using single centimetres. Questions: If this is 1cm (share 1cm straw), what might these lengths be (show straws)? How do you know? If I make a line 20cm long, how many 2cm will I need? How many 5cm will I need? What if you make a line 40cm long, or 60cm long? 	Discussing and establishing lengths of green, yellow and red straws. Exploring relationship between 20cm, 10cm, 5cm and 2cm. Making lines 20cm, 40cm and 60cm long and finding out	Reinforcing the use of composite unit, restriction of counting in single unit. To make links with standard units of measure.	Making lengths as multiples of 10cm encourages consideration of multiplicative relationship as learners will need to establish how many red straws are needed. They are then asked to work out how many yellow and green straws are needed to make the same lengths. For the	Learners knew that length was typically measured in cm though some were not able to suggest what 1cm looked like. They quickly established 2cm and then suggested 3cm or 4cm for the 5cm straw but then showed awareness of why it was 5cm. They could

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
	how many 2cm, 5cm and 10cm straws are equal to these lengths.		longer lengths (40cm and 60cm, insufficient numbers of green and yellow straws are available so learners will need to predict).	then predict the length of the red (10cm straw) *It's ten centimetres* *The little yellow ones. Five and five makes ten.* Learners referred to the straws by their lengths and calculated how many 2cm and 5cm straws were in 20cm, 40cm (and one 60cm) and wrote these as multiplicative relationships. Learners complained about the 'little' 2cm straws being annoying, which supported a point about the use of bigger units.
C2a.3b How much medicine? Questions: My dog needs 10 millilitres of medicine each day. This spoon is worth 10 millilitres. I want to find out how many spoons worth of medicine is in this bottle. How could I do that? Is there a quicker way than counting spoons?	Suggesting ideas for a quicker way of counting spoons. Recognising an intermediate unit	To reinforce the notion of an intermediate unit. To make links with	5 10ml spoon = 1 50ml bottle. There are 9 50ml bottles worth in the big bottle. Relationship between spoon and	There was a suggestion of understanding of intermediate unit use being quicker: *Oh I've got it! We work out it to there, pour some in there, then*

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
	(bottle) could help. Finding out how many spoons worth are in the bottle by using the intermediate unit.	standard units of measure.	little bottle is established as a group. Learners are then asked to find how many spoons worth are in the big bottle, but the use of the spoon is restricted, necessitating counting the bottle as equal to 5 spoons.	*work out how many in there.* Learners were able to work out how many little bottles were in the big bottle and how many spoons that would be – through counting in fives but showing recognition of composite unit. *I've done forty-five* There was an option of developing the concept of millilitres further (one spoon was introduced as 10 millilitres) and then reinforcing a little bottle was 50ml but this was not explored with the groups.
C2a.4a Exploring relationships between different masses Question: How many 1g weights are the same as these weights? How many 5g weights are the same as these (10g, 20g). In partners use type of weight each time (each partner to use a different weight but only one type) to make the scales balance.	Exploring relationship between 5g, 10g and 20g masses. Recognising that it is easier to	To reinforce use of composite unit. To establish relationship	1g will only be used to introduce what 1g feels like as a mass/weight. Once the relationship between 1g and	Learners were keen to use the pan balances and seemed to enjoy exploring their use and the use of the masses (weights).

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
	weigh in multiples of 5g, 10g or 20g.	between 1g, 5g, 10g and 20g.	other weights is established, its use will be restricted.	They needed encouragement to keep the same colours (and therefore encourage the multiplicative relationship). The balancing of scales using did encourage multiplicative relationships. *'I have two twenty grammes'* *'Five, ten, fifteen, twenty, twenty-five, thirty, thirty-five, forty' and I have 8 five grammes'* (And they are equal) As a starter activity this worked well in encouraging multiplicative relationships, learners began to record their findings of

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
C2a.4b How many portions of pasta? Questions: 10g of pasta is needed for one portion of pasta soup. How many portions of pasta coup could be made from these bags? How could you find out? 	Suggesting ways to find out how many portions of pasta can be found. Recognising that the weight can be established through use of a composite unit.	To use composite units as a measure.	The use of 1g will be restricted. Learners are restricted to using one particular composite unit (5g, 10g or 20g each time).	multiplicative relationships. Learners struggled a bit with this task initially. One learner tried to count the pasta despite the demonstration of working out what 10g and 20g of pasta looked like. Though some learners recognised the need to find out many 10g there were, they initially suggested portioning into 10g portions to work out how many people could be fed. Nevertheless, learners did use the multiplicative relationships, e.g. *'OK I am putting in three twenty 'gallons' (meant grammes) and*

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
				now I am going to try with bag B' *'It will feed six people'*
C2b1a Using straws to measure (Similar to C2a.1b) Questions: Here are some straws – red straws and yellow straws. Do you notice anything about the relationship between the straws? If you measure with the red straws and then also measure with the yellow straws, how will your answers be different? Can you measure these sticks with both the red straws and the yellow straws? Could you predict what the number of yellow straws would be if you knew the number of red straws? 	Discussing relationship between yellow and red straws. Showing awareness that the yellow straws will give a larger number than the red straws. Possibly being able to predict that the number of yellow straws will be double the number of red straws.	To assess learners' understanding of concept of unit.	Acting as an assessment of learners' understanding of relationship between unit and referent number in a measure. Red straw measures 10cm and yellow straw measures 5cm. All sticks multiples of 10cm. Restrict number of red and yellow straws available to necessitate iteration and possible prediction of yellow straws. Ask one partner to	Learners quickly suggested that the smaller (yellow) straw would give a higher referent number. *Learner 9: The yellow sixteen* *RW: So I used one straw eight times and I used one straw sixteen times* *Learner 9: The small one sixteen* One learner did suggest the opposite: *Learner 10: I think it's the red straw because the red straw can go on sixteen, because it's longer*

Task	**Summary of expected learner activity**	**Purpose**	**Design notes**	**Learner responses (overview)**
			use red straws and other to use yellow.	Although learners did seem to recognise relationship, they did not explicitly articulate it: *Learner 9: Obviously a yellow straw would be four because two of them would make...* *RW: Ah so two of the yellow straws make the same length as the red straw don't they* *All learners try to speak* *Learner 10: And one...* *Learner ?: And that's four* *Learner: The yellow straw is bigger than that yellow straw* *Learner 10: And one red straw is four* *Learner: Eight, four, four* *Learner 10: So this is four, because if that*

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
C2b.1b Making lengths without using single centimetres. (Similar to 2a.3a) Questions: If this is 1cm, what might these lengths be (show straws)? How do you know? If I make a line 20cm long, how many 2cm will I need? How many 5cm will I need? What if you make a line 40cm long, or 60cm long? 	Discussing and establishing lengths of green, yellow and red straws. Exploring relationship between 20cm, 10cm, 5cm and 2cm. Making lines 20cm, 40cm and 60cm long	Reinforcing the use of composite unit, restriction of counting in single unit. To make links with standard units of measure.	Making lengths as multiples of 10cm encourages consideration of multiplicative relationship as learners will need to establish how many red straws are needed. They are then asked to work out how many yellow and green straws are needed to make the same lengths. For the longer lengths (40cm and 60cm, insufficient numbers of greed and yellow straws	*was eight, this would be four… If that was...six, that would be three and if that was ten, that would be five. Half of everything is the yellow straw* When asked how learners worked out the 'other' colour straw: *RW: So Learner 12, tell us how you knew how to get the right number of yellow straws* *Learner 12: I just, I just doubled seven* Learners showed awareness of

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
	and finding out how many 2cm, 5cm and 10cm straws are equal to these lengths		are available so learners will need to predict).	standard units for length, although initially unsure what '1cm' might look like. They were able to suggest lengths of straws e.g.: Green (2cm): *Learner 10: Um half of this is one centimetre* It took longer to establish the length of the red (10cm) straw, even though learners recognised it would be equivalent to 5 green (2cm) straws. Once the length of the 10cm straw was established, learners suggested the yellow straw would be 5cm:

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
				RW: How could you find out Learner 11? *Learner 11: Um* *Learner 12: I know. I know a good way. You get two of these and put them right next to each other.* *Learner 11: Yes.* Learners worked on finding relationships though did make comments about the straws moving about, e.g. *Learner 11: One wrong move can ruin this* Learners worked quite confidently with the 2cm, 5cm and 10cm straws. *RW: That's forty centimetres isn't it.*

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
				So how many red straws. How many green straws and how many yellow straws. That's it Learner 10. *Learner 12: I'm going to need to get twenty.* In commenting on what they had learned, learners commented on learning what a cm looked like.

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
C2b2a Exploring relationships between different masses (Similar to 2a4a) Question: How many 1g weights are the same as these weights? How many 5g weights are the same as these (10g, 20g). Can you identify multiplicative relationships between them?	Exploring relationship between 5g, 10g and 20g masses. Recognising that it is easier to weigh in multiples of 5g, 10g or 20g.	To reinforce use of composite unit. To establish relationship between 1g, 5g, 10g and 20g.	The use of 1g will be restricted. Learners are restricted to using one particular composite unit (5g, 10g or 20g each time).	Although some learners were able to suggest grammes and kilogrammes as a measure of mass/weight, they were surprised at how light a 1g was: *Learner 9: Really light* *Learner 10: Light* *Learner 9: I can hardly feel it* Once the use of pan balances was established (some learners seemed familiar with them and others did not), learners worked to establish relationships, e.g.: *Learner 10: Right, I'm doing five* *RW: So now how many five grammes is*

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
				*the same as twenty grammes?* *Learner 9: Miss you could put...uh..four five grammes* *RW: Right so Learner 10, forty grammes, you make forty grammes again, so you make forty grammes again, so you make forty grammes first...is equal to how many five grammes?* *Learner 10: Um...* *Learner 12: Wait, wait take some out...it's equal* *Learner 11: And how much is that?* *Learner 12: That is...one, two, three...eight*

Task	**Summary of expected learner activity**	**Purpose**	**Design notes**	**Learner responses (overview)**
C2a4b How many portions of pasta? Questions: 10g of pasta is needed for one portion of pasta for a baby. How many babies could be fed from these bags? How could you find out?	Suggesting ways to find out how many portions of pasta can be found. Recognising that the weight can be established through use of a composite unit.	To use composite units as a measure	The use of 1g will be restricted. Learners are restricted to using one particular composite unit (5g, 10g or 20g each time).	Some learners suggested counting pasta to find out what 10g might look like. When asked how they might find out how many 10g portions were in a bag, learners suggested portioning: *Learner 9: And then we could put like, try and put one portion there, and portion there, so we know that's one portion for one baby and the other portion for another baby and then we could like keep on doing that* *RW: Ah, you could keep on doing it*

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
				*Learner 9: And count all of the bags* *Learner 13: You can take this out and then we pour some more in here because we know that's ten grammes, so wait until it gets equal again, then put that into a pile and then leave that into it and then pour a bit more in until it reaches the middle again and keep on doing that and then we could find out how much groups* It took some time and input to suggest finding the mass of the bag of pasta.

Task	Summary of expected learner activity	Purpose	Design notes	Learner responses (overview)
				Once this had been established learners did use the masses to work out how many portions of pasta, e.g.: *RW: So how many babies will that feed?* *Learner 12: Nine* *RW: Do you agree Learner 11? How many babies does that feed?* *Learner 11: Nine* *RW: And how much does it weigh?* *Learner 11: It weighs ninety grammes*