

Improving students' mathematical understanding by engaging with the mathematical structure

Fiona Richards

Supervised by:

Dr Jane Waters–Davies

Dr Susan Jones

Professor Anne Watson

Submitted in partial fulfilment for the award of the degree of PhD

University of Wales Trinity Saint David

2023

DECLARATION

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed

(candidate)

Date 11th August 2023

STATEMENT 1

This thesis is the result of my own investigations, except where otherwise stated. Where correction services have been used the extent and nature of the correction is clearly marked in a footnote(s). Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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STATEMENT 2

I hereby give consent for my thesis, if accepted, to be available for deposit in the University's digital repository.

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Acknowledgements

I met Dr Howard Tanner at the Institute of Mathematics Pedagogy in 2015 and he encouraged me to undertake this degree. It was his belief in me and my project and his encouragement that enabled me to start. His death, early in the project, was a tragedy and I miss his input and his approach greatly. Thank you to Dr Jane Waters-Davies for supervising me from the start to the finish of this project. Your patience and support have been invaluable. Thank you to Professor Anne Watson for stepping in as my advisor after Howard's death. You are an inspiration in the field of mathematical education, and I feel privileged to have worked with you. I have learned so much from all the three of you. Thank you too, to Dr Jan Barnes who started as my second supervisor and encouraged me greatly. I am grateful to my PhD support group Helen Denny, Libby Chancer, Rachel Wallis and Charlie Hosking for all their support. Our face-to-face days made such a difference to the quality of my research. The project would not have been possible without the support of the leadership, staff and the students of the two study schools.

Last, but definitely not least, I need to thank my family for their patience while I have been writing the PhD. I am grateful to them for listening to me explain my ideas and proofreading sections.

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Abstract

This study investigates whether teaching using the underlying structure of mathematical concepts improves students' understanding. This case focused on fractional magnitudes. For this study, instantiations are defined as mathematical concepts considered from the viewpoint of the learner. Mathematical structure consists of the instantiations alongside their properties.

The pragmatic approach explores the research aims using qualitative and quantitative methods. A series of interventions was designed with Year 7 students. The interventions focussed on teaching students about fractional magnitudes; an area of mathematics vital for much of the secondary mathematics syllabus. The approach was cyclical, with three interventions. The design for each intervention was informed by the specific mathematical ideas that the students had misunderstood in the previous intervention.

Although students' understanding of fractional amounts had improved at the end of the first intervention, the post-test and interview indicated a persistence of incorrect responses, which were automatic and accompanied by mantras. The second intervention included tasks to try to address these intuitive responses. The third intervention focussed on intuitive responses and language around fraction magnitude.

The first intervention investigated the mathematical thoughts and understanding of students. The second and third interventions allowed for in-depth investigation of specific incorrect intuitions and perceptions of mathematical structure from the viewpoint of these learners. This cyclical approach exposed specific incorrect intuitive responses that appeared to be persistent.

The study demonstrates a process of designing teaching episodes interpreting both the mathematical structure and the student errors through the students' viewpoints. The results indicate ways that teachers can address intuitions in their classroom. The findings suggest that the use of mantras, helpful in teaching early mathematics, may be a basis for intuitive errors that persist. Exploration of a possible link between a learner's general mathematical difficulties and dependence on intuition could be an area for future research.

Chapter 1

1 Introduction

1.1 Motivation

Whilst teaching, I have watched lessons and used published resources in which each concept, each part of mathematics, is considered discretely. Links between topics are explicitly made, but often as an afterthought, more ‘what a coincidence’ than ‘this is the structure of mathematics’. An example is the drawing of pie charts. In my experience students are often offered a method to finding the angle in the chart which does not explicitly tell them that what they are doing is using direct proportion. Sometimes the way the specific idea is taught also masks the connections. For example, when teaching reverse percentages¹ it is often taught with its own method rather than the inverse of finding a percentage increase. Therefore, for example, to find an increase of 12% requires multiplying by 1.12 but to find the original amount before the increase requires use of a unitary method find 1% and then use that to find 100%. To a competent mathematician the two methods are the inverse of each other, but unless students specifically engage with that fact it is perhaps not obvious.

Askew *et al.* (1997) reported that students made more learning ‘gains’ where the teachers held a strongly connectionist orientation. These teachers concentrated on making links and connections between mathematical ideas and methods.

¹ Reverse percentage questions are ones where students are asked to find the value before a given percentage change. E.g. Sally buys a house. She sells it for £300 000 having made a 20% profit. What did she pay for the house?

As the National Council of Teachers of Mathematics (NCTM) reported:

When students connect mathematical ideas, their understanding is deeper and more lasting, and they come to view mathematics as a coherent whole (NCTM, 2000, p. 4).

1.2 The school

During the course of this study, I have worked in two schools. When I started the study (February 2016) I was second in the mathematics department in School A. School A is a smaller than average secondary school in a rural area. The school is in an area of low deprivation, classified as being in bottom 10% of deprived schools. Only 17% of students attract the pupil premium, the funding paid to schools based on the numbers of students eligible for free school meals. However, a large percentage of the students live in families subsisting just above the poverty line. When the local authority released funding for holiday support for those families who were experiencing hardship, but who were not eligible for free school meals, a further 23% of the cohort were eligible for that funding. The school also lies between two large army bases and over 40% of the intake are students from service families. 29% of the students are considered to have special educational needs and or disabilities (SEND). The majority of the intake describe their ethnicity as White-British and the vast majority have English as their first language. The area has a grammar school system and therefore the intake is skewed towards students who have attained less well in their Key Stage 2 tests. Every year the results in Year 11 are near the government's floor standards, meaning that attainment and progress of the Year 11 cohort is close to the point where the Department for Education becomes concerned and considers whether to intervene. For school years 2015/16 to 2017/18 the results were measured by the progress 8 (P8) measure and were well below the national average, putting

attainment and progress in the bottom 10% of schools nationally. P8 compares the progress of students in a school against a national picture. Average progress gives a P8 of 0. An overview of how it is calculated is given in section 1.7. An Ofsted (the national schools inspectorate) visit in June 2019 categorised the school as 'requires improvement', this was based on the results from the three previous years, although when the results came in from the exams taken during the Ofsted visit, a marked improvement meant that the school's results were broadly average. Two years of disruption due to the COVID-19 pandemic meant that there were no published results for 2019/2020 or 2020/2021. In June 2022, exams were sat again for the first time since 2019 and the results were in line with national average; P8 was -0.08. During the time of the study there have been six Head Teachers with another Head Teacher due to start in September 2023. A recent (February 2023) Ofsted visit still classified the school as 'requires improvement'. Because the school has received two successive 'requires improvement' classifications the school is now classed by the Department of Education as a 'Coasting School', and we have had to assure the Department for Education that there was a plan to ensure that our next Ofsted visit will result in a good classification. If these assurances do not work, then the Department for Education has the power to move the school from our current trust to another one that they believe would be more successful.

In September 2016, I moved as Head of Mathematics to school B. School B is a larger school in a coastal area. It is situated in an area of high deprivation, with 40% of its students attracting pupil premium funding from the government. Also, in a grammar school area, the Year 7 intake had a similar academic profile to the students at school A. It had been a school under significant support from the government, due to failing to meet educational standards and having poor results

but had improved the results to the point where P8 was at the national average in 2016. School B had a rising roll, and, in most year groups, was full (180 students per year) with a waiting list.

In December 2016, school A lost its Head Teacher and the Acting Head Teacher asked me to return to lead the mathematics department and to take up a position on the senior leadership team. Therefore, in April 2017, I returned to school A. As the study had been designed initially for school A this did not mean a disruption to the study. The only difference was that the assessment instrument was piloted in a different school. During the school years 2016/17 and 2017 /18 the mathematics department was not fully staffed; many lessons were taught by supply teachers and a higher level teaching assistant (HLTA). The teaching assistant had been solely allocated to the mathematics department as she had a mathematics A level. She had no further teaching or mathematics training. I mentored her through the year and encouraged her to undertake more training. From September 2018, I was no longer Head of Mathematics but had been promoted to Assistant Principal and, in January 2019, to Vice Principal. I still taught in the classroom, but my timetable was much reduced, and my lessons focused on Key Stage 4 (Year 10 and 11) and not Year 7, the cohort my intervention was aimed at. After the departure of the Head of Mathematics in July 2019 and a failure to recruit a replacement, I temporarily took on the Head of Mathematics role on top of other roles. Although this was intended to be short term (three months), financial difficulties and a falling roll meant that the Head of Mathematics was not replaced until September 2022.

School A has changed during the course of the study both in size and in outlook. In 2018/19 the school roll was just over 400 students (nearer 650 at the start of the project). At the start of the project there were five classes in each year group. In 2018/19 this had fallen to three. By this time each year group had under 100

students. During 2016/17 staff, student and parental confidence in the school dropped. The school's reputation in the local community became poor. Parents moved their students to other schools and new entrants into Year 7 dropped from over 120 to under 70. There was a change in Head Teacher in September 2017 and the reputation of the school in the community improved again, though this was a slow process. Despite the improvements to behaviour, culture and academic achievement, the school still had a falling roll with fewer than 400 students in 2021. At the start of 2021 the school trust board announced that the sixth form would be closing at the end of the academic year. Students in Year 12 were found places at other sixth form providers and, in September 2021, the school became an 11-16 school. In May 2022, the multi-academy trust that the school was part of was taken over by a larger multi-academy trust, so that there were 17 schools in the trust with the school A being the smallest of three secondary schools. The make-up of the senior leadership team changed significantly over the months surrounding the takeover, including four different Headteachers. In February 2023, the Acting Head of School was stepped down and I was given daily operational control of the school to work alongside the Executive Headteacher. In January 2023, the school had 450 on roll and is projected to exceed 500 in September 2023.

Before the start of the project the results in mathematics had dropped significantly over time. In 2016/17 only 41% of students gained at least a grade 4 in mathematics. In 2017/18 this had increased to 58% - much closer to the national average of 65% (Department of Education, 2018). As a school there was a focus on improving the results to bring them in line with national average. To do this there was a focus on year 11 – ensuring they were making the progress towards

their exams. Other year groups could not be ignored, however, and improvements needed to be made to teaching and learning in all year groups in all subjects.

Within this context, my project had to be allied with the school's aims. I believed that by improving the mathematical understanding of students there would be a long-term gain for students and for the school. Despite all of the institutional changes and difficulties I created ways to give time, effort and thought to my research.

In the Year 7 for 2017/18 there were three classes with set 1 being the one for the highest attaining students. This was a very small year for the school, only just over half the size of year 9. This was the year group of the participants in my first intervention.

1.3 Initial Thinking

As a teacher at the start of my career, I always wanted to ensure that my classes knew more than just how to do something. I had come to the realisation that this was what I wanted in my classroom, from my own experience as a mathematics student in school. As a student, I rarely felt that I had to learn mathematics, because I usually worked things out. For example, I did not learn the matrix transformations I needed to use. Instead, I used a unit square to work them out. Classmates would say that it was hard to remember everything. My thought, as a student, had been that if I knew concepts with a deeper understanding then I would have less to learn. One girl getting ready for her mathematics exam at age 16 declared that she had never been taught how to find the area of a triangle. At the time I thought that this was a failing of the education system, failing to see that it was the same system that had taught me how to find the area of a triangle. I thought that she had not been taught it, rather than the more likely, that she had

been taught it: but had forgotten. It became clear to me, that perhaps I had a 'thing' for mathematics and later I realised I had a very good memory. I remembered individual lessons and so, if I remembered the lesson and why we were doing something, the 'how' tended to follow. I realised that my experience of school mathematics was different from many students'.

When I took up my first teaching post, I found that behaviour management and motivation were factors for which I had not prepared enough. I trained at a grammar school and my first job in a comprehensive (secondary modern) was going to require a different approach. I worked with the mathematics specialist for Wiltshire, and we developed the first few lessons for my classes. We developed lessons with lots of cutting up and seeing how things worked then applying them to the mathematics, sometimes using unusual manipulatives to illustrate ideas. By any measure, my first few lessons were a disaster. The students were not interested in what I was offering and were more likely to talk, get up and wander round the room. My classroom looked like a disaster area each afternoon. The advice of another teacher in the department was that I was trying too hard to make the mathematics enjoyable. This did get better as I learned more behaviour techniques but still my lessons were not as calm as other teachers' were and it was clear to me that the children did not value my lessons as much as they did the other teachers'. I observed other lessons and talked to the children about their experiences. A typical lesson that they enjoyed was often, 'Here is an idea (for example finding the percentage of an amount). Multiply the percentage by the amount and divide by 100.' Questioning was limited to the practice of the same method. The students worked in silence and did 50 questions which were all correct, the summary discussion at the end of the lesson tended to reinforce the lesson and the children left feeling they had learned a lot. But in my observation,

nothing 'stuck' and it was accepted by the teaching staff that it was all going to have to be retaught again and again but that was considered to be acceptable because the school had a spiral curriculum in which, ideas were frequently revisited.

I still felt this was wrong. I took over a class who vociferously told me that they had never seen an idea before and yet they had done the exact same exercise in their books three weeks before with their previous teacher and were amazed when I pointed this out to them.

One teacher said that I was trying too hard and that you needed to teach 'how' before 'why', and from the 'how' the 'why' evolves. Although the literature demonstrates that conceptual knowledge supports the development of procedural knowledge, this is not uni-directional. There is literature also agreeing with my colleague's statement that procedural knowledge can also support the development of conceptual knowledge (Rittle-Johnson *et al.*, 2001, 2015).

My lessons improved and I managed the motivation and the behaviour but still the students appeared to find the lessons with other teachers more interesting and engaging. In these lessons, the teachers appeared to concentrate on procedural competency and fluency. I had one particular class that proved to be quite difficult, and my head of department said to me that I should not be despondent. She believed that the reason they were difficult for me was that I made them think and she believed that they did not want to work that hard. She asked me to observe them with the other mathematics teacher who also taught them, and I saw that with him they were told a method and practised it. Their lessons were not cognitively challenging, but their behaviour was good and there was a veneer of learning.

I received an email, out of the blue, from an initial teaching training (ITT) student who had been asked to contact me by her university tutor who believed that what he had seen when visiting me with another PGCE student was a desire to teach relationally rather than instrumentally. At the time I did not know these terms so a quick Google search brought up Skemp (1976) and this was my first introduction to mathematics education literature and a feeling that my thoughts about teaching might be validated.

I realised there was something being missed when students in Year 11 (aged 15 or 16), having had years of using numbers and decimals, were still unable to answer questions in exams that I considered should have been simple. Asked to write $\frac{3}{5}$ as a decimal many would write 0.35 or 3.5. Pearson Edexcel is one of the accredited GCSE examination boards in England, and the GCSE exam often asks students to write a fraction as a decimal. An analysis of the examination reports from Edexcel says that the type of mistake I regularly saw was common nationally. In two separate years, candidates in the Edexcel GCSE paper were asked to write $\frac{1}{4}$ as a decimal, with common answers being 0.4, 1.4, 2.5 and 25. Quotes from the examiner's reports illustrate the point.

About 50% of candidates gave 0.25 as the decimal equivalent to $\frac{1}{4}$ in part (a). Incorrect answers included 1.4 and 0.4. (Pearson, 2011)

the most common incorrect decimal given for $\frac{1}{4}$ was 0.4 Other incorrect answers involved the incorrect placement of the decimal point giving 2.5 or 25 and some candidates gave 1.4 (Pearson, 2012)

Each year, when I have reviewed mock papers with year 11 students, they have often made similar mistakes. When talking to them about these answers a very simple line of questioning showed the mistake. Either students looked at the answer and asked, 'Why on earth did I write that?' or, if they did not see the

answer immediately, I asked them to compare both numbers to 1. These responses were one of the reasons why I have used the students' own response to classify errors in this study (see section 7.1.5). Whilst I am happy to accept that this error could have occurred because of the stresses caused by an examination situation, there still remained the question of why it was *that* answer and why that answer so commonly recurred.

1.4 MSc

My memories of my own schooling, the first years of teaching and the attitude that students did not appear to want to think were very unsatisfactory to me. As my ITT had been a Graduate Teaching Programme (GTP), effectively learning to teach on the job, and not a Post Graduate Certificate in Education (PGCE) my academic learning in the GTP year had been restricted to five days in university and two essays. I felt I wanted more and so I applied for the Masters in Learning and Teaching. I then felt that I was part of a community that believed, similarly to me, that there was more to mathematics teaching than just transmitting a method. I also felt, though, that everyone else was far further down this journey than I was. Their descriptions of lessons appeared to include motivated students who were fully involved in exciting conversations about mathematics. I recognise now that this probably said more about my attitudes than it does about their lessons.

My MSc brought out three particular areas of interest for me, 1) the importance of scaffolding and the timing of the removal of support 2) how discussion affects the learning in the classroom and 3) how the different representations and moving between them aids a deeper understanding of the mathematics being taught. This final point was the focus of my second-year project where I looked at three different representations of trigonometry and how a student can select or move between the representation giving them an overall better understanding.

1.5 Topics in mathematics

During the 1970s the Concepts in Secondary Mathematics and Science (CSMS) project investigated children's errors in mathematics (Hart, 1981). In all, about 10 000 children were tested and then a small sample was interviewed about the tests. The tests were repeated for three years and so had a longitudinal element. The purpose of the study was to investigate understanding and errors of students in various mathematical topics from age 11-16. The topics that were selected were:- measurement; number operations; place value and decimals; fractions; positive and negative numbers; ratio and proportion; algebra; graphs; reflections and rotations; vectors and matrices (Hart, 1981).

The Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) study is an on-going study that has compared the results of CSMS with similar tests 30 years on and is now in a phase where schools are teaching using materials and lessons supplied by the ICCAMS team and these students are being tested to see if there is an improvement in attainment.

In reviews of mathematics teaching, it has been noted that teachers often teach to the test and the curriculum is covered in a piecemeal way such that there is little focus on the connections and links in mathematics (Brown *et al.*, 2012). Building connections and investigating the 'why' as well as the 'how' for mathematical methods was the key message from the report considering lessons in 2005-2007 (Ofsted, 2008). The focus in lessons on teaching procedurally is not new as can be seen from the Cockcroft report in 1982.

I know that I should not be teaching in this way, and I would much prefer not to do so, but I know also that I have a responsibility to do all that I can to make it possible for my pupils to obtain a CSE grade. (Cockcroft, 1982, p. 131)

This teacher's quote from the Cockcroft report resonates closely with me and reflects staffroom conversations I have had with many mathematics teachers, some 40 years after the report. It is part of my motivation for this project.

1.6 Research Questions

The title of the thesis is 'Improving students' mathematical understanding by engaging with the mathematical structure'.

My research question is:

Can the mathematical understanding of students be modified by encouraging students to engage with underlying mathematical structures?

I investigated whether selecting tasks which are designed to encourage students to interact with underlying mathematical structures help students understand the mathematics they are using more effectively. Rather than taking concepts and topics in mathematics as discrete parts of learning, I aimed to recognise and use the different links and representations between and within concepts to improve the students' understanding.

Increasingly through my teaching career I have wondered if the teacher, when planning teaching episodes, explicitly makes the links between different concepts and representations and provides the students with specific tasks which require the students to work with, compare and contrast these differences, then the students' understanding will improve. To be able to research this question it is important to clarify: -

- what structure is;
- what understanding is; and

- whether an improvement in understanding can be measured.

Although I felt I knew the answer to each of these, the concepts were vague and needed to be more carefully defined. I reviewed the literature on structure in mathematics and applied a novel definition which fits particularly with what I wanted to achieve. This is explored in section 3. The idea of understanding was more difficult to clarify as it is used in much of the research but not as a consistent idea. In my experience teachers will use the word 'understand' when a child can demonstrate a particular task: 'Jimmy understands Pythagoras' usually means 'Jimmy can answer some questions using Pythagoras' theorem'. It is not entirely clear how this is measurable, so I needed to investigate the literature surrounding understanding of mathematical concepts. This is reviewed in section 2.4.

The idea of using structure to teach mathematics is broad and I needed to narrow down the focus for my research. I selected areas of mathematics that I considered fundamental to the Key Stage 3 curriculum and that are taught in year 7. In particular, I selected topics which, from my experience, if not fully comprehended, lead to errors and misconceptions that persist through Key Stage 3 and Key Stage 4.

Another factor in the choice was how much mathematical structure could play a part in teaching these topics. Each topic can be understood superficially and can have algorithms applied but each also has an underlying structure. In addition, each topic is a foundation stone for more advanced work in the secondary mathematics curriculum. The areas I selected were a subset of the topics in the CSMS study (Hart, 1981). I selected place value and decimals, basics of algebra, basic fractions, and multiplicative reasoning.

1.7 National Context

School attendance (or education otherwise) is compulsory in England until age 16. From 16 to 18 young people should be in education or training, although this can include paid work with a significant element of training. The national context described here is for students who do attend school. Broadly, schools are split into primary, for children aged 4-11 years and secondary, for students aged 11-16 years. Each school year has a name: foundation, year 1, year 2 etc, with the last compulsory year of school being year 11.

Education, in England, is split into Key Stages, with the end of primary coinciding with the end of Key Stage 2. At the end of Key Stage 1 and Key Stage 2 there are national Standard Assessment Tests (SATs). Key Stage 3 is the first three years of secondary: years 7,8, and 9, and Key Stage 4 is years 10 and 11, when, in general, students are being prepared for public exams. The public exams are called the General Certificate of Secondary Education (GCSE) although there are vocational alternatives for some subjects. Students have some freedom to select which subjects to study and will typically take 8-10 GCSEs. Mathematics and English are compulsory and need to be taken by all students. GCSEs are graded 9-1 with 9 being the highest grade, awarded to approximately 3% of the cohort nationally. GCSEs are passed at either level 1 (grades 1-3) or level 2 (grades 4-9). There are two pass boundaries for a level 2 pass: grade 4 is a standard pass and grade 5 is a strong pass. If students do not pass English and mathematics at a grade 4, then they must also be offered courses in English and mathematics at Key Stage 5 (age 16-18) whether the students are in education, training or work. This focus on English and mathematics is also reflected in the fact that in the Government tables published to assess schools' performance and hold them to account, progress in English and mathematics GCSE is double weighted. Also, the

percentage of students that pass both English and mathematics at a grade 5 and at grade 4 are separate accountability measures for schools and are published nationally and schools are ranked on these scores. These measures mean that schools face high stakes accountability in terms of their students' mathematics attainment. This is considered further below.

The mathematics curriculum is mandated nationally by the government. It has three overarching aims for students: -

- to become fluent in the fundamentals of mathematics,
- to reason mathematically by following a line of enquiry,
- to solve problems by applying mathematics (Department of Education, 2013a).

Students in secondary schools in England are commonly taught in mathematics classes that are grouped by attainment, these groups are usually called sets and in general set 1 will be the highest attaining class. The aim is for students to be taught in a grouping with children of a similar level of mathematical attainment. It is not clear, however, that this grouping is fair or genuinely addresses the problems that it is designed to solve. Some schools will give students a test, will then rank them and then place them in the appropriate set. Evidence shows that once a student is placed in a set, they are very unlikely to move sets. Boaler notes a statistic from Dixon (2002, cited in Boaler (2005, p. 125)) that 88% of 4-year-olds placed in sets are still there at the end of their schooling. Without specific intervention, my experience also shows me that students with poor literacy or who are financially disadvantaged also frequent sets for lower attaining students. It appears to be common practice that lower sets are frequently taught by non-

specialists. These teachers may not be able to teach some of the more abstract concepts in higher GCSE or A-level. Although schools commonly use the phrase 'set by ability', Francis *et al.* (2017) discusses how problematic this phrase is. I will use the phrase 'set by prior attainment' as this is more indicative of the actual process though, in my experience, the setting process is frequently more affected by a judgement about the effort of the student. Francis *et al.* (2017) point to a general lack of clarity of how setting is arranged. Taylor *et al.* (2017) lists the detrimental effect on lower attaining students, both in terms of academic achievement and confidence. Although there is evidence to support mixed attainment teaching, many schools still practise attainment grouping. At the school I work in, students are initially placed in sets based on their Key Stage 2 (KS2) SATs results. The students are ranked from highest KS2 result to lowest and the first 30 are placed in set 1, the next in set 2, etc. One of the aims of this is to ensure that the class with the lowest attaining students is a smaller class. Up until about 2020, we would have ensured that this class had the support of a teaching assistant in all lessons. Due to financial restrictions this is no longer possible. All students study the same curriculum. Each half term (about every 6-8 weeks), the lists of students in each class are reviewed and students move sets if their attainment indicates a different set would be more appropriate. Teachers and students are involved in the discussions about whether students should move sets.

Not all schools in England are required to follow the national curriculum. However, all state schools are held to account by measures which inevitably lead them to follow the national curriculum and ensure that the majority of students study certain combinations of subjects. This leaves schools with little freedom to decide the curriculum for the students in their school. The two major government

accountability measures in schools are 'progress 8' (P8) and the 'basics' measure. P8 is the progress of a certain combination of each student's exam results measured against their Key Stage 2 test scores and against the progress of all other students nationally in that year. For each school the P8 score is the average of all their students' scores. A score of 0 (progress neutral) is, therefore, an indication that, on average, the students in the school made the same progress as other students with the same starting point made nationally. A positive P8 score says that the students, on average, made more progress than students nationally. The figure is calculated once all the national data has been collected and adjusted for the results of appeals. Schools, therefore, do not know their P8 figure officially until almost 6 months after exam results are known.

The basics measure is the percentage of students gaining a pass in both English and mathematics. The English result used is the better of the student's English language and English literature result. The basics measure is the percentage of students who have passed one English GCSE and mathematics GCSE at a grade 5. There has been much discussion over the fact that students need to pass at grade 4 to move onto a level 3 sixth form but schools are measured at grade 5. This measure is an absolute measure and so takes no account of the attainment of students when they entered the school and therefore the progress made by the students.

Schools are placed into league tables and the headline data is widely published, making the data available to parents to help them choose their school places.

There are significant pressures on schools to show that their results, in particular the mathematics and English results, are good or getting better. Within schools there is a significant amount of pressure on mathematics and English departments and subject teachers to ensure that the results in year 11 are good. In addition.

when judging schools Ofsted make judgements at least partially informed by these accountability measures.

1.8 Structure of the thesis

The thesis is presented to follow my personal journey with this project. Each intervention included a refinement of the work for the next intervention. Cycles 1 and 2 start with a review of literature, each cycle has an intervention, Cycles 1 and 2 have student interviews and each cycle has a reflection on what I have learnt from that intervention and what unanswered questions are left, which then feed into questions for the next cycle. There are three interventions, each with a narrower focus than the one before.

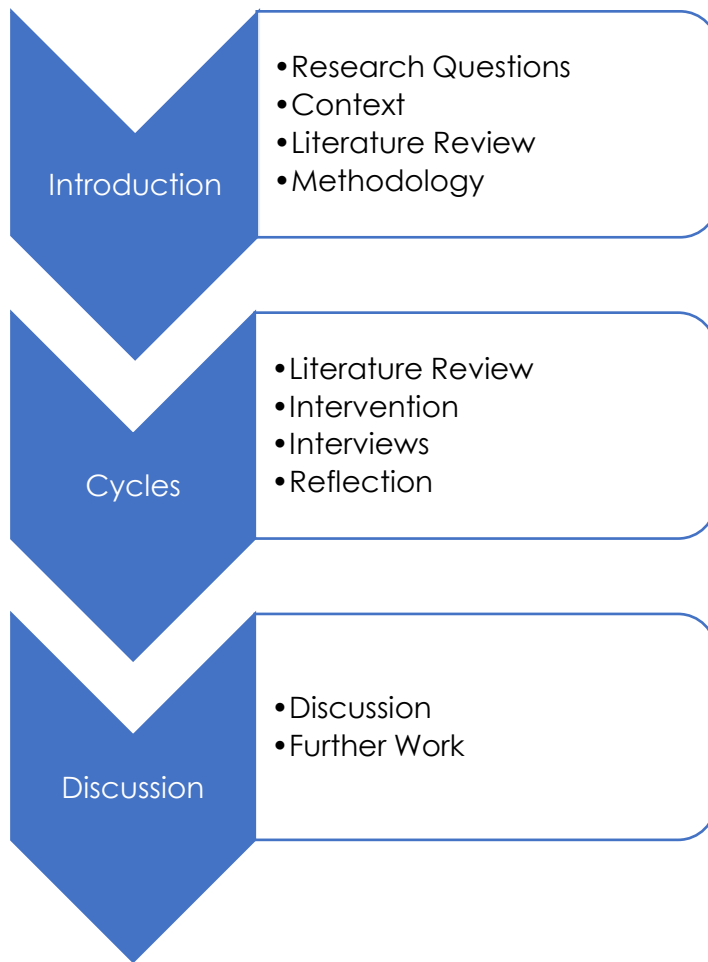


Figure 1-1 Structure of the thesis

1.9 Key Terms

Natural numbers are any whole, positive number excluding, unless otherwise stated, zero.

Integers are any whole number. It is the extension of natural numbers to include zero and negative whole numbers.

Rational numbers are any quantity that can be expressed as $\frac{p}{q}$ where p and q are both integers and $q \neq 0$.

Real numbers can be thought of as points on an infinitely long number line. The points may not be rational (e.g., $\sqrt{2}$) but can be placed on the number line between two rational numbers.

The word fraction has many different meanings. In everyday English it can relate to a part (usually a small part) of something. For example: 'Jim only paid a fraction of what it was worth'. Rarely in everyday speech would a fraction relate to more than a whole. In school mathematics in the UK, fractions usually mean common fractions i.e. a fraction expressed in the form $\frac{p}{q}$ where p and q are both natural numbers and $q \neq 0$. Some authors use fractions to refer to both common fractions and decimal fractions (see below).

Decimals are fractions with powers of 10 as their denominator. Rather than express in the form $\frac{p}{q}$ place value is used for the representation. $0.3 = \frac{3}{10}$ and $0.03 = \frac{3}{100}$

Decimals can be referred to as decimal fractions. In the National Curriculum the title of the section on Fractions is "Fractions (including decimals)." (Department of Education, 2013b, p. 26).

The decimal representation of a real number is a base ten representation, where every digit represents, successively powers of 10. Examples are $321 = 3 \times 10^2 + 2 \times 10^1 + 1 \times 10^0$ and $0.783 = 7 \times 10^{-1} + 8 \times 10^{-2} + 3 \times 10^{-3}$. The latter could be expressed with fractions $0.783 = 7 \times \frac{1}{10} + 8 \times \frac{1}{100} + 3 \times \frac{1}{1000}$. Whilst 0.783 is a fractional amount and can be referred to as a fraction or a decimal fraction, in this thesis, common to UK classrooms, I will refer to a decimal representation of a fraction as a decimal.

The experience, in school, for students studying these fractional numbers is very much dependent on how they are represented. I therefore need to be able to clearly differentiate between them. I will use the term fractions to mean common fractions, those expressed in the form $\frac{p}{q}$ and the phrase ‘fractional notation’ to describe this representation. I will use decimals to describe anything in the form 0.34. Where authors use these terms differently, I will distinguish their meaning where it is important.

The **number line** is a graphical representation of the real numbers, where points on the number line correspond to real numbers and the position on the line can be considered an ordering, where the position of the points will also correspond to the value of the real number. Points to the right are larger than those to the left. The number line, as a teaching tool, is a representation of part of the line, often subdivided so represent certain points and often with some numbers marked.

A **decimal representation** has two parts – the whole number part and the decimal part. For example 4.356 has 4 as its whole number part and 0.356 as its decimal number part.

The **length** of the decimal part is the number of digits explicitly shown. In the case of 4.356 this has decimal length of 3, so would 4.350, or 4.300. This is not a term I would use with students but is used in the thesis to explain certain misconceptions.

Larger decimals are dictated by their magnitude and hence their place on the number line so $4.35 > 4.278$.

Longer decimals are decided by the number of decimal places so 3.456 is longer than 3.62. Again, I will be using this term to describe misconceptions rather than within teaching.

Ragged decimals (Steinle, 2004b, p. 473) these are decimals of different lengths. It is common for texts and curricula to restrict consideration of decimals to a certain number of decimal places. Steinle used the term 'ragged decimals' to describe working with decimals of different lengths.

1.10 COVID-19

In March 2020, the UK government, in response to the COVID-19 pandemic, introduced legislation giving them far-reaching powers. The effects of this included national lockdowns, social distancing and mask wearing. My study was affected by COVID. In Appendix 1, I detail general background to the COVID pandemic and how it affected the UK education system and my school in particular. In this section I talk about how it specifically affected my study.

The COVID pandemic started towards the end of intervention 1. I had intended to re-interview the students whom I had interviewed the year before, so that I could investigate longer-term effects of the intervention and persistence of misconceptions. Having re-interviewed one student, the school was partially closed, and I was not able to continue the interviews. Although I only collected data from one student, I used this to help me alter the design of intervention 2. Intervention 2 was taught whilst some of the strictest COVID rules were in place in the classroom, including not allowing group work and teachers not being allowed to sit with students. Students were also unable to share resources. I concluded the interviews for intervention 2 on the final week of term immediately preceding the third lockdown where schools were again partially closed.

Personally, my role in school was altered significantly by the COVID pandemic. I was responsible for much of the school's COVID response, including risk assessments and for the move to online learning as well as leading on the assessments that replaced GCSE examinations for two years. These internal assessments still followed the basic requirements of the GCSEs.

COVID added delays and forced changes to my study; however, any teaching in a school comes with complications and alterations due to the school, the cohort and many other factors. Other than reducing the number of second interviews I managed to do with the students from the first intervention, there were no effects of COVID that meant that I was not able to engage with the research at the level I had intended to or to investigate what I wanted to find out.

Chapter 2

2 Literature Review

2.1 Introduction

The literature review is in three parts: -

Firstly, a review of the areas of mathematics which were taught in the intervention (sections 2.2 to 2.6). Secondly, errors in mathematics – how they arise and how teaching should be adjusted to avoid them (if, indeed, avoiding errors is deemed to be fruitful) and recover from them (section 2.7). Finally, I consider how the literature impacts on designing teaching episodes (section 2.8).

2.2 Areas of Mathematics

2.2.1 Introduction to the areas selected.

The initial scope of the project, and therefore the literature review, included four areas of mathematics: place value and decimals; fractions; basic algebra and multiplicative reasoning. My teaching experience and discussions with other teachers indicated that these areas of mathematics appear to be difficult for teachers to teach and / or for students to learn. These were also areas of mathematics that were researched by Hart *et al.* in the late 1970s (Hart, 1981).

The aim of the literature review was to find out why these areas proved to be difficult for students to learn; what the features of the areas were and whether there was advice about how to teach them. After the pilot of the pre-test (section 6.1.1), I excluded investigating algebra and multiplicative reasoning from the project and so these areas are no longer included here in the literature review.

Considering fractional amounts, the literature pointed towards understanding of fractional amounts to be a key area to focus on to improve mathematical

achievement. In the US, where teaching of number appears to be largely similar to the UK, the National Mathematics Advisory panel (reported in Siegler *et al.*, (2011)) concluded:

the most important foundational skill not presently developed appears to be proficiency with fractions (including decimals, percent, and negative fractions). The teaching of fractions must be acknowledged as critically important and improved before an increase in student achievement in algebra can be expected. (Siegler *et al.*, 2011, p. 275)

Siegler *et al.*, (2011) found a significant relationship between the knowledge of fraction magnitudes and mathematics achievement even after controlling for arithmetic knowledge. Siegler *et al.*, (2011) further speculate why knowledge of fraction magnitudes might be so significant for accurate fractional arithmetic. They suggest that either accurate estimation strategies allow children to reject or accept arithmetic results or that students who have accurate fractional arithmetic strategies will become better at judging and estimating fractional magnitudes.

The focus of the literature review is, therefore, about place value and fractional numbers, and how these differ from whole numbers.

2.3 Number

2.3.1 Natural Numbers compared to rational numbers.

Natural Numbers

The natural numbers are whole positive numbers, 1,2,3 etc. (Wolfram, 2022a).

The natural numbers can be considered, therefore, as the counting numbers: whole numbers 0 and above. There is no general agreement as to whether zero is a member of the natural numbers. Because of this it is important to state whether or not zero is intended to be included when considering the natural numbers.

When considering the properties of numbers, zero has significantly different properties from other numbers. For example, if two integers, each greater than

zero, are multiplied the product will be equal to or larger than both of the original numbers. This is not true when zero is considered. Because of these differing properties I will use natural number in this thesis to include all positive whole numbers strictly greater than zero, unless otherwise stated. The set of natural numbers (whichever definition is adopted) is denoted $\mathbb{N}$.

Rational Numbers

A rational number is any number that can be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$ (Wolfram, 2022b). This means that any fraction, and any terminating or recurring decimal is a rational number. The set of all rational numbers is denoted by $\mathbb{Q}$.

Real Numbers

The real numbers ($\mathbb{R}$), extend the rationals to include numbers that cannot be expressed as a fraction. Any real number can be placed on the number line. They include all rational numbers, but also non-terminating, non-repeating decimals, which are called irrational numbers. Examples of irrational numbers include π and $\sqrt{2}$. However, consideration of real numbers is really outside the scope of this thesis, because any decimal approximation considered (e.g. $\pi \approx 3.142$) will necessarily have a finite or recurring expansion and will, therefore, be rational. In addition, there are technical differences between rational numbers and fractions but for the scope of this study the two terms can be used interchangeably.

2.3.2 Properties of natural and rational numbers

There are significant differences in the structure and use of natural numbers and rational numbers, these are important to note when considering how children learn to use numbers.

Density of Numbers

Between any two given natural numbers there is a finite number of natural numbers. Between 3 and 7 there are 3 natural numbers not including the end points (4,5,6). However, between any two rational numbers there is an infinite number of rational numbers. Between 0.1 and 0.2 there are, for example 0.11, 0.111, 0.1111, and so on. Rational numbers are therefore described as dense, whereas natural numbers are discrete.

Representation

There is one standard symbolic representation for each natural number (assuming we are working in base 10) and an infinite number of equivalent representations.

However, for rational numbers there are two different ways to represent each number and an infinite number of equivalent representations for each number.

Consider $\frac{1}{10}$. It can be represented as $\frac{1}{10}, \frac{2}{20}, \frac{3}{30}, \dots$ or as 0.1, 0.10, 0.100, ...²

Operations

When working with natural numbers the outcomes of operations are more predictable than rational numbers. For example, in ‘student speak’, adding and multiplication always makes answers bigger, subtraction and division always makes the answers smaller. This is true whatever elements are selected from the natural numbers. For example: the following is true $xy \geq x, \forall x, y \in \mathbb{N}$. The same is not true when you consider the real or even the rational numbers. $xy < x$ can

² Other representations exist such as $0.1 = 0.0999 \dots$ however, this is outside the scope of this thesis

never be true for x, y being natural numbers, but it can be true if both x and y are fractional amounts less than 1. For example: $\frac{1}{2} \times \frac{1}{4} < \frac{1}{2}$.

Unique Successor

Every natural number, N , has a unique successor $N+1$, and whilst it is always possible to order rational numbers and say that for a rational number x , there will be a rational number y , such that $y > x$. It is not possible to select the smallest rational number y that is larger than x .

2.3.3 Number Line

The simple elegance of the number line is that it can represent all real numbers. A basic representation can start with the positive integers, but then can be extended by adding decimals, fractions, and also irrational numbers, as well as extending into negative numbers. The number line can be used to compare the size of numbers and also for simple arithmetic. By extending the number line to two dimensions all complex numbers can also be represented. The number line can be used to test understanding of magnitude and linearity of number (Siegler *et al.*, 2011). Some younger children perceive numbers to have more of a logarithmic nature, rather than linear and this can be tested by using a number line.

2.3.4 Fractional amounts

In terms of structure, fractions have a duality that can, in itself, be hard for students. Let us consider a fraction as any number that can be written with a part written as $\frac{p}{q}$ where p and q are both integers and $q \neq 0$. A fraction is also an operator, so for example $\frac{1}{4}$ of 8 can be considered as $\frac{1}{4}$ operating on the integer 8. The ability to appreciate this duality is in the UK National Curriculum mathematics KS3 programme of study (Department of Education, 2013a).

In addition, any fraction is equivalent to a division sum, and any division is equivalent to a fraction. $\frac{2}{3}$ is therefore two thirds and also two divided by three. It could be 2 thirds of a cake or two cakes shared amongst 3 people. Each fraction can also be written in an infinite number of equivalent fractions. The link between fractions and decimals is strong because each terminating or recurring decimal can be written as a fraction. 0.6 is $\frac{6}{10}$; 0.783 is $\frac{783}{1000}$. Hart found that when comparing decimals students were not always aware of this (Hart, 1981).

Gabriel *et al.* (2012) explores the conceptual nature of fractions: a combination of the “general properties of rational numbers”, “understanding the roles of the numerator and denominator” and the “understanding of global fraction magnitudes” (Gabriel *et al.*, 2012, p. 138). They conclude that it seems likely that there is an interaction in improvement between procedural and conceptual competence.

most of the difficulties in fraction learning stem from the fact that pupils have a poor representation of the global magnitude of fractions and that they do not grasp the conceptual meaning of fractions (Gabriel *et al.*, 2012, p. 144).

2.3.5 Place Value

Understanding place value is not straight forward (Ross, 2002; Thompson, 2003).

Our denary number system has four properties.

1. Additive property. The quantity represented by the whole numeral is the sum of the values represented by the individual digits.
2. Positional property. The quantities represented by the individual digits are determined by the positions that they hold in the whole numeral.
3. Base-ten property. The values of the positions increase in powers of ten from right to left.

4. Multiplicative property. The value of an individual digit is found by multiplying the face value of the digit by the value assigned to its position. (Ross, 2002, p. 1)

Having a thorough understanding of the place value and thereby the magnitude of a number would entail a student having to be engage with each of these four properties.

Several authors, e.g. Ross (2002), point to the fact that vocabulary in English does not support understanding place value, particularly around the teens. The order of place value changes in the names of the number. Twenty-three is two tens and three ones, whereas fifteen is 5 ones and 1 ten. In Asian languages this is not the case. In Mandarin, for example, 15 is said as ten five, 50 is said as five ten; giving the student immediate access to the place value (or at least the quantity value (Thompson, 2003)) of the number. The English language is even less helpful when it comes to naming decimals. Compare the names for 5623 and 0.5623; when the digit 2 in the former is enunciated, it is '20' and it gives the hearer some sense of magnitude, the 2 in the latter, however, would be exactly the same whatever its place in the decimal, meaning that it derives its magnitude by its position in a sequence of numbers. In my experience, students often arrive at secondary school incorrectly naming decimals. They will say that 0.23 is nought point twenty-three, potentially reinforcing the whole number bias that can lead to decimal errors.

Thompson (2003) distinguishes between understanding column value and quantity value. An example of the former is that 634 is 6 100s, 3 10s and 4 1s – whereas the latter is $600 + 30 + 4$. He believes that young students (year 2), have an appreciation of quantity value but not column value.

This leads to recognising that understanding place value is not universal to the end of primary school, although by the end of year 6 the National Curriculum states that students should be taught to identify the digits up to three decimal places (Department of Education, 2013b, p. 40).

It is also important to understand that each decimal place of a number is combination of tenths, hundredths etc. so that 0.6 is $\frac{6}{10}$ and 0.06 is $\frac{6}{100}$ and 0.35 is $\frac{3}{10} + \frac{5}{100}$ or $\frac{35}{100}$.

Place value includes being able to understand that 8732, is 8 thousands, 7 hundreds, 3 tens and 2 ones (or units). The CSMS study found that there were misconceptions around this understanding particularly with numbers more than thousands (Hart, 1981, p. 49). In my teaching, I have noticed that students will often name the place value columns as ones, tens, hundreds, thousands, millions, making the error that the next column is the 'next' name.

2.4 Development of Number Concepts

Young children are first introduced to natural numbers. Natural numbers are positive, whole numbers. Various theories of acquisition of number exist.

Siegler *et al.*, (2011) discuss the contemporary theories of development of number, there are privileged domain theories which consider the acquisition of whole number understanding to be privileged. These state that the acquisition of competence and understanding with whole numbers is a barrier to later learning about other types of number. Evolutionary theories consider development of whole number understanding to be 'biologically primary', whilst the development of other number structures is 'biologically secondary'. Although Conceptual Change theories do consider the development of fractions, they discuss more how a whole

number bias interferes with learning about fractions. All of these explain that the acquisition of an appreciation of natural numbers interferes with the acquisition and understanding of rational numbers.

An integrated theory of whole number and fractions development was proposed by Siegler *et al.*, (2011). This theory states that there is a progressive broadening of what can be understood. The conceptual mental structure for the whole numbers is initially a number line and this number line is extended to be able to represent and manipulate all types of number. This is a smoother approach than the theories which consider that children learn about natural number first, such as through counting, and then have to acquire an appreciation of rational number separately. Siegler *et al.*'s integrated theory states that knowledge about fractions is part of the process of an increasingly sophisticated understanding of number rather than being separate, therefore understanding natural number does not *per se* create a barrier to understanding rational number.

Siegler *et al.*, (2011) investigated students' understanding of magnitude of rational number, fractional arithmetic and general mathematical attainment. They also looked at self-reported strategies for answering questions and found that fraction magnitude representations are not accessed as automatically as whole number magnitudes are. They describe the fact that students are more likely to resort to using strategies when estimating fraction magnitudes than they are in estimating the magnitudes of whole numbers. An indication of this is that whole numbers are estimated no less accurately under time pressure than without pressure (Siegler & Opfer, 2003). One example of a strategy used for placing a fraction on a number line, reported by Siegler *et al.*, (2011) include turning the fraction into a percentage of the length of the number line (e.g. $\frac{4}{5}$ would be 80% across the gap between 0 and 1).

Children's initial appreciation of number may not be the same as an adult's appreciation. In fact, Siegler *et al.*, (2011) found that children start with a logarithmic understanding of numbers and this later develops into a linear understanding. This can be tested by asking children to mark the magnitude of several different numbers onto blank number lines and then later overlaying or comparing them. This could potentially be thought of as the law of diminishing returns; if you need something then the first one matters more than the tenth when you already have nine. It could also be that the smaller numbers are those that children are more used to seeing.

Siegler *et al.* (2011) and Siegler and Lortie-Forgues (2014), Obersteiner *et al.* (2013), Torbeyns *et al.* (2015), Resnick *et al.* (2016) all focus on being able to assess the magnitude of a fractional amount. Siegler *et al.* (2011) then link the aptitude of assessing fraction magnitude with a general mathematical attainment.

Steinle and Stacey (e.g. (Stacey, Helme, & Steinle, 2001; Steinle, 2004b; Steinle & Stacey, 2003, 2004) have investigated misconceptions around assessing the magnitude of decimal representations of fractional amounts. They classified misconceptions into two major categories, *longer-is-larger* (e.g. $0.47 > 0.6$ because 0.47 is a longer decimal), and *shorter-is-larger* (e.g. $0.8 > 0.47$ because 0.8 is a shorter decimal). Longer-is-larger can be a result of whole number bias. . 47 is greater than 6 and so 0.47 is greater than 0.6. Investigating the reasons for a shorter-is- larger bias Steinle considered further subclassifications such as *reciprocal thinking*, *negative thinking* or *denominator focussed thinking* (Steinle, 2004b)). Reciprocal thinking is where students make an analogy with fractions. In this case 0.3 is larger than 0.4 because $\frac{1}{3}$ is bigger than $\frac{1}{4}$. Negative thinking will give the same answer, but this time because students have made an analogy with the structure of the negative numbers. Denominator based thinking reflects that

fact that 1 230 is greater than 999 and this can be seen because the first number has a digit in the 1000s column. With denominator-based thinking students think that any number with only digits in the tenths column will immediately be larger than any number with digits in the hundredths column. They would therefore correctly select 0.5 as larger than 0.3 but would incorrectly state that 0.55 was smaller than 0.3. Stacey, Helme and Steinle (2001) also found that some people do not understand zero. In their studies students were likely to assign some of the properties of 1 to 0. They selected 0.6 as smaller than 0, potentially due to considering that 0 is a whole number and 0.6 is a decimal and thinking that decimals are smaller than whole numbers. In fact, 13% of the preservice teachers in their sample made this error. The theme that misconceptions around fractions and decimals are not only held by school children continues in other studies (Vamvakoussi *et al.*, 2012).

Whether the theories consider that rational number thinking is disjoint from natural number thinking or if the former is a later development of the latter, all believe that children first acquire natural number thinking. Torbeyns *et al.* (2015, p. 6) have shown that students then overgeneralise their understanding and try to apply it to fractional amounts less than one. This in turn leads to many of the problems with students ordering fractions and performing fractional arithmetic. Examples where students may overgeneralise can be considered in students' understanding that $0.3 > 0.22$, even though $3 < 22$ and students realising that multiplying by fractional amounts results in a smaller answer rather than multiplication always making the answer bigger. Both of these ideas oppose the student's natural overgeneralisation. An example of an overgeneralisation from Steinle (2004a) is how a student with "denominator focussed thinking" (Steinle, 2004a, p. 208), recognising that tenths are larger than hundredths would believe that any number

of hundredths is smaller than any amount of tenths. The example given in Steinle (2004a, p. 208) is " $4.82 < 4.3$ ". Although this is an incorrect generalisation, it is not very different to a child using rules for their mathematics from the correct assumption that $382 > 45$, solely because there are hundreds in the first example.

In the example above a student would be able to compare magnitudes of a decimal representation if they had a clear understanding of the concept of place values in decimals or if they had acquired a robust algorithm for comparing. The algorithms that I have seen commonly used in an English classroom are 'add zeros' and 'compare digits from the left'. An example of 'add zeros' would be if comparing 0.451 and 0.5, to make 0.5 the same length by adding (redundant) zeros, i.e. 0.500. Now it is possible to compare 0.451 and 0.500. This calls on the same rules as when comparing integers. An example of 'compare digits from the left' would be to look successively at the left-hand digit and compare. For 0.451 and 0.5, the left-most digit for both numbers is 0. Next consider the second left-most digit. For 0.451 this is 4 and for 0.5 this is 5. The process can now stop. 5 is bigger than 4 so 0.5 is bigger than 0.451. Used properly, these algorithms are complete. These algorithms are referred to and detailed in Steinle (2004a, p. 168) as *annexe zero* and *full left-to-right comparison*. The former has a potential undesired effect of promoting natural number thinking, whilst the latter works with the structure of decimal representation. The former also has limitations when considering decimals with an infinite number of decimal places. These algorithms are discussed further below in how teaching can help to address misconceptions. Steinle and Stacey (2001) also discuss partial algorithms which include truncating and rounding decimals which can lead to errors and misconceptions

Another tool that helps comparison, though not in itself a complete algorithm, is benchmarking (Roche, 2010). Comparing 0.47 and 0.6 a student might be able to

see that 0.47 is less than 0.5 and 0.6 is more than 0.5, where the student already has a clear understanding of the magnitude of a few key fractions. A similar idea around key fractions is discussed in Siegler *et al*, (2011).

The longer-is-larger misconception is very common but Steinle (2004a) notes that this misconception is often outgrown, but can be replaced by the more persistent shorter-is-larger misconception. This development is potentially explained by the confusion caused by the introduction of negative numbers and negative powers in the secondary curriculum, or could be that students are overcompensating for their earlier misconception.

A big step for students in understanding fractional magnitude is the realisation that all the properties of whole number development will not apply to rational and other real numbers (Siegler *et al*., 2011, p. 275), as detailed in section 2.3.2. This resonates with my conjecture that students' understanding will be improved by asking them to engage with the structure of numbers, not only their symbols, and will be revisited in more detail in section 3.

Siegler and Lortie–Forgues (2014) describe an integrative theory of numerical development. This theory discusses the broadening of understanding of the magnitude of numbers. Children start with a good understanding of small whole numbers. The numbers under scrutiny both become bigger and extend to different types of numbers.

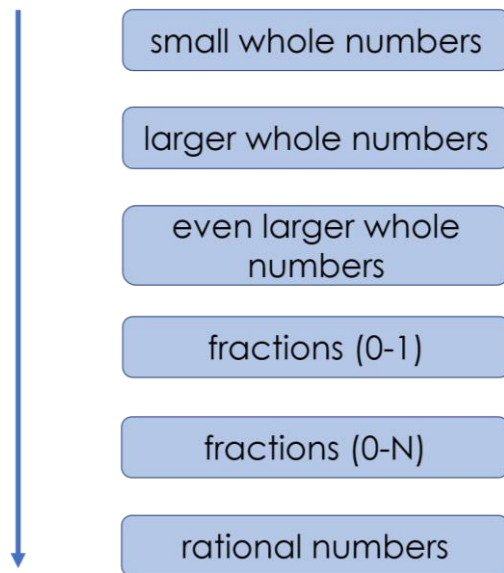


Figure 2-1 - integrative mode of number understanding

Figure 2-1 shows the development of number understanding in the integrative model (adapted from Siegler and Lortie–Forgues (2014, p. 146)).

The mental number line suggested for understanding number magnitude by many authors is adapted in Siegler and Lortie-Forgues (2014, p. 145) to be dynamic with the number line extending towards larger numbers, backwards through zero and then magnified to consider fractions and decimals with a greater number of decimal places. Current technology allows teachers to work with a dynamic number line with children.

The literature usually focusses either on decimal representations (eg. Steinle & Stacey, 2004a) or on fraction representations (e.g. Siegler & Lortie-Forgues, 2014). The findings from both ideas are similar; with students having difficulty in accessing the magnitude of fractional amounts however they are presented.

2.5 Observed Difficulties

2.5.1 Introduction

Whatever the mechanism by which number sense is acquired, it is worth noting is that there is an observable natural number bias (e.g. (Obersteiner *et al.*, 2016; Vamvakoussi *et al.*, 2012; Vamvakoussi & Vosniadou, 2010). That is, a preference for working with whole numbers that creates and maintains misconceptions in students and adults alike. Although, as described above, the cause is in debate there appears to be consensus that the differing properties of natural numbers from rational numbers plays a large part in the maintaining the errors. Students appear to be far more accurate when dealing with rational numbers when they can treat them the same way as they can natural numbers (Vamvakoussi *et al.*, 2012). Some of the differences between natural numbers and rational numbers are described above in section 2.3.1. The observable natural number biases reported in research are summarised as four dimensions by Obersteiner *et al.*(2016). These were taken from a study by Vamvakoussi, Van Dooren and Verschaffel (2012) where natural number bias in adults was investigated.

Dimension 1 Size.

To investigate size the participants are asked to compare decimal magnitudes (e.g. 0.23 and 0.1). This strategy is also used in Steinle and Stacey's decimal comparison test (DCT) (Steinle & Stacey, 2003a). In some cases, a natural number bias would result in a correct answer. For example, $0.27 > 0.1$, the natural number bias would consider the question: is $27 > 1$?. However, in the case of $0.7 > 0.34$ the natural number bias would give the incorrect result.

Dimension 2 Density.

Some students believe that rational numbers have successors – so that, for example $\frac{4}{5}$ follows $\frac{3}{5}$ or 0.2 follows 0.1. Vamvakoussi and Vosniadou (2010, p. 189) states “students would often answer that there is a finite number of intermediate numbers and, in particular, that there is no other number between two pseudo-successive numbers like .005 and .006, or $\frac{1}{3}$ and $\frac{2}{3}$.”

Dimension 3 Representation

Obersteiner *et al.* (2016) point out that students, on seeing an x , will tend to immediately only consider natural numbers. This certainly resonates with my experience when asking for a solution for $3x = 10$, students often say it is not possible, and when I ask for as many solutions as possible to $x + y = 7$ students, will write down natural number (plus zero) solutions and then ask whether they can include negative or decimal answers.

Dimension 4 Operations

Obersteiner *et al.* (2016) investigated students’ responses to operations with rational numbers, looking at accuracy and speed of answering. Questions were asked about the truth of statements, such as whether $4 < 4 \times x$ could be true. Questions that could be answered correctly by considering natural numbers were answered more accurately and more quickly than those that required a different reasoning for rational numbers (see section 2.5.2).

The CSMS studies (Hart, 1981) found that students were generally not competent in working with decimals and with fractions. Hodgen *et al.* (2009) shows that, 30 years on, students in the UK are perhaps more competent with decimals but less so with fractions. Although it is not clear why there has been an improvement they offer two possible reasons: one the emphasis on improving the attainment of

borderline students (level 4 at KS2 and a C at GCSE, using the grading systems in place in 2009 and not currently) and also the cultural shift towards decimals and away from fractions, including the greater familiarity with decimals in everyday life with the greater adoption of metric measurements and the increased use of calculators (Hodgen *et al.*, 2009, p. 9).

2.5.2 Whole Number Bias

As referred to in section 2.4, theories of whole number acquisition often refer to whole number bias resulting from inaccurate analogies being made between whole numbers and fractional amounts; this is a possibility that Siegler *et al.* (2011) discuss in their integrated theory. Obersteiner *et al.* (2016) further studied the effects of such a bias on expert adult mathematicians to see if it is possible to be unaffected by the bias. For the study Obersteiner *et al.* considered adults with a masters or doctorate in mathematics working within a university mathematics department.

The study by Obersteiner *et al.* (2013), considered congruent and incongruent items. Congruent items are those where answering the question for natural numbers will give the same answer as for rational numbers. For example, considering if $x + 4 > x$ is true does not rely on any thinking to differentiate between natural and rational numbers and so is congruent. Adapting an example from Obersteiner *et al.* (2016, pp. 551–552) anyone, when asked, whether $4x < 4$, could be true, would have to use a different thought process when considering natural and rational numbers, meaning that this is an example of an incongruent item. The study showed that although expert mathematicians performed equally well on both congruent and incongruent items, the expert mathematicians took longer to answer the incongruent items. Obersteiner *et al.* (2016) suggests that expert mathematicians could escape the bias by “relying on their skilled intuitions

about the solvability of equations and inequalities” (Obersteiner *et al.*, 2016, pp. 551–552). When asked, for example, if $ax < b$ could be true, the expert mathematician might consider instead whether $x < \frac{b}{a}$ could be true, bypassing the natural number bias. They noted, though, that this strategy was not available to novices.

2.5.3 Decimal Comparisons

One concern noted by Steinle and Stacey in their study up to tenth grade (year 11 in the UK) was that the students exhibited a low rate of expertise in comparing decimals, for example, students may not know that “0.453 is near 0.45” (Steinle & Stacey, 2004a, p. 547). This resonates with my experience of teaching classes of all attainments up to GCSE. It seems likely that this is not a localised problem.

Steinle and Stacey (2001, 2004) used a decimal comparison test (DCT) to find out if students were able to compare the size of two decimals. They found that errors were identified as longer-is-larger and shorter-is-larger. Participants were classified as *apparent-experts*, holding a longer-is-larger misconception, a shorter-is-larger misconception or unclassified. They could not label students as ‘expert’ because, by using one of two expert strategies, a student could appear to be expert without fully understanding the magnitude of the decimals. The two rules they discussed are full left-to-right comparison and annexe zero, adding zeros to the shorter decimal (Steinle & Stacey, 2001). To be able to decide that a student is an expert more testing would be needed, for example being able to pinpoint the decimals on a number line (*ibid*). The Decimal Comparison Zero Test (DCT0) (Steinle & Stacey, 2001) required students to select the greater between a pair of decimals. The DCT avoided comparisons with zero, whilst the DCT0 specifically tested this type of question, for example, 0.8 and 0.80. Both of these types of

question were in the CSMS testing (Hart, 1981). The DCT0 considered 0 where it had different roles and discussed them as visible and invisible zeros. Examples of pairs were: 0.8 and 0.08; 0.45 and 0.045; 1.347 and 1.0347; 0.8 and 0.80; 0.6 and 0 (Steinle & Stacey, 2001, p. 437).

Steinle (2004a) shows that in longitudinal testing the prevalence of the longer-is-larger misconception decreases. However, the shorter-is-larger misconception, whilst less prevalent than the longer-is-larger, is more persistent with reasonably similar proportions of students displaying this at all grades and, after grade 7, this becomes more prevalent than the longer-is-larger.

My experience in the classroom tells me that a common misconception is longer-is-larger; students will state that $0.42 > 0.6$ because $42 > 6$. This is an example of *whole number thinking* (Steinle & Stacey, 2003) and is an example of the whole number bias. As described in section 2.4, while this misconception is being unpicked sometimes a shorter-is-larger misconception occurs. The reasoning could be, 'I know my thinking was wrong for 0.42 and 0.6 so now that I am considering 0.3 and 0.65 it must be that 0.3 is bigger' or it could be due to the expansion of the curriculum to negatives and negative powers.

An example of whole number thinking might be that $0.4 < 0.34$, because $4 < 34$. Reciprocal thinking appears to be the result of an analogy being made between decimals and larger denominators in fractions leading to the assumption that decimals of equal length are larger if the number is smaller. An example might be that $0.5 < 0.4$ because $\frac{1}{5}$ is smaller than $\frac{1}{4}$. Reverse thinking is where a student might read "0.123 is read as something like 321 thousandths" (Steinle & Stacey, 2004, p. 546).

Many of the types of answers show incomplete understanding or application of a rule. For example a student might correctly order to 2 decimal places and then ignore any further decimal places, this is classified as *money thinking* (Steinle & Stacey, 2003, p. appendix). Money thinking resonates with my experience. The advice I was given by many teachers more experienced than me was to use money as an example when using decimals. However, it is possible that this allowed me to engage students with decimals up to two decimal places but not with ragged decimals. It appears likely that the reason students were able to work well with decimals in terms of money was that they were thinking in terms of whole numbers of pence.

Roche (2005) used the Steinle and Stacey's DCT in an interview situation and discussed choices with those who performed well: finding out their strategy. Strategy 1 was to convert to fractions (i.e., hundredths and thousandths). Strategy 2 was to add zeros to the end of the decimal to make them the same length and then treat as whole numbers. Roche (2005, p. 12) found that students who used strategy 2 were less likely to be able to perform other tasks involving the magnitude of decimals, potentially indicating that Strategy 1 led to students have a better understanding of the underlying concept. The other advantage of using fractions is that the language describes the size of the fractional amount. '72 hundredths' describes the size of the 7 more than '0.72' does. This is also discussed in Roche (2010).

2.5.4 Applying Decimals

Although appreciation of fraction magnitude is an important part of working with rational and real numbers, there are other elements that need consideration. Revisiting the properties of fractional numbers that differ from real numbers (see section 2.3.1), one aspect is how the four operations (add, subtract, multiply,

divide) change between the types of number. Part of the CSMS study was to consider how students were able to use the four operations. Hart (1981) finds that the focusing on rules and algorithms rather than understanding is successful up to a point but usually fails. So, for example, although the ‘to multiply by 10 you add a nought’ rule works for integers it fails applied to a decimal. Hart even found that a “move the figures” rule was misapplied by students. This rule is where multiplying by each power of ten moves the figures one column to the left. The CSMS study also finds that there is a persistent misconception that multiplication makes numbers bigger, division makes numbers smaller and that you can only divide a big number by a smaller one (Hart, 1981). My experience shows me that these misconceptions, consistent with a natural number bias, still persist in students 40 years later.

A preliminary finding from the ICCAMS project (Hodgen *et al.*, 2009) was that between 1976 and 2008 students became better at using and understanding decimals. This was unusual in that all other areas were either similar or slightly worse. It was noticed during an interview that students were less likely now to express a decimal as a fraction but were more likely to express a fraction as a decimal. Hart (1981) too considered that the more frequent use of calculators might shift the emphasis in teaching from acquiring procedures to be able to understand decimals. I find that in my classroom students frequently convert to decimals, even when a fraction result would be more straight forward (e.g. in the solution to $7x = 3$). I have encountered mathematics teachers with a similar bias: one in particular who told me he thought that students should only be taught decimals and not fractions and appeared to instil a similar view in his students. I noted this when I took a class over from him and every time I suggested they

might find something easier working with fractions they would counter that Mr. Smith had told them to work with decimals.

When four operations questions are presented to students in the context of a worded problem this adds a further layer of complexity to the question because students now also need to select the operation as well as perform the calculation. Greer (1994) found that students have difficulties solving worded problems unless the numbers in the problem are easy. Even when students are working on questions where all they had to do was to select the appropriate operation the success of the students is dependent on the numbers in the question. Even more interesting, was the fact that given two questions identical, but for the numbers involved, students did not see the connection. Greer (1994) found that students do not generalise from the domain of integers to questions with decimals and fractions. I will refer to this later under the heading Structure in section 3.

When students are faced with having to select the correct operation, they can then commit to an interpretation without applying critical thought. Siegler *et al.* (2011) noticed that by selecting the incorrect arithmetic, the students in the study arrived at answers that did not comply with basic principles of arithmetic, for example that adding two positive numbers should result in a greater number. The student, though, did not recognise the conflict. This resonates with my experience where students, fully aware that $\frac{3}{5}$ is less than one will write it as 3.5 when asked to convert to a decimal. This is recognised in both Zaslavsky (2005) and Zazkis and Chernoff (2008) which show that students can accept conflicting concepts. At this point Zaslavsky (2005) labels these as potential conflicts. If they are to be used in teaching, then it must be ensured that the student's attention is drawn to the conflict.

2.5.5 Fractional Comparisons

Natural number bias is not only present in students. A&W (an American burger restaurant chain) produced a burger, that consumer tests indicated was better than McDonald's quarter pounder for the same price. It was larger and was sold as 'the third pounder'. It did not sell though, and A&W employed focus groups to investigate why it had not sold. The consensus of the focus group was that the burgers were at least as good as the McDonald's quarter-pounder but that over half the participants felt they did not want to pay the same price for a smaller burger (Conradt, 2016). This is an example of a natural number bias and whole number thinking becoming a barrier to understanding fractional magnitudes; the reasoning is, 3 is smaller than 4 and so $\frac{1}{3}$ must be smaller than $\frac{1}{4}$.

Interestingly, although students might be able to say that $0.3 = 0.30$ they are less likely to be able to express that $\frac{3}{10}$ is the same as $\frac{30}{100}$ (Steinle & Stacey, 2004). This could be due to the frequent teaching of the annex zero method of comparing decimals, so that students become more used to the equivalence of 0.3, 0.30 and 0.300 but this might not transfer to them realising that $\frac{3}{10}$ is equivalent to $\frac{30}{100}$ and $\frac{300}{1000}$.

2.5.6 Fraction Magnitude

Understanding fraction magnitude could be said to be inherently harder than understanding decimal magnitude, just by the nature of the symbolic representation of the numbers and because students have to understand the relationship between two quantities (Nunes & Bryant, 2009). Watanabe (2002, p. 461) describes the different roles that the numerator and denominators play in a fraction "The numerator counts while the denominator partitions the whole to create the unit to be counted". He points out that too often students are taught that

both numerator and denominator are about counting; the number of parts in the whole is counted and this gives the denominator. He advocates, instead, that the denominator should express the fractional unit. When considering whether $\frac{5}{7}$ is greater or smaller than $\frac{2}{3}$ the algorithms to decide this procedurally are more complex than the ones for decimals. To compare fractions with different denominators, students need to consider a common denominator – in the case above, 21, convert both numbers to a fraction out of 21 and then look to find the bigger. Not only is fractional magnitude inherently difficult, the algorithms to order fractions and to apply fractional arithmetic are difficult (Siegler *et al.*, 2013). This appears to be a universal problem. Siegler *et al.* (2013) state that a possible problem is that students appeared to think of fractions as part of a whole rather than realising that it can have any magnitude. For example, a student considering $\frac{4}{5}$ might have a visualisation of 5 blocks with 4 shaded in, rather than visualising $\frac{4}{5}$ as a number in itself.

Resnick *et al.* (2016, p. 753) found that fourth and some fifth and sixth grade students estimated fractions as “small” and improper fractions as “just more than one”. This approach fails when trying to estimate on a number line. They point out that estimating the magnitude of fractions not only depends on the values of the numerator and the denominator but also on attending to the order in which they appear. Resnick *et al.* (2016) found that understanding mixed numbers develops later than the understanding of proper fractions. This agrees with the development of number in the work of Siegler and Lortie-Forgues (2014) as shown in the diagram Figure 2-1 above.

2.5.7 Fractional representation

There are many different ways to represent fractions. Watanabe (2002) discussed different ways of representing fractions for teaching at primary level and mentions the difference in how the child and the teacher perceive the model being provided. In one lesson where I was teaching students about how to convert mixed numbers to improper fractions, I presented a diagram, like the one in Figure 2-2 below. The aim was for the students to see that each whole could be considered as 4 quarters; and so, the picture represented $1\frac{3}{4}$, instead many students thought that the diagram represented $\frac{7}{8}$. This is a perfectly reasonable interpretation and one where both sides will consider they are right without much recourse of checking their interpretation with the other.

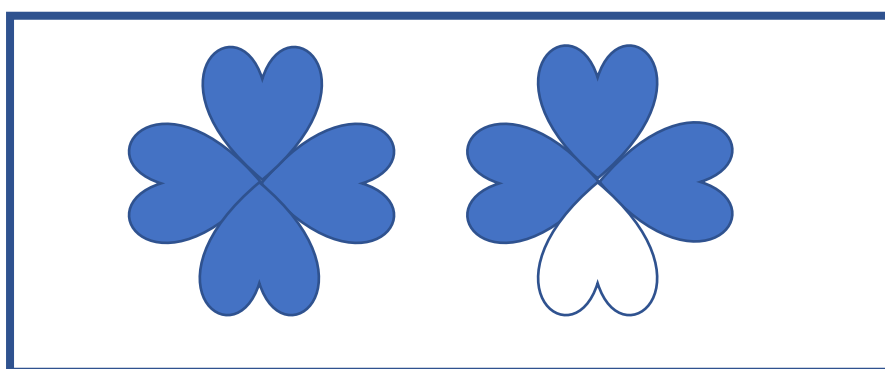


Figure 2-2 a pictorial model for teaching mixed numbers.

As well as two-dimensional representations of fractions, such as Figure 2-2 above, the number line is the representation that would be the most accessible to link fractions with real numbers. Wong (2013), among others, suggests that creating a number line can support development of understanding fractions, though Watanabe (2002) posits that the number line is not a useful representation for fractions unless the student already has a concept of a fraction as a number.

Primary aged students mix up the number $\frac{1}{2}$ with a measurement (Hallett *et al.*, 2010, 2012). When asked to place the number $\frac{1}{2}$ on a number line students in the study marked halfway along the number line whether or not the line was 0 to 1 (Wong, 2013). Such an error would not be noticed if students are only ever given a number line with ends 0 and 1. Mitchell and Horne (2008) express a similar view. However, this would potentially be a misconception covered up rather than avoided. Placing a fraction on a number line is not easy for primary aged children. Gabriel *et al.* (2012) pose the question of whether the numerical value of a fraction is available for adults to process as a whole.

Mitchell and Horne (2008) found three misconceptions in students using fractions on number lines: 1) students marking the fraction of the line rather than as a number (e.g. marking $\frac{2}{3}$ at 6 when the number line went between 0 and 9) this is similar to the misconception described by Wong (2013) above. 2) counting pips not spaces and 3) decimalising the answer (e.g. $\frac{5}{6}$ was recorded as 0.5; each space was 0.1)

Language is a significant part of understanding fractions and can promote, or inhibit, different models of fractions (Wong, 2013). Considering $\frac{2}{3}$ as two out of three links with the part-whole model, but $\frac{3}{2}$ does not make much sense in this way. When considering quantity then two-thirds makes more sense. It seems likely that students look at two-dimensional fraction representations as a part-whole model and that they are counting both the numerator and the denominator. Roche (2005) suggested using colouring-in questions with visual distractors to see if students are treating fractions questions as a counting task. An example would

be to show a student a picture of a circle split into 4 equal parts and ask them to colour in $\frac{1}{3}$ as shown in Figure 2-3.

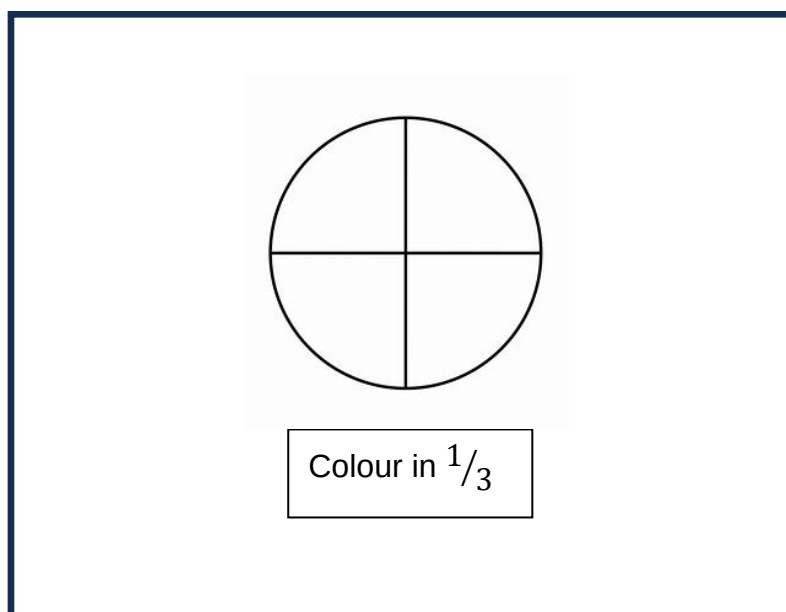


Figure 2-3 fraction with distractor - colour in one third

Sielger *et al.* (2013, p. 15) find that students at a young age appear to have the ability to judge fractions in the continuous case – but errors creep in when the same question is presented in the discrete. Using tasks such as the one below in

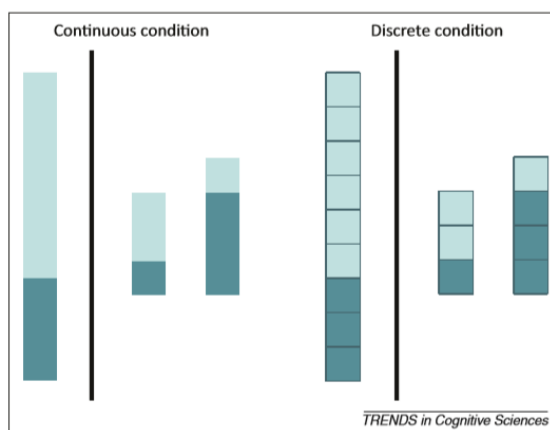


Figure 2-4 continuous vs discrete fractions (Sielger *et al.* (2013, p.15))

Figure 2-4, where students were asked to judge a fractional amount either with a continuous bar or a discrete bar, they find that students are able to judge the fractions by scaling, but errors are produced when they count. My experience

suggests that this is not the case with older students who appear less ready to work with the continuous case.

2.5.8 Applying Fractions

Students often show a procedural knowledge of calculating with fractions and Forrester and Chinnappan (2010) showed that preservice teachers displayed a similar understanding. Both students and teachers know that multiplying fractions, for example, could be seen as 'multiply the tops and multiply the bottoms'. This procedural approach can be classified as instrumental understanding (Skemp, 1976).

Calculating with fractions causes issues, particularly if it is taught as a set of rules. The rules for adding fractions are not intuitive: multiply the denominators, restate the fractions with their common denominator, then add the numerator but the denominators do not change. Each step makes sense to someone who knows why it is being done. For someone who does not understand the 'why' then it could appear to be a list of unconnected operations. A common question from students when multiplying or dividing fractions is 'Is this the one where we flip the second fraction?' or 'is this the one where we add or where we multiply?'. The rules appear to stick but as unconnected parts of a method, this is highly indicative of instrumental understanding (Skemp, 1976).

Errors and problems include trying to treat fractions as whole numbers (Hart, 1981) leading to erroneous thinking, such as 'multiplication makes bigger', although it is noted that $\frac{1}{2}$ is understood differently and treated differently by students than other fractions (Hart, 1981).

It is also important to note that fraction arithmetic is not always governed by a singular set of procedural rules. For example, consider exam fractions; Bill has two

papers returned as part of the same exam – on paper one his mark was $\frac{27}{50}$ and on the second paper his mark was $\frac{14}{50}$. Although these are displayed as a fraction and the two papers are added together for his final score – that sum is $\frac{41}{100}$, not $\frac{41}{50}$.

2.5.9 Decimals and Fractions

Although there are different obstacles in understanding decimal and fraction magnitude there are themes that are consistent. One is the persistence of difficulties when comparing the magnitude of decimals. Stacey *et al.* (2001) found that although the longer-is-larger misconception was outgrown by students the shorter-is-larger misconception was persistent; the prevalence was 10% to 20% of students throughout secondary school and Steinle and Stacey (2004) found that 3% to 8% of preservice teachers hold the shorter-is-larger misconception. Siegler and Lortie–Forgues (2014) found that adults attending community college were only 70% accurate when comparing fractions whilst almost 100% accurate when comparing natural numbers. It appears that these difficulties are, to some extent, inherent and universal. Whilst expert mathematicians might be expected to be free from these misconceptions, it does not mean that they are free from a whole number bias. In Obersteiner *et al.*'s (2013) test of university mathematicians the time it took to differentiate between solutions indicated a whole number bias (see section 2.5.2).

Features that individuals find harder are the size of the numbers and the distance between them. This is so for decimals, fractions and whole numbers. It persists whether the representation of the number is symbolic or non-symbolic. Siegler and Lortie–Forgues (2014) found that the greater the absolute magnitude and the smaller the distance between the two numbers, the harder it becomes to identify

the larger one. Roell *et al.*, (2019) showed that it takes adolescents and adult longer to assess that “ $0.6 > 0.131$ ” than “ $0.612 > 0.157$ ”, due to the differing lengths of the decimals. Comparing 0.6 to 0.131, where the first number has one decimal place, and the second three, is harder than comparing 0.612 and 0.157 where both have the same number of decimal places. Siegler and Lortie–Forgues (2014) noted that both whole number and fraction magnitude understanding are distance and magnitude sensitive, with both having effects on how accurately and how quickly people can judge magnitude comparisons. The smaller the numbers are (magnitude) and the further apart (distance) the more quickly and easier magnitudes can be distinguished. An example might be that it is easier to order 7 and 9 then 177 and 179, and it is easier to order 3 and 25 than 21 and 22. These effects are seen with symbolic and non-symbolic representations and they persist (Siegler *et al.*, 2013; Steinle & Stacey, 2004).

Although these studies represent a variety of different countries, Torbeyns *et al.* (2015, p. 9) found that the difficulties with decimals were not the same in different countries. Chinese and Belgian (Flanders) students displayed significantly better aptitudes at assessing the magnitude of fractional amounts than their US and UK counterparts.

2.5.10 Place Value

Although ordering numbers gives an idea of how well students understand place value, more can be seen from how students apply algorithms for arithmetic operations. Hart (1981) found that “carries” and “borrows” in addition and subtraction appeared randomly distributed and showed little understanding of place value. Even where place value is well understood with integers there are a significant number of misconceptions that appear when considering place value in decimals.

Thompson (2000) concludes decimal place value in the UK is not well understood by students at the end of primary school and calls for a reappraisal of teaching place value in the UK. He focuses his discussion on acquiring two-digit place value and how this is done as soon as students meet two-digit numbers, rather than when they need to be able to partition numbers for formal written techniques. Thompson reflects on the difficulty with vocabulary in English. Consider the English words for 23 and 14. The first word in 23 is twenty, this describes the value of the tens column, whilst the first part of 14 is four, which describes the unit part of the number. He contrasts this with European teaching which focuses on numbers up to 20, before moving on rather than numbers to 10. Although Thompson wrote about Year 2 students in 2000, the comments are still relevant as the Year 2 programme of study in the current National Curriculum still requires students to “recognise the place value of each digit in a two-digit number” (Department of Education, 2013b, p. 11).

2.6 In the Classroom

2.6.1 National Curriculum

In England, the teaching of mathematics in primary schools is dictated by the National Curriculum and the related programmes of study (Department of Education, 2013b). In each school year from Year 1 to Year 6 both place value and fractions are taught. As early as Year 1, students are expected to recognise a half and a quarter of a shape and the non-statutory guidance says that this should be “fractions of discrete and continuous quantities” (Department of Education, 2013b, p. 8). In Year 2, this extends to being able to find, recognise and name $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{2}{4}$ of shapes and quantities (Department of Education, 2013b). The non-statutory guidance suggests that students should be able to count up in a given

fraction past one, e.g., $\frac{1}{2}$, $1, 1\frac{1}{2}$, ... so that students can see that fractions are not limited to numbers less than 1 (Department of Education, 2013b).

The non-statutory guidance indicates that fractions should be taught with continuous quantities, that fractions can take on any value and, by Year 3, that students connect tenths with place value (Department of Education, 2013b). In Year 6, the statutory guidance specifically links fractions with the result of division (Department of Education, 2013b). What is interesting is that at no time are decimal places discussed in the place value section of the programme of study. In fact in Year 4, Roman numerals are specifically taught and the development of the number system over time to include the need for and the introduction of zero (Department of Education, 2013b) but decimals are not included. By Year 6, students are expected to understand place value up to 10 000 000. (Department of Education, 2013b). The word 'number', through every part of the place value sections of the programme of study appears to be limited to integers, although within the fractions parts of the curriculum students are expected to link place value with tenths etc.

At Key Stage 3 the National Curriculum states that children should be able to

- understand and use place value for decimals, measures and integers of any size.
- order positive and negative integers, decimals and fractions; use the number line as a model for ordering of the real numbers; use the symbols =, $\neq$, $<$, $>$, $\leq$, $\geq$ (Department of Education, 2013a, p. 5)

No order of teaching is specified in the current programme of study for KS3, but it is interesting to note that place value in KS1 and 2 refers only to numbers and, by

implication, these numbers are integers and that for KS3 place value includes decimals. Here is also the first mention of real numbers.

2.6.2 Problems

There are specific problems identified within the literature, for example Steinle and Stacey (2004) comment on how teaching can lead to reinforcing some of the common misconceptions, and how some rules can cover over misconceptions. An example of this is that if teachers only consider decimals of the same length this can reinforce that decimals form a whole number type system. One conclusion from Steinle and Stacey's work is that some students are "receiving teaching that is covering over rather than overcoming misconceptions". (Steinle & Stacey, 2004, p. 545).

Another problem is caused by the representations that are used to explain decimals to children. Roche (2010) discusses that the disadvantage of the 100-square, used by many text books and activities, is that it only discretely represents hundredths. I found in the piloting that the students were counting 25 mini squares and that there did not seem to be any meaningful connection between this and $\frac{25}{100}$.

Another representation is the number line. This is used widely for explaining the magnitudes of numbers and also for teaching arithmetic. Murphy (2011) compared the use of the empty number line in England and in the Netherlands. It appeared to be more successful in the Netherlands, where it was used as an exploration of the number line and the structure of addition, rather than in England where it was more likely to be used as only a method.

Although it can appear that a purely procedural approach to the magnitude of fractional numbers can lead to overgeneralisation and misconceptions being developed, there have been studies which consider both procedural and

conceptual competence e.g., (Siegler *et al.*, 2011, p. 275). Gabriel *et al.* (2012) also discuss both the procedural and the conceptual understanding that are needed for a thorough understanding of fractions. They recognise that both conceptual and procedural competence are needed. Resnick *et al.* (2016) discuss the development of number understanding and that children need to be aware that what they have learned for whole numbers does not necessarily hold for fractions. Their study concludes that fractional magnitude understanding is central to mathematical development. This is a similar finding to that in Siegler *et al.* (2011). Hallett *et al.* (2010) and Hallett *et al.* (2012) looked at students' answers to questions and whether the answers showed that procedure predates conceptual understanding, conceptual understanding predates procedure or whether the two go hand in hand. This paper discusses the fact that it may be the case that it is down to the individual child. Conceptual knowledge appears to be 'knowing that'; it is an interconnection between different ideas and the ability to see the connection. Procedural knowledge is 'knowing how' to do something. Procedural competency allows algorithms which are independent of meaning. The method in the project was to identify which questions are conceptual and which are procedural. Procedural questions are what can be tested and answered without much understanding. Defining a question as such can only be done with reference to a child's experience and what has been taught. Hallett *et al.* (2010) overcome this by talking to the class teacher about what has been taught.

Another issue with the teaching of fractions, according to Siegler *et al.* (2011) is the reliance on the part-whole model of interpreting fractions. The US and, in my experience, the UK rely heavily on such an interpretation. This has the advantage that it is easy to represent concretely. It does not, however, respond to the continuous nature of fractions or the magnitude.

Gardiner discusses “backward-looking” and “forward-looking” methods (Gardiner, 2016, pp. 11–12) in his book. Essentially a backward-looking method is expedient to help solve the particular problem at the time, whereas a forward-looking method allows the student to progress. Backward-looking methods may be easier in the short term but may also make progress harder later. Conversely, a forward-looking method may seem harder at the time but will still work when the work becomes more complicated. When I was a student at school, I always disliked the fact that, at many stages of learning, I was told that what I had learned before had to be ‘unlearned’. On the other hand, I appreciated the teacher in sixth form who told me that my way of writing a d (∂) was going to cause issues at university as it looked too much like a partial derivative. As a teacher, I have always tried to teach in a way that means that students can build on what I have taught in coming years rather than have to unlearn it. Simple examples would be not teaching that to multiply by 10 you add a zero; this fails when you multiply 4.7 by 10 (seen in Figure 6-12 where a student thinks that 4.90 is 10 times greater than 4.9). A more pedantic example of a forward-looking reply is where I will always tell a class that you cannot find a real number that is the square root of a negative number. It seems likely that it is important whether our representations of fractional amounts are forward-looking or backward-looking.

2.6.3 Decimat

Students encounter models and representations of fractions and decimals which are two-dimensional spatial models. The 100-square, often used to teach decimals, is a proportional model, where each small square is $\frac{1}{100}$ of the area of the model but also represents $\frac{1}{100}$ of the value of the larger square. This allows the students to work with the structure and relative value of place value. Wright (2004) discusses how the representations of decimals should allow students to appreciate

the structure and allow students to manipulate strong mental images. He advocates the use of the decimat developed in the Numeracy Development Projects in New Zealand (Ministry of Education, 2003, cited in Wright, 2004, p.3.). The strength of the decimat is that it is a two-dimensional, visual representation of a decimal number, which, like the 100-square, is a proportional model. An example of a blank decimat is at Figure 2-5.

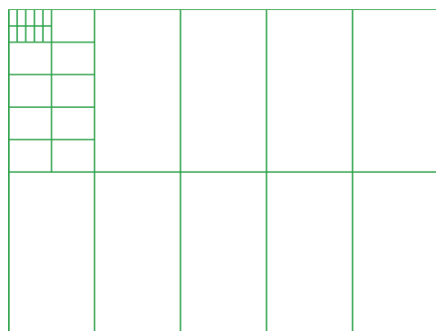


Figure 2-5 Decimat (Roche, 2005)

This is the form of an adapted form of the decimat used by Roche (2005, 2010) where I first became acquainted with the model. The decimat is a rectangle split into 10 smaller rectangles each representing $\frac{1}{10}$ or 0.1. One of the smaller rectangles is further split into 10, each of these smaller rectangles represents $\frac{1}{100}$ or 0.01. Again one of these smaller rectangles is split into 10, each of these rectangles represents $\frac{1}{1000}$ or 0.001. Any decimal of 3 decimal places can be represented in this way. It can be seen how this representation could be extended to any number of decimal places. This representation reinforces the denary system and constantly splitting into 10. The decimat depicts the structure of decimal numbers, with each smaller rectangle being $\frac{1}{10}$ of the area of the next larger. The decimat extends to representing thousandths and students can envisage further decimal places, by partitioning successively one of the smallest

rectangles into 10. Comparing the decimat to the 100-square the decimat has many advantages:-

- Only one rectangle needs to be coloured in for 0.1, 0.01 or 0.001 allowing conversations about equivalence but not requiring the student to be fluent with equivalence before using the model.
- The decimat clearly represents visually the proportional values of each decimal place. Although the 100-square is also proportional there are further arithmetic steps that would need to be taken to access the proportionality. Roche (2010, p. 5) describes how the decimat “makes explicit the ‘ten-ness’ of our base ten place value system”, meaning that the students can visually access that each successive decimal place is ten times smaller than the preceding one.
- The decimat allows students to envisage the infinite nature of a decimal representation by a continual partitioning of the smallest rectangle. Although portioning of the regions of the 100-square would be possible it is not the intended structure of the representation.
- The 100-square possibly leads to confirming whole number thinking (Roche, 2005) because the students rely solely on counting to find the decimal representation.

2.6.4 Suggestions

Authors have made suggestions as to what might be more effective in improving teaching of decimal and fractional amounts

- Roche (2005) suggests that teachers avoid rules which encourage whole number thinking (e.g. adding zeros to compare decimals of unequal length), encourage use of fractional language to describe decimals (*‘2.75 is “2 and*

75 hundredths” rather than “2 point 75” Roche (2005, p. 16) and to use representations such as the decimat (see section 2.6.3)

- Steinle and Stacey (2004) found that, by being involved in the testing with the DCT, 10% more students were assessed as being expert when they repeated the test, even without extra teaching. The study suggested no special teaching, although some teachers did. Steinle and Stacey (2004) conclude that this indicates that it should be relatively easy to improve decimal understanding. Steinle and Stacey express that the remedy for all of these misconceptions is basically the same: to expose the underlying place value structure of the number system and to make explicit connections between its many different facets.
- Siegler *et al.* (2011) suggest that children should be encouraged to build correct analogies between fractions and whole numbers.
- Siegler and Lortie–Forgues (2014) suggest that acquiring magnitude understanding of fractions requires the understanding of which properties of whole numbers apply to fractions (e.g. having magnitudes) and where they do not apply (e.g. unique successor).
- Obersteiner *et al.* (2013) find that teaching should focus on understanding the magnitude of fractions.
- Gabriel *et al.* (2012, p. 145) concluded that “Teaching methods should focus more in the common characteristics between fractions and whole numbers, such as both fractions and whole numbers have magnitudes that can be ordered, compared and represented on number lines”

2.7 Errors in mathematics

2.7.1 Introduction

Students, and their teachers, make errors in mathematics. It appears to be a commonly held belief that the beauty of teaching (or learning) mathematics is that there is 'only one right answer'. Leaving aside questions and tasks which require the student to work with an incomplete set of information and therefore require them to make assumptions, many textbook and exam questions have one answer or one clearly identifiable set of answers. It is, therefore, easy to decide whether the final answer is correct. What is harder to decipher is, for example, whether the answer has been reached by correct mathematical thinking, what caused any errors, and whether a student with a correct answer actually understands the mathematics or has just followed a procedure. Teachers need to be able to address these and it is therefore useful to consider the cause of an error.

From my reading, experience and discussion with other teachers, errors could be due to: -

- Slips
- Lack of understanding
- Literacy obstacles
- Pedagogic effects
- Misconceptions
- Notational obstacles
- Epistemological obstacles

Ryan and Williams (2007, p. 13) discussed the fact that as well as slips, errors can be diagnosed as “developmental errors”. The non-development errors include “jumping to conclusions” and “cognitive overload”, (Ryan & Williams, 2007, p. 13). Students may have performed one part of a two-part operation or may have read the question incorrectly.

When marking a task or a set of books it sometimes becomes clear that the same error has been made time and time again. An error being repeated systematically by students working independently indicates that the students might have employed the same, wrong, thinking. This is likely to indicate a misconception. Misconceptions are widely discussed in both the professional and academic literature. Often used by teachers to mean a mistake, misconception, by looking at its name is actually a mistake formed by a misapplied concept. Tirosh and Stavy (1999, p. 669) refer to an “alternative conception”.

Here I will use the term misconception to refer to errors formed by a conscious incorrect application of a conception. A misconception is an error where the student has applied a logic which, while inaccurate, makes sense to them. It is considered good practice to expose misconceptions, rather than to try to teach students to avoid them (Bell, 1993). However, exposing misconceptions is not enough in itself to ensure such errors are not repeated. Both Zazkis and Chernoff (2008) and Zaslavsky (2005) discuss that some learners can hold conflicting concepts. An example of this from my teaching is where a student is asked to write $\frac{3}{5}$ as a decimal and they will frequently write 3.5. The reply is so frequent that there must be some sense to it in the mind of students. Any of these students would be able to say $\frac{3}{5}$ was less than 1 and 3.5 was more than one, but it is only when their attention is drawn to the conflict that they become aware of it.

Ryan and Williams (2007, p. 13) categorise developmental error as

- “modelling”- where a real-life situation is incorrectly modelled
- “prototyping” – students hold somehow typical examples
- “overgeneralizing” - where students find a rule for the specific cases they have seen but incorrectly generalise to all cases
- “process-object linking”

Their concept of the process-object linking is similar to the “procept” in Gray and Tall (1994, p. 2). A procept is where notation represents both a concept and process. Students recognise an equals sign as a process (‘the answer to the question is’) but an equals sign also represents the concept of equality. Being able to embrace the ambiguity of notation and work flexibly with procepts allows a student be more flexible and have a better mathematical understanding (Gray & Tall, 1994) but difficulties with procepts increase the divide between more and less ‘able’ students (Gray & Tall, 1994).

Procepts are not the only notational issue. Notation and conventions can make mathematics harder for students. Consider $2a$, 29 and $2\frac{1}{4}$. Each of these has the same apparent form, $2 *$, but $2a$ means $2 \times a$, $2\frac{1}{4}$ means $2 + \frac{1}{4}$ and 29 it means $2 \times 10 + 9$. These problems lead to misconceptions and errors, such students who asked to find $2a$ when a is 4 give the answer as 24. Backhouse (1978) considers this difficulty, discussing AB being $10A + B$.

It is important to be able to identify misconceptions so that they can be addressed. Ryan and Williams (2007) discuss that using tests to diagnose errors is problematic and it is far better to talk to students and have them discuss their

reasoning. Errors in tests may well occur due to the stresses of tests, rather than any sensibly applied logic. Discussions are useful because misconceptions that will be evident during discussions whereas careless slips will often be corrected.

Steinle indicate that her analysis provides evidence that many students are receiving teaching that *covers over* rather than *overcomes* their misconceptions. (Steinle, 2004a, p. 3). The evidence for this statement was that 20% of children in their study, having been marked as experts on one test, fail to be marked as expert on the next. Although it may be the case that this has been caused or exacerbated by poor teaching it might be the case that there is another layer to consider.

Steinle (2004a, p. 3) refers to “latent” misconceptions and the failing of an incomplete algorithm. The error or misconception that Steinle refers to here is where the decimal portion of a number is treated as a whole number. Its prevalence across countries and teaching styles, and its persistence across ages, certainly points to an underlying cause rather than merely poor teaching.

In my experience students sometimes answer questions where they are not even in the position to over-generalise. To the onlooker, it appears that the student has guessed. To all intents and purposes this is an unanswerable question for the child. An example might be: Billy looks at a picture of a triangle where he is asked to find the angles. The question requires the sine rule to answer it. Billy, however, has not even been taught basic trigonometry. He tries to apply missing angle rules and fails as there is not enough information. He is unaware that there is even a rule he could have applied. In this situation he does not know what he does not know.

An error that might look similar to a teacher would be where a student has forgotten work. He has been taught the mathematics to answer the question and he knows that there is some mathematical tool he should be able to call on but cannot remember it. In this situation he knows what he does not know but it does not help him to be able to tackle the question.

The difference between these two circumstances is that under the right conditions a student, knowing what they did not know, could go back and work it out. They might be able to ask questions that will guide them towards the correct mathematical tool. In the first instance though they will be unaware that mathematics even exists to help him. Ryan and Williams (2007, p. 16) discuss these errors relating to the “limitations in their mathematical concepts”. As a teacher the warning from these types of error is to ensure that students are equipped with the knowledge to solve the problems they are being given.

Questions in mathematics lessons sometimes relate to the specific way mathematics is taught in school and the standard expectations on a student to interpret instructions in a certain way (Ryan & Williams, 2007, p. 17). This can lead to a social disadvantage because students from the middle classes are more likely to understand the ‘game of school’.

In my experience students with misconceptions will tell me that this is what they have been taught. Examples of this that I have encountered frequently are ‘a square is not a rectangle’, ‘you can’t take a bigger number from a smaller number’, ‘1.49 rounded to the nearest whole number is 2, because the 9 rounds the 4 to a 5 and that rounds the 1 to a 2’. Each of these happens too regularly for there to be no common cause. There are different possibilities for these misconceptions: -

- The previous teacher has taught them incorrectly. This, though, is not likely to keep happening with students coming from different schools and different teachers.
- The previous teacher taught them correctly considering a slightly different concept and the students have overgeneralised in applying what they have been taught.
- The students have misinterpreted what they have been taught.

As a student teacher I was talking to a class about the fact that $-x^2$ when $x = -3$ was -9 . The students said that Mr Wiltshire, their class teacher, had said it was 9. Mr Wiltshire was a very precise mathematician and, I knew that the most likely situation was that he had not taught them that, but that they had applied his teaching of $(-3)^2$ incorrectly. Avoiding these errors is closely linked to ensuring that teaching is “forward-looking” (see 2.6.2) and discussions around the use of mantras (see section 9.4.3).

2.7.2 How teaching can reinforce misconceptions.

As discussed in 2.5.5, there are two complete algorithms for comparing decimals: annexe zero and full left-to-right comparison.

Using annexe zero the student recognises that adding a zero to the end of a decimal has no effect on its magnitude: so $0.3 = 0.30$. This allows students to restate all the decimals as decimals of equal length. However, if these algorithms are incomplete, students can revert back to erroneous thinking. Steinle and Stacey (2004) found that apparent experts were prone to reverting. If using the left to right comparison of digits students reverted to erroneous thinking if one of the numbers ran out of digits.

Although the annexe zero algorithm is complete within itself, it fosters whole number thinking. Once the zeroes are added the student can completely ignore the decimal point. This means that they may well be able to order accurately but they may not appreciate the magnitude of each decimal place.

By careful selection of examples, it is possible, as teachers, to ensure that misconceptions are avoided. By only presenting decimals of a given length at a time (Steinle & Stacey, 2004, p. 547) students do not see the need to move from comparing 5.34 and 5.44 to comparing 5.434 and 5.43. If only decimals of the same length are considered, then students do not need to move from whole number thinking and do not develop their understanding of decimal magnitudes.

My own teaching in the past, and reports from many teachers, tell me that by putting everything in the context of money, and students will be more successful working with decimals. It is possible that money thinking is only a further example of whole number thinking. There is no requirement to interact with the decimals as the students can think in pence. Also, as discussed in Roche (2005), money is not good for representing place value. A 20p piece is not 10 times larger than a 2p piece. Other issues with money thinking are thrown up when the result of a money calculation on a calculator is, for example, 0.4. Students are as likely to read this as 4p instead of 40p.

When children develop a feel for numbers, 7 has a sevenness. They start by not linking the sevenness of a set of objects to the sevenness of another set of objects (Siegler *et al.*, 2011) then they grow to the point where sevenness is itself a constant. Seven is always more than 6. But with fractions in particular, it is important to ask fraction of what? Is $\frac{1}{2}$ always greater than $\frac{1}{4}$? That depends on whether the question specifies the answer to $\frac{1}{4}$ of what? Fractions therefore have

this dual meaning, similar to the procept (Gray & Tall, 1994), as a scalar operator and as an interpretation of a rational number that can be placed on a number line. On the number line $\frac{1}{2}$ is always greater than $\frac{1}{4}$ but $\frac{1}{2}$ of 8 is not greater than $\frac{1}{4}$ of 16.

In Siegler *et al.* (2011), students in the study were asked to estimate the magnitude of different fractions on a number line. There were two main strategies they used. One was to transfer the fraction to an easier number to deal with, for example considering that $\frac{5}{9}$ was just a little more than $\frac{1}{2}$. The second was segmentation of the number line by considering the denominator.

Siegler *et al.* (2011) states that refraining from encouraging whole number thinking does not mean that analogies between whole numbers and fractions should not be drawn.

2.7.3 Summary

Teaching can reinforce misconceptions by :-

- Using strategies, such as annex zero and using money as an example which reinforce whole number thinking.
- Teaching concepts that do not generalise into the realm of fractions, for example, multiplication makes the answer bigger.
- By not considering whether the concepts and the methods they are teaching are forward-looking.
- By selecting examples where the misconceptions can be avoided rather than tackled. For example by only asking students to compare the magnitude of decimals of the same length.

2.8 Implications of the literature on the current study

The literature review focused on types of number, in particular the different properties of natural numbers and rational numbers and how the structure of these numbers makes it hard to transfer ways of thinking about natural numbers to ways of thinking about rational numbers. In particular, there are difficulties with the density of numbers, the representation and the more complicated algorithms to calculate with rational numbers.

The literature showed that there are persistent misconceptions around the representation of fractional magnitudes, in either a fraction or a decimal format and that these misconceptions persist, sometimes well into adulthood. Even where misconceptions are not held then the literature shows that there is a natural number bias which exists, even in mathematics postgraduate students, making judgements about the magnitudes of rational numbers harder than judgements about natural numbers. Likely misconceptions are caused by trying to apply natural number thinking to rational numbers, even when this is not appropriate. Although some level of natural number bias appears to be unavoidable, misconceptions can be addressed and accounted for. Suggestions from the literature to help students include: -

- avoiding rules which encourage whole number thinking (Roche, 2005),
- making explicit connections and analogies between natural number and rational number (Gabriel *et al.*, 2012; Siegler *et al.*, 2011; Siegler & Lortie-Forgues, 2014; Steinle & Stacey, 2004)
- focusing teaching on the magnitudes of fractions (Obersteiner *et al.*, 2013)
- ensuring that students have to activate difficult thinking rather than selecting intuitive answers. (Obersteiner *et al.*, 2016)

Chapter 3

3 Mathematical Structure

3.1 Introduction

I have worked in five different mathematics departments in my career, working with approximately 50 different teachers. In addition, I frequently attend mathematics CPD events across schools and am in various social media groups for teachers. In my experience, there is a feeling amongst these teachers that mathematics is often taught in UK secondary schools to ensure that students can recite and apply procedures rather than to become mathematical. Although the following quote is old it does not feel as if there is much that has changed in the years in between.

A growing body of research that suggests that children's mathematical instruction in school results in learning rules and procedures without developing the essential conceptual understanding (Wood *et al.*, 1991, p. 588)

More up to date though, the Ofsted review of English mathematics education in 2021, calls for limited use of informal procedures and classifies declarative knowledge of facts as “conceptual understanding” (Ofsted, 2021).

Whilst describing the bidirectional relation between conceptual and procedural knowledge Rittle-Johnson *et al.* (2015) consider whether there is an ideal balance or sequence between conceptual and procedural, but find that this is still an unanswered question. They do, however, state:

Mathematical competence rests on developing both conceptual and procedural knowledge, and it is widely agreed that conceptual knowledge often supports and leads to procedural knowledge. (Rittle-Johnson *et al.*, 2015, p. 594)

Research and the profession alike both claim that both procedural and conceptual competence together is the ideal way to learn mathematics. My experience shows me that many students learn mathematics with a procedural approach and conceptual competence is not as widely achieved. In my experience, and in my analysis of the literature, the approach to becoming mathematical is through structure and through engaging with the connections between concepts.

We start from the position that mastering procedures is an important component of taking advantage of opportunities to make mathematical sense, but that it is of little value to learners if it is simply a procedure, because as the number of procedures increases, the load on memory and retrieval becomes more and more burdensome. (Mason *et al.*, 2009, p. 11)

Although there is an increasing demand for fluency in the professional literature, I do not believe that this needs to replace mathematical thinking. Such thinking is characterised by “Specializing”, “Conjecturing”, “Generalizing” and “Convincing” (Burton, 1984, p. 36) and also “Refutation” (Richards, 1991, p. 31). Activities and tasks that include these do not preclude practising and attaining procedural competence.

Bell (1993, pp. 9–10) discusses the principle of connectedness. He finds that “connected bodies of knowledge” are better understood and more easily retrieved.

As Byers wrote: -

Being able to connect different representations ... of the same object has been shown to be a strong indicator of a college student’s mathematical knowledge and ability (P. Byers, 2010, p. 1)

To me, this connectedness is part of the underlying structure of mathematics. In this I found support for looking to the underlying structure (Askew *et al.*, 1997; Mason *et al.*, 2009; Skemp, 1976; Swan, 2006a). My personal experience of mathematics is to investigate the beauty of how many connections there are and to consider the similarity and the differences between different mathematical

concepts and this is the approach that I have wanted to pass on to the students I teach.

I began this research with a vague concept of structure. For me, it was how mathematical ideas were built on other mathematical ideas and the links between them. I needed to define structure carefully for the purposes of this study. I do this below starting with *instantiations* and *properties*.

From a pedagogical point of view, mathematics is usually taught using concepts. Each concept might be a mathematical relationship or a model. For example, students might be being taught about the multiplicative relationship or the partitive model of division. Each of these is something that the student will need to conceptualise. Numbers, shapes, relations, sets, and functions are all examples of such clusters of concepts. The concepts and structures in mathematics are objective and do not alter because of who is considering them. It is clear though, from teaching, that students' understanding of the concepts and the structure are not always the same as the generally accepted concepts or even as the teacher intended them.

This was considered by Tall and Vinner, who differentiated between the "concept definition" and the "concept image" (Tall & Vinner, 1981, p. 152). The concept definition is the generally accepted formal definition of a mathematical idea or concept. The concept image is the sum of the experiences and conceptions of the student. It is individual to the student. The concept definition and the concept image are rarely the same and this is where errors can occur. A part of a teacher's job is to help the student move their concept image towards a better match of the concept definition. I have had to recognise in my own teaching that my concept images did not always coincide with the concept definition.

Schools nationwide (the study school included) have started to become increasingly interested in Cognitive Load Theory (Sweller *et al.*, 1998) and Rosenshine's Principles of Instruction (Rosenhine, 2012). Although the ideas around Rosenshine's principles have been around for many years, it appears to be this particular article that has had the most influence recently in schools. There has, in schools, been a resurgence of discussion about the formation of schema in long term memory as described by Piaget (1952). It seemed to me that a schema is similar to a concept image. It is the sum and organisation of all experiences around the concept. I elaborate the idea of schema in section 7.1.3.1 and consider the similarities and differences between schema and concept images at section 7.1.3. In the end though, both describe how the student sees the mathematical world.

In defining structure, I started with concepts and the relations and connections between them, viewed from the point of view of the learner.

3.2 Instantiations

Mason *et al.* (2009) outline a theory of the use of mathematical structure in the classroom. In defining structure for the paper, they state,

We take mathematical structure to mean the identification of general properties which are instantiated in particular situations as relationships between elements. (Mason *et al.*, 2009, p. 10)

I adapted the idea of instantiated properties to help me define mathematical structure for the purposes of this study. Firstly, I defined instantiations and considered properties of each instantiation. This gave me a concrete idea of how to work with my concept of structure. An instantiation is an instance of a concept, each concept might have many instantiations. Each will be connected, as they are

instantiations of the same concept, but to the learner they might look very different. It is best described by examples.

Consider, for example, the multiplicative relationship $ab = c$. This relationship has many instantiations, for example, different models of multiplication and division. Students in a classroom might work with an instantiation of a sweets in b bags. A student in secondary school might consider two amounts (a and c) in direct proportion, with the constant of proportionality, b . These are possible instantiations.

In this thesis, I will use the term mathematical *instantiation*³ to be the mathematical focus of a teaching excerpt. Instantiations can be considered to reflect the viewpoint of the learner. To illustrate this, let us consider solving linear equations of the form $ax = b$. My experience shows me that in a class of twelve-year olds, their experience means that $3x = 9$ is an easy equation that they barely need to think about, but that they would consider the equation $3x = 10$ unsolvable. I would, therefore, say these are different instantiations, though to a group of students more competent with solving equations, both are examples of the same instantiation, the solution to $ax = b$, being $x = \frac{b}{a}$. Both are examples of the general class of proportional relationships and, written as $ax = b$, are represented in the same algebraic form.

Students, in the classroom, encounter various concepts; they meet them through representations. A representation, in this sense, is a way to describe or symbolise a situation. Students act upon those representations and are likely to consider this

³ My use of instantiation is specific for this thesis, although also has other uses within mathematics education literature.

combination of representation and action to be 'the mathematics' rather than considering the underlying mathematical concept. Any representation of a concept is likely to be an instantiation but not all instantiations will be representations.

3.3 Concepts and properties

Mathematics may be experienced conceptually as a series of disconnected ideas. However, mathematics is many-layered. These layers or conceptual developments can be experienced by different instantiations. Each mathematical concept may be itself experienced as an instantiation of a deeper, more abstract mathematical object. To use the example already given: partitive division is an instantiation of division which is, in itself, an instantiation of the multiplicative relationship which could be seen as an instantiation of a polynomial relationship. Broadly, the mathematics can be considered to be more abstract the further through the layers travelled. The details of the mathematics and the depth of the possible instantiations are beyond the scope of this thesis. This thesis will focus on the instantiations being discussed and the next layer of structure.

For each concept there is a set of properties that apply to all instantiations. Two such instantiations might be, in some sense, isomorphic to each other; exactly how this isomorphism is defined is considered further below.

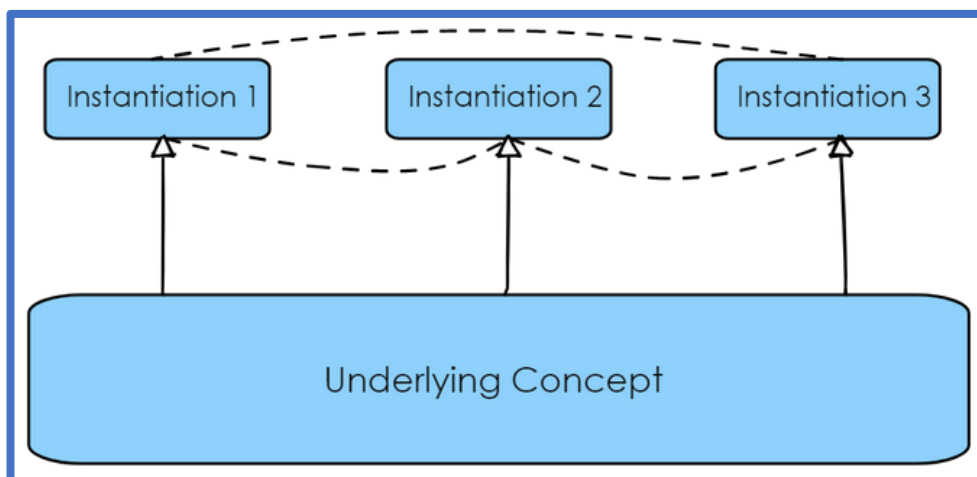


Figure 3-1 a visual representation of concepts and instantiations

Figure 3-1 is a visual representation of how structure is defined here, the solid arrows indicate an instantiation of a concept and the dotted lines the properties that hold between different instantiations.

Structure is an everyday term. The dictionary refers to “arrangements of and relations between ... elements” (OED, n.d.). This resonates with the definition above from Mason *et al.* (2009, p. 10). In this thesis, I take *structure* to mean the relationships between a concept and its instantiations and properties that are maintained and differ from one instantiation to another. This definition fits in with both OED and Mason *et al.* above.

For example, consider a linear relationship between x and y .

This could be expressed as: -

- the set of ordered pairs $\{(x, mx + c) : x \in \mathbf{R}\}$
- the equation (formula) $y = mx + c \forall x, y \in \mathbf{R} \ m, c \in \mathbf{R}$
- the equation $ax + by + c = 0 \ a, b, c, x, y \in \mathbf{R}$
- the graph depicting the straight line
- a function mapping $f(x) \rightarrow mx + c$

Each of these can be used to describe the same linear relationship.

The instantiations, however, might appear different. The graph is visually different from the algebraic or set notation representations. $y = mx + c$ is, in terms of GCSE mathematics, a formula whereas $ax + by + c = 0$ is an equation. The function could be expressed in terms of function notation, as above, or could use the idea of function machines; first multiplying by m then adding c . Only one of these does not directly interact with m (the gradient of the line) and c (the y intercept) but in

the case $ax + by + c = 0$ algebraic manipulation soon reveals that the gradient is $-\frac{a}{b}$ and the y intercept is $-\frac{c}{b}$

3.4 Application

Mason *et al.* (2009) discuss developing an awareness of mathematical structure in both teachers and students to transform mathematical thinking. They conjecture that this could promote a more “relational” and less “instrumental” (Skemp, 1976) approach to teaching mathematics.

Each example of an underlying concept is also itself an instantiation of another concept. By generalising or restricting the focus, the structure and the properties that can be applied across the instantiations can be specified. For example, by considering the field of real numbers one property of multiplication is that it is commutative, but by extending the structure to the multiplication of square matrices, the property of commutativity is lost. Teachers, therefore, need to be very clear about the focus of their interest.

For example, the linear relationship above could be considered to be a specific instantiation of a quadratic relationship $y = ax^2 + bx + c$, in the special case where $a = 0$. In this case where $a = 0$ the graph will be a straight line rather than a curve, c will still, however, be the y intercept. When we consider $y = mx + c$ clearly shows us that the gradient is m , the generalisation required to reveal the gradient of $y = ax^2 + bx + c$ is to differentiate y with respect to x . This property holds for the specific case of the linear relationship and still reveals and the gradient to be the coefficient of x .

When considering the linear relationship, restricting x to natural numbers would result in the topic of linear sequences and n th term which can be represented by

pictures and by algebra in the form $u_n = an + b$. Some properties, but not all, hold between this and the equation of a straight line. An example would be the graphical representation. The graph of $y = ax + b$ would be a straight line with gradient a and y-intercept b . A graphical representation of $u_x = ax + b$ would be discrete points from the same line, where x was an integer.

In primary school, teachers look at division as either partitive or quotitive: items are either shared out in equal parts (partitive) or are shared into groups of equal amounts (quotitive). 24 sweets can be split into 6 groups of 4 sweets or can be shared amongst 6 people, each getting 4 sweets. Both of these models allow children to engage with a practical interpretation of division. They might seem very different to a child, but a teacher is aware that these are instances of $4 = 24 \div 6$. The two are similar, they are instantiations of the multiplicative relationship $a = \frac{b}{c}$. By considering what happens to a if b increases or b as a increases it can be seen that the structure of both is the same. However, close as these two instantiations appear to be, they are not isomorphic, there are slight structural differences. The result of the above partitive division is measured in number of sweets, whereas the quotitive result can be seen as a ratio.

A third instantiation of division would be scaling. Scaling is an idea encountered at secondary school that does not fit into the partitive or quotitive models. Increasing a shape using a scale factor of $\frac{3}{2}$, making every side 1.5 times bigger, is an example of multiplicative reasoning. Reversing the process to find the original shape after such an enlargement would require division by $\frac{3}{2}$. At this point the primary school models of division no longer appear to work but the relationships are still the same. The structure of division and these three instantiations is shown below at Figure 3-2

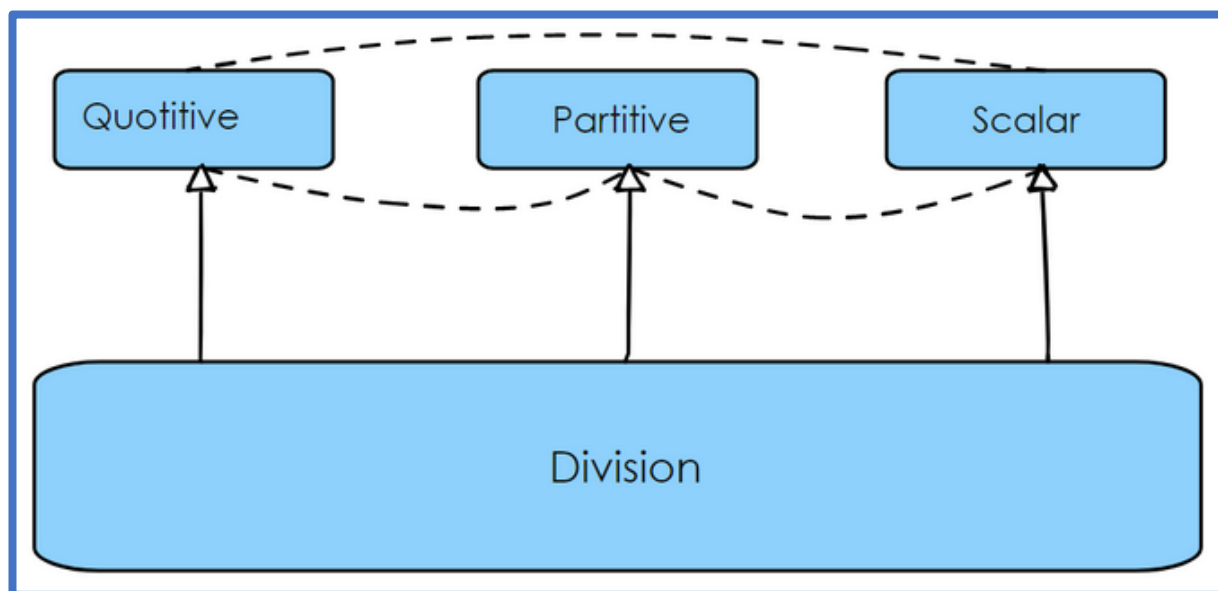


Figure 3-2 A visual representation of division and three instantiations.

My conjecture is that by selecting tasks that require students to work with different instantiations and to move between the instantiations they will develop an appreciation of structure and will improve their mathematical thinking and understanding. I have based the conjecture on my experience and on the work of, for example, Askew *et al.* (1997), Skemp (1976), and Swan (2006a). Skemp, when discussing the inclusion of certain unifying parts of mathematics in the school curriculum, says

Unfortunately the benefits which might come from teaching them are often lost by teaching them as separate topics, rather than as fundamental concepts by which whole areas of mathematics can be inter-related (Skemp, 1976, p. 24)

There has been significant work on the benefits of offering students multiple representations of a concept to build conceptual understanding across a variety of different mathematical areas (Brenner *et al.*, 1997; P. Byers, 2008, 2010; Hodgen *et al.*, 2014; Swan, 2008). Each of these considered multiple representations in teaching different concepts. P. Byers (2010) compared high school courses and textbooks treatment of the multiple representations of trigonometry as preparation for college courses. Swan details using multiple representations to “foster

conceptual understanding” (Swan, 2008, p. 1) in teaching about algebraic expressions. The study described by Brenner *et al.* (1997) taught algebra to six classes; three using traditional methods and three focusing on the use of multiple representations. The students in the intervention group were found to be more successful at problem representation tasks. All three found that considering multiple representations improved conceptual understanding. ICCAMS aimed to improve understanding and uses “connectionist teaching, multiple representations and collaborative work” (Hodgen *et al.*, 2014, p. 168). The preliminary findings from the ICCAMS team (Hodgen *et al.*, 2014) indicate that there may be a substantial positive effect in terms of the rate of learning.

3.5 Necessary and essential properties

Any concept under consideration has properties. The properties are either *necessary* or not. For the purpose of this study I define, *necessary properties* as ones that must hold true for a given instantiation of the concept.

Taking, for example, a square it has a necessary property that it has four right angles. This is usually part of the definition of a square. The diagonals of a square are equal and bisect at 90° . This is a necessary property that does not usually form part of the definition but could, along with other properties, form part of an alternative definition. Necessary properties are those that, with other necessary properties, could form an alternative definition. This definition of necessary properties is similar to that of Hewitt (1999), who defines necessary properties as those that must hold true mathematically, rather than those which are conventions or could be replaced. In my definition necessary properties hold true in a given instantiation and therefore, could include Hewitt’s arbitrary properties. An example might be that the sum of the interior angles of all triangles is 180° . For Hewitt, the constancy of the angle sum is necessary but the value of 180° is arbitrary because

other measures of angles could be used. For my work it would be a necessary property of any triangle that the angle sum was 180° . The fact that the measurement used an arbitrary unit would be something I would make clear to the students but I would not differentiate here between Hewitt's arbitrary and necessary properties. Measuring in degrees would be one instantiation and measuring in radians would likely be another, the unit of measure would be a necessary property for the instantiation.

I define *essential properties* as necessary properties which hold across all the instantiations which are under consideration. It is not necessary to consider whether these properties are sufficient; there is no intention to build the concepts from the properties, just that moving from one instantiation to another the properties which hold in both can be identified. The instantiation or the representation may introduce properties which are not necessary for the underlying concept. For example, an instantiation which introduces a concrete representation might introduce restrictions that a more visual or abstract representation does not. In this case the instantiation would have necessary properties which would not hold for the underlying concept. An example could be that when using Cuisenaire rods to represent an algebraic expression coefficients may well be restricted to integers, this would become a necessary property of the instantiation but not of the mathematical concept that is being represented. A representation or instantiation may not be flexible enough to describe all possible outcomes.

Consider different instantiations of trigonometry. Four instantiations might be: -

- trigonometric ratios as functions on the real numbers $f(x) = \sin x$ etc
- the graph of the trigonometric functions

- trigonometric ratios as applied to right angled triangles.
- trigonometric ratios as defined by the (x, y) coordinate on the unit circle.

The graph and the right-angled triangle are standardly met in schools in England in Years 9 to 11. The unit circle might be introduced in KS4; but trigonometric functions in the real numbers would not be met until Key Stage 5.

Each instantiation can be used to explain what happens to one variable when another changes. By considering one instantiation it is possible to focus on one particular aspect of the concept. By considering, for example, trigonometric ratios as resulting from right-angled triangles, the domain is restricted to the functions in x such that $0 < x < \frac{\pi}{2}$, thus never realising, for example, that trigonometric ratios can be negative or that the tangent function is not continuous everywhere or that trigonometric functions are many to one. By restricting the focus, properties have been introduced (e.g. $\sin(x) > 0$) which are not essential properties, they are necessarily true for this instantiation but are not true for the others or for the underlying concept. The four instantiations are only isomorphic to each other on a restricted domain ($0 < x < \frac{\pi}{2}$).

Careful thought needs to be taken when defining properties which are to be considered essential: namely the necessary properties for the underlying mathematical concept that is necessary to each of the instantiations. Properties which are non-essential are either not necessary properties for the instantiation or are not necessary conditions for the underlying concept.

Consider the trigonometry example above: -

- $\sin(x) \leq 1$ is necessary property of each of the instantiations used in the example,

- $\sin(x) > 0$ is a necessary property of the right-angled triangle instantiation but does not hold for the other instantiations.

Therefore, only the first is an essential property. My own experience of the classroom would tell me that sometimes concepts are introduced with necessary properties which students hold onto, in error, when new concepts are introduced. It is also important to realise that by focusing on properties which only hold for the instantiations in question students might be led to overgeneralise (e.g., they might decide that $\sin(x)$ cannot be negative). This overgeneralisation from properties which are not essential could be a cause of misconceptions. For example, when students first learn about multiplication, then they can use a property that multiplication makes larger. This property holds for all natural numbers. It does not hold for fractional numbers with magnitude less than 1.

Consider the solutions to $x^2 = a, x \in \mathbf{R}$. An essential property of this class of equations is that there are no solutions unless $a \geq 0$. If x is extended to the complex numbers, this is no longer an essential property as this is not a necessary property in the complex numbers.

Necessary properties depend on the instantiation and its domain of application. Essential properties depend on which particular instantiations are in focus. By designing tasks that require students to focus on the essential properties to move between instantiations or to reflect on which properties mean that instantiations are not isomorphic, students are focusing on the links and connections in mathematics. Many authors state that making connections is beneficial in terms of improving the understanding of students for example: NCTM, (2000), Askew *et al.*, (1997), Mason *et al.* (2009), and Siegler *et al.* (2011) also discuss ensuring that differences are explicitly considered. Moreover, by explicitly making links between

new concepts and instantiations that students have already understood, students can draw upon previous knowledge and apply it as they learn. Building mathematical competence requires that learners can build upon their existing knowledge and mathematical experience (Hodgen *et al.*, 2018).

Tall discusses the how awareness of properties in changing contexts enhances learning.

To support meaningful learning of mathematics over the long term, the aim is for teachers and learners to become aware of properties that remain supportive through several changes in context to give a stable foundation for new learning. (Tall, 2019, p. 6)

Vale *et al.* (2011) consider the mathematical development of teachers and the models of mathematical knowledge that frame teacher development.

These models allow teachers to connect their own understandings with more complex concepts, identify general properties, and blend their understandings into more encompassing models of mathematical ideas and hence begin to appreciate the importance of structure in mathematics and for mathematics learning (Vale *et al.*, 2011, p. 210)

When solving a question, the technique that is used must be related to the structure of the question. When solving $3x + 5 = 2x - 1$ a standard procedure is to transform the equation until it is in the form $ax = b$. From this point the solution is relatively straight forward. $3x + 5 = 2x - 1$ is an example of a linear equation; an instantiation of a more general polynomial equation. Considering solving $x^2 + 3x = 6 - 2x$ students often employ a similar strategy. There is logic in this - it is a thought-out response; however, this is an overgeneralisation.

Finally, I defined structure as: -

the relationships between a concept and its instantiations and properties that are maintained from one instantiation to another.

Chapter 4

4 My Study

This study aimed to put teaching via instantiations and their properties into practice in the classroom. In a small-scale intervention, students were taught by a teacher who had specifically engaged with the structure of what is to be learnt. The students themselves were not taught about structure in general, but were taught about the properties of the specific instantiations.

4.1 Structure in my study

The first stage of defining structure was to select the concepts that would be taught in the intervention and then to consider the instantiations that would be taught. Because the instantiations are the different ways a concept can be viewed, through the lens of a learner, not all instantiations were possible to anticipate. Although many instantiations might be anticipated by a teacher, learners might be aware of instantiations unacknowledged by the teacher. An example I used earlier (see section 3.1) is the idea that learners might view $3x = 9$ as a different instantiation of a linear relationship than $3x = 10$ but a more advanced student or a teacher might consider them to be the same. This means that a teacher might perceive fewer instantiations of a concept than their students would. If we take the example above, I would consider $3x = 9$ to be isomorphic to $3x = 10$ and I would have considered them to be examples of the same instantiation. The student who might consider them to be different instantiations, also does not see them as isomorphic. In the student's eye, the instantiation for $3x = 9$ held the necessary property that x was an integer, whereas I had intended that x was a real number. In section 2.5.4 above I referred to Greer (1994), who stated that students do not generalise from the realm of integers into the realm of rational numbers.

4.2 Structure of numbers

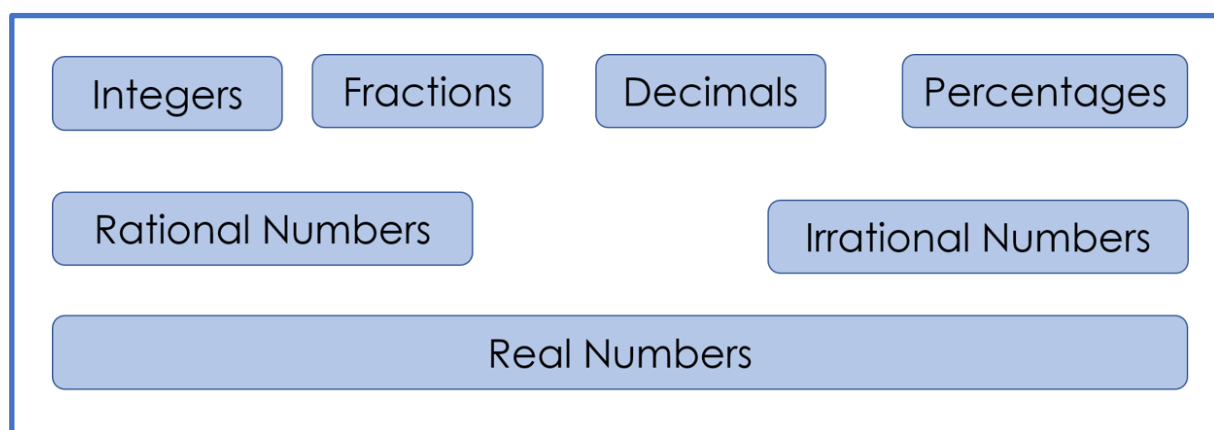


Figure 4-1 Structure of the real numbers

Consider the real numbers. One instantiation of the real numbers is the number line. As seen in section 2.3.3, the number line can be considered as a good mental representation for the real numbers. Other instantiations, such as formal definitions, of the real numbers also include the rational and irrational numbers. As described before, the irrational numbers are not particularly useful for this study so I will focus on the rational numbers. Most properties of real numbers also hold for rational numbers, but the two sets are not isomorphic. In terms of this study, it is important for teachers to know that rational and real numbers are not the same, however, the fact will have little to no impact on the study, because any decimal or fractional representation of an irrational number would be an approximation and therefore, by definition, finite or recurring and therefore rational. Any representation of an irrational number that was exact (e.g. $\sqrt{2}$) would not be a decimal or fractional representation.

One instantiation of the rational numbers is the integers, and the integers have an instantiation in the natural numbers. Table 4-1 shows some of the key properties of each instantiation.

Property	Rational	Integers	Natural Numbers
Has a magnitude	true	true	true
Can be ordered	true	true	true
Has a unique successor	false	true	true
$x + 3 > x$	true	true	true
$x + y > x$	sometimes true	sometimes true	true
$xy \geq x$	sometimes true	sometimes true	true

Table 4-1 Properties of the rational numbers, integers and natural numbers

Where properties are marked as ‘true’ these are necessary properties for that instantiation. The essential properties are the ones that are true in all three instantiations. The fact that numbers can be ordered is a necessary property of all three types of number, it is therefore an essential property. This would not be the case if we extended to the complex numbers. Having a unique successor is a necessary property of integers and natural numbers but not of rational numbers. This would be an essential property if only the natural numbers and the integers were being considered but not if all three were under consideration. Siegler *et al.* (2011) advocated teaching explicitly where these properties held and where they failed to hold.

Rational numbers can be expressed, by definition, as the division of two integers, with the divisor not equal to 0. This is an essential property that holds only for integers, fractions and rational numbers. Up to a point decimals and fractions can be treated as isomorphic. The majority of necessary properties that hold for one, hold for the other. In fact, the only property that does not hold for decimals that

does for fractions is how they are represented, the fact that fractions are represented as a ratio of integers. My conjecture is that by focusing on where the structure is the same and where it is different, students' understandings of real numbers will improve. As a technical point, fractions can only represent the rational numbers, whereas unbounded decimals are able to represent all real numbers (DeWolf *et al.*, 2015). However, for every decimal a student would be able to write, there is an exact fractional representation. Students looking at decimals are therefore really only considering the rational numbers.

One of the important aspects of the numerical development of students is learning that the properties which hold for natural numbers do not necessarily hold for real or rational numbers. Siegler *et al.* (2011) detail this in their integrated theory. As I explained in section 2.3.2, students need to learn that rational and real numbers do not have unique successors, are dense and that the result of calculations is not as predictable. For example, multiplication; $xy \geq x$ is true for all x, y if x and y are natural numbers but not if they are real. This resonates with my approach, where I intended to consider the essential and necessary properties of different instantiations of the real number line. My intention was to start with all the different instantiations that I was aware of, consider the similarities and the differences and try to bring some instantiations together. Siegler *et al.* (2011) base the development of number around the understanding of each different instantiation of the real numbers. This needs to be a focus of the intervention.

One way of thinking about this pair of developments (learning the properties shared and not shared by different types of numbers) is as a gradual change from initially conceptualizing numbers in terms of characteristic features (salient properties of whole numbers that are not necessarily properties of other types of numbers) to later distinguishing between defining features that apply to some but not all classes of numbers (Siegler *et al.*, 2011, p. 275).

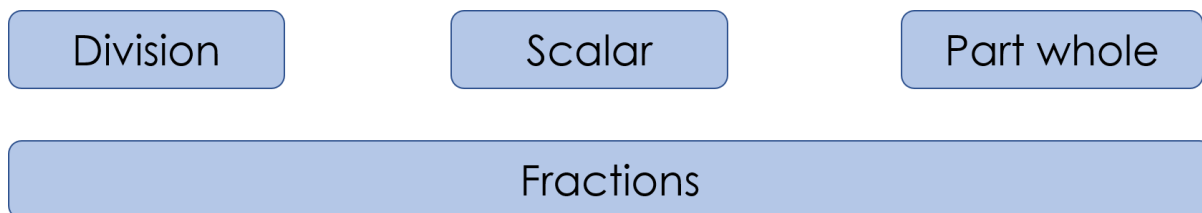


Figure 4-2 Instantiations of fractions

Figure 4-2 shows diagrammatically some of the instantiations of fractions.

Fractions can be thought of via the part-whole model, as a division of quantities (discrete or continuous) or as a scalar operator. In terms of increasing sophistication, the part-whole is met first in the curriculum with division next and scalar operators third. The part-whole model is used in the UK and the US extensively when teaching fractions. Siegler *et al.* (2011) criticises the model for the following reasons. Whilst $\frac{3}{4}$ meaning 3 parts out of 4 (e.g. 3 pieces of a pizza cut into 4) is an accessible idea, there are some drawbacks with the model.

Cutting a pizza into 4 does not in itself guarantee that the pieces are equal, $\frac{778}{893}$ is not so accessible, mixed numbers are not so easy to explain; it is not so easy to explain $\frac{7}{4}$ as 7 pieces of pizza from a pizza cut into 4 and this does not represent negative fractions well.

4.2.1 Applying structure to the study

The concept underlying my study is to select and design tasks which require students to work with the necessary and essential properties of different instantiations so that they engage with the structure of the numbers, avoiding the overgeneralisation of the properties of integers to decimals and fractions, which could become barriers to good understanding.

The starting point is ensuring that the students work to improve their understanding of the magnitudes of each of the different instantiations of real numbers (Siegler *et al.*, 2011) and “to expose the underlying place value structure of the number system and to make explicit connections between its many different facets” (Steinle & Stacey, 2004, p. 548).

4.3 Number Lines

In my experience, most students arriving at a comprehensive secondary school are able to order natural numbers, for example, they can state that, 67 is greater than 54. On the other hand, many state that 6 is obviously smaller than 54 and so 0.6 must be smaller than 0.54. This type of thinking is classified by Steinle and Stacey (2004) as longer-is-larger. It is very common thinking in the development of number thinking. This is an example of a natural number bias.

Maybe as a reaction to the longer-is-larger misconception students sometimes appear to develop a shorter-is-larger misconception. For example, although $11 > 3$, $-11 < -3$ and with fractions $\frac{1}{11} < \frac{1}{3}$, students appear to overgeneralise this thinking and apply it to an inappropriate set of circumstances. Steinle and Stacey (2004) even found that the shorter-is-larger misconception led to students selecting 0.3 as larger than 0.4, again potentially showing reciprocal thinking.

An issue for both students and teachers is that the incorrect assumptions or the incorrect conceptualisation will sometimes produce correct results. For example, 0.545 is greater than 0.4 and 0.43 is greater than 0.343. Obtaining a correct answer reinforces the students' conviction that their method is correct. Although there is always a possibility when designing tasks that correct answers can be found using incorrect reasoning, it is particularly the case when considering decimal magnitudes because of the overlap of decimals that are longer and

decimals that are larger. Take, for example 0.54, any decimal greater than 0.54 is larger but any decimal with three decimal places or more is longer. Within the set of answers that are longer than two decimal places, any decimal which is greater than 0.54 will also give a correct response. Just asking a student to decide which of two decimals is larger is not going to give the teacher the security that the student is applying the correct logic. Several questions would need to be asked.

Whole numbers have other properties that students internalise and would have to counter to have a good understanding of the real number line. As discussed before, whole numbers, for example, have a unique successor and predecessor. This is not the case with real, or specifically, rational numbers.

4.4 Awkward fraction rules

Next to algebra, 'fraction arithmetic' is the topic students come into my classroom with the biggest dread towards their lessons. They have memories of confusion and so many rules to learn. If we consider mathematics instruction as a list of rules and procedures, then the rules around fractions seem particularly bewildering. The following is a list of common fraction rules which might result from a procedural exploration of fractions.

- As the denominator gets bigger the fraction gets smaller but as the numerator gets bigger the fraction gets bigger.
- To find equivalent fractions, multiply the top and the bottom by the same number.
- When multiplying a fraction by a whole number, only the top changes.
- When you add fractions, you need a common denominator but when you multiply them you do not.
- When you add fractions, you have to use multiplication.

- When you divide fractions, you have to flip one fraction but not the other.
- When you add mixed numbers, you can think of the whole number separately from the fractional part.
- When you multiply mixed numbers, you cannot think of the whole number separately from the fractional part.
- When you add fractions with a common denominator the denominator does not change.

If these are only ever taught as rules, then it is unsurprising confusion occurs.

4.5 The end goal

I asked a mathematics graduate how they knew 0.83 was bigger than 0.7 and they told me that the first decimal place was greater. I had heard such an answer from a student; this might be classed by Steinle and Stacey as an apparent expert (Steinle & Stacey, 2004). Both gave the same answer, however this answer, by itself, does not demonstrate full understanding, because either could be using a full or partial algorithm to decide. Only further questioning can determine this. I pursued their reasoning with both. With the adult, I asked how she knew, and she said well 0.83 is 83 100ths and 0.7 is 70 100ths and that makes it clear that she understood. When I asked the student, she was not able to explain further. It is clear that an end goal is needed. It needs to be clear what an expert would know or be able to do that an apparent expert would not. It needs to be considered in what way “expertness” can be demonstrated. Stacey (2005) discusses this: -

For the purpose of our wider work on teaching and learning about decimals, full understanding means that they should be able to interpret a number such as 17.373 in terms of place value in several ways (as 17 + 3 tenths + 7 hundredths + 3 thousandths or as 17 + 373 thousandths, etc) and to appreciate that it is less than halfway between 17 and 18, close to 17.4 but with an infinite number of numbers between it and 17.4 (Stacey, 2005, pp. 20–21).

I think that this is a good starting point for a definition of being an expert. I would probably also want them to be able to locate it on a number line, consider an infinity of fractions equivalent to it and to recognise it as the result of the division sums represented by the fractions.

4.6 Intervention design

The underlying conjecture of the intervention is that students' understanding of mathematical ideas will improve if they specifically engage with the structure. This is not to say that they were taught about structure itself, but that the tasks required them to engage with the structure. Tasks were provided which required students to interact with the different instantiations, considering the properties of the instantiation and moving from one instantiation to another in order to appreciate the magnitude of the numbers given the representation they were currently using.

The essential property across all of the instantiations of fractional numbers is that they have a magnitude and can be ordered. Tasks which will allow students to interact with magnitudes are:

- Colouring of shapes into fractions and decimals, e.g. 100-squares and decimats (Roche, 2005)
- Colouring in shapes with perceptual distractors (Swan, 1983)
- Placing numbers on a number line. (Siegler *et al.*, 2011)
- Benchmarking with known points for example, bigger or smaller than 1, $\frac{1}{2}$, 0.5, (Roche, 2005)

While requiring the students to move from one instantiation to another, it is important to consider the necessary properties of each instantiation and the essential properties across all instantiations under consideration.

- An essential property is that all numbers have a magnitude and that this is how a student can move from one instantiation of a real number to another.

Tasks can consider some of the following properties

- All integers can be written as a decimal or fraction.
- All rational numbers can be written as a result of a division.
- All terminating and recurring decimals can be written as a fraction.
- An appreciation that there might be decimals which did not terminate or recur and that these might not therefore be able to be represented as a fraction.
- Transferring from decimals to fractions (e.g. Gabriel *et al*, 2012)

It was important to consider where necessary properties fail to hold and to ensure that the students engaged with this. One example of this is the property of numbers having a unique successor. Whilst all natural numbers have a unique successor, this is not true for rational or real numbers. To appreciate that there is no unique successor unless the numbers under consideration are restricted to integers, students could be asked to select two numbers and then to find a number in between. This is relatively straight forward with two integers but becomes harder when the numbers being considered are real numbers, particularly if they are close together. An example of this can be seen from the analysis of question 6 in the post-test data (see section 6.2.1.2)

In all this, I needed to be mindful to avoid reliance on whole number thinking (Roche, 2005). I believe that by moving from one instantiation to another I followed the advice from Siegler and Lortie–Forgues (2014) suggesting that acquiring magnitude understanding of fractions requires the understanding of which properties of whole numbers apply to fractions (e.g. having magnitudes) and where they do not apply (e.g. unique successor).

A more comprehensive guide to how to use the instantiations to plan for the use of appropriate tasks is included at section 9.1.

4.7 Overview of the project

Time	Cycle	Event
September 2016	1	Pilot of the diagnostic test (CSMS based questions). School B. Year 7 set 2
January 2017	1	Pilot of the multiple choice version of the test. School B year 7 set 5.
September 2017	1	Pre-test for comparison group using the original diagnostic test. School A. Year 7. 52 students did both tests.
School year 2017 - 2018	1	No intervention
April 2018	1	Delayed Post-test for comparison group (same questions as the pre-test). All Year 7 students were invited from a variety of sets. School A.
May-June 2018	1	Some students piloted tasks. Year 7 set 2. School A.
September 2018	1	Pre-test of intervention group. DCT also given in first lesson. Year 7 set 3. 20 students. School A.
September 2018	1	Intervention 1 (Teaching episode 1). Two weeks (8 lessons). Year 7 set 3. School A.
January 2019	1	Intervention 1 (Teaching episode 2). Two weeks. Year 7 set 3. School A.
April 2019	1	Delayed Post-test for intervention group 1 (same as pre-test). Year 7 set 3. School A.
May/June 2019	1	Interviews (15 minutes). Year 7 set 3. School A.
March 2020 2020	1	Second interviews. Year 8 (same students as with Year 7 above). School A. Only one student was interviewed as the school was then closed because of COVID.
September 2020	2	Pre-test (DCT) Year 7 set 3 School A.
September 2020	2	Intervention 2. One week. Year 7 set 3 School A.
September 2020	2	Post-test of intervention group (DCT). School A.
December 2020	2	Interviews. Year 7 set 3 School A.
March 2021	3	Intervention 3. Two lessons. Year 7 set 3 4 students – all of whom had been interviewed in December. School A.
May/July 2021	3	Post-interview task. Same 4 students. Decimat question. School A.

Table 4-2 - Overview of the project

Chapter 5

5 Methodology

Research Question

Can the mathematical understanding of students be modified by encouraging students to engage with underlying mathematical structures?

I investigated whether selecting tasks which are designed to encourage students to interact with underlying mathematical structures helps students understand the mathematics they are using more effectively than tasks that focus on processes and facts. Rather than taking concepts and topics in mathematics as discrete parts of learning, I aimed to recognise and use the different instantiations to improve children's interactions with structure and hence with understanding.

As I planned this research my "hunch" (Aspers & Corte, 2019, p. 152) was that if the teacher, as the expert in the classroom, explicitly makes the links between the different instantiations and provides the students with specific tasks which require the students to work with and compare the different instantiations and properties of each, the understanding of students will improve. I have offered justifications for this hunch in sections 1.3, 1.4 and 2.8.

5.1 Design of research and theoretical framework

5.1.1 Social Constructivism

The theoretical perspective adopted in this study is social constructivism (Ernest, 1994). Within this perspective reality is socially constructed and shared; that is knowledge and understanding is shaped by the social context of individuals.

An individual student has subjective mathematical knowledge. This has been formed from their experiences, mediated by cultural tools, such as manipulatives and language. Their understanding may, or may not, reflect the agreed understanding of the mathematical community. Where an individual has overgeneralised, for example, there can be misconceptions. A student might believe that because addition and multiplication are both commutative so are subtraction and division.

A theory of learning does not translate directly into a theory of teaching (Ernest, 1994). My experience of teaching and observing lessons informs my choice of theoretical framework. I have observed that although some aspects of a lesson can be predicted there are too many variables to be able to decide how a lesson will work. You can never be sure, as a teacher, that a particular approach or explanation will be successful every time for every or any learner. Each lesson will depend on the interactions between the students and between the students and the teacher. I know that when I have tried to repeat a planned lesson, or when I have seen someone else teach from the same plan, the results have been very different. In this, my experience resonates with the theory that knowledge is socially constructed.

The work of Piaget (1952) and Vygotsky(1978) led to theoretical understanding of learning moving from a transmission model to that of constructivism (von Glasersfeld, 1983, 1989), a model where students are no longer considered an empty vessel to be filled with facts and ideas but where they construct their own understanding.

The social interaction of the students and the teachers will shape the learning (Bauersfeld, 1992). Students' assumptions and interpretations will shape

the learning process as will the interpretation of the teacher. The outcomes from a classroom cannot merely be defined by the input. The norms (and socio mathematical norms) established in a classroom will also affect the learning (Yackel & Cobb, 1996). Sociomathematical norms are the normative understandings of the mathematical ideas In a classroom, for example, “what counts as an acceptable mathematical explanation” (*ibid*, p. 461).

5.1.2 Pragmatic Approaches

I adopted a mixed method, pragmatic approach in this study. This allowed me to explore the richness of the interactions in the classroom and to ascertain the impact of the intervention.

For many years the two major research paradigms were quantitative or qualitative. Both have elements to recommend them. Quantitative research can evaluate the results of intervention, allowing a level of objectivity for comparisons and qualitative research can describe in depth the interactions present in the classroom, allowing for a richness of analysis. Pring (2015, p. 73) points out that the dichotomy between the two approaches is false that that “the quantitative methods can be suggestive of differences to be explored in a more interpretive mode” and that “qualitative methods can clear the ground for the quantitative”. Dissatisfaction with these extreme positions allowed a third major paradigm to emerge: that of mixed methods (Flick, 2023). This is the pragmatic approach, the one that entails researchers using both quantitative and qualitative methods, selecting those that are most relevant to their aims. Pragmatism has been described as the third major research paradigm (Johnson *et al.*, 2007).

The Oxford English Dictionary defines pragmatic as

dealing with things sensibly and realistically in a way that is based on practical rather than theoretical considerations (OED, n.d.).

There is no single definition for a pragmatic approach. It is intended to amalgamate the best of qualitative and quantitative approaches (Denscombe, 2008). It can be 1) a fusion of approaches, 2) mixed methods, 3) a new “orthodoxy” where a mixed approach is desirable or 4) an approach used for expediency (Denscombe, 2008). Although the pragmatic approach can be expedient it should not be considered that an “anything goes approach to research” is acceptable (Denscombe, 2008, p. 274). One advantage of the pragmatic approach is the ability to triangulate evidence (Johnson *et al.*, 2007). Research can triangulate, for example, different data or different perspectives.

In this study, I aimed to assess and improve children’s understanding. It is difficult to assess levels of understanding via quantitative methods because it is never clear whether the answers have been answered via a thorough understanding or the use of a potentially incomplete algorithm. Like Mercer (2007), I do not want to reduce the richness of the interaction in the classroom to quantitative analysis. I have used a limited amount of quantitative analysis which signposted me towards the misconceptions and other problems that would be best addressed in a small group intervention. Qualitative analysis has allowed me to interrogate the thinking of individuals and in this “meanings and understandings replace proof” (Cohen *et al.*, 2011, p. 220). Qualitative analysis allowed me to focus on the perspectives of the learners in the study (Flick, 2023).

5.1.3 Action Research

I have adopted action research as the research design for this study. Action research embraces the constructivist stance taken and provides me with the framework to explore my own practice systematically.

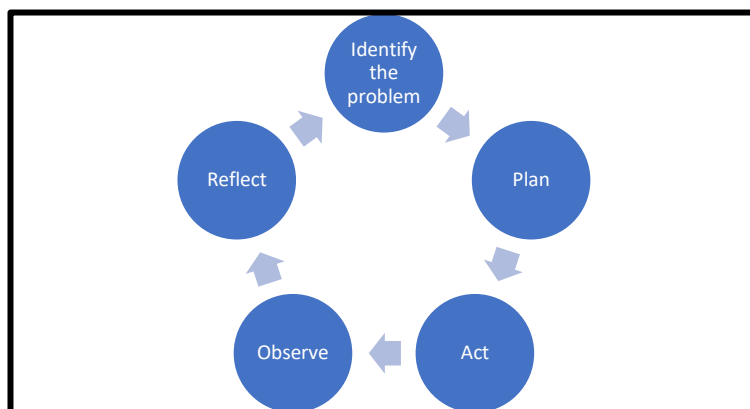


Figure 5-1 Action Research Cycle (adapted from three models in Cohen et al. (2011, pp. 353–355).

Figure 5-1 shows the action research cycle. Although represented by a cycle it is more of a helix with each reflection leading to a refinement of the initial problem or identifying a new problem. Since refinement and identification of problems arises through classroom interaction, constructivism is compatible with using action research to meet the aims of my study.

“There is no single definition for action research” (Cohen et al., 2011, p. 345) It is a process where the researcher identifies a problem or a need, plans how to change, acts on the plan, observes the results, reflects and then adjusts the plan. This is an extension of the standard mode of working of a reflective teacher. By using the action research model, the reflection and adaptations can be more analytical than mere reflection. It is defined as “ a small scale intervention in the functioning of the real world and a close examination of the effects of such an intervention” (Cohen et al., 2011, p. 345). Although a teacher’s own reflective practice has some of the hallmarks of action research, McAteer (2013) notes that this is not significant enough to be classed as action research and that action research is the product of “planned deliberation” and that “crucially, they [the reports] should also be subjected to discussion and debate of scholars and practitioners in the field” (McAteer, 2013, p. 24).

5.2 Ethics

The students in my project were all under the age of 18 and, therefore, specific ethical considerations related to research with young people were paramount. Ethical guidance is provided by Mason (2001), British Educational Research Association (2011,2018) and also UN (1989). The university's own guidance was followed, and approval was sought from the Ethics Committee before the commencement of any data collection.

British Educational Research Association (2011) refers specifically to the ethical considerations that follow a dual relationship. I was both researcher and teacher and, as teacher, had a responsibility first and foremost to my students to teach them to the best of my ability. If a situation were to arise where I had to prioritise one role over the other, ethically my role as teacher would have to prevail. As this study was planned before the update to the British Educational Research Association guidance, the 2011 guidance was current at the time. However, reviewing the requirements of British Educational Research Association (2018) the study also meets these requirements.

For the pilot study, for use of the data, and for all follow up interviews consent was sought from all parties and the right of withdrawal has been explained to all students and parents. Students, as participants in the study, needed to give full and informed consent to be part of the study and had the right to withdraw at any stage. These rights were explained to students and teachers taking part and to the parents of the students taking part. One child asked not to be part of the pilot of the assessment instrument. Many of the activities under scrutiny took the form of normal classroom lessons and students did not have the right to withdraw from these. They did, however, have the right to have their data excluded from my study and one child asked for this and their data have been excluded.

Where the activities of the study are outside the normal scope of the classroom (the assessments and the interviews) then separate written permission was sought from parents and students were given a free choice about whether to take part. Example letters can be seen at Appendix 8. I also sought approval from the head teachers at both schools. Students were informed that any information about them gathered during the study would be anonymised. I could not guarantee total anonymity as I was able to identify them from their work, but I explained the extent to which their data would be shared and that before being shared it would be anonymised. I assured them that where they could be identified it would only be by me. To this end all student names within this thesis are pseudonyms.

As I conducted this project my standing in the school changed and this was likely to alter how the students perceived me and reacted to me. When I started the project, I was a Second in Department at school A, and my aim was to work with the classes that I taught. For the pilot of the assessment, I was the Head of Department in school B. For the main study, I was a Senior Leader at school A. By this time my timetable was reduced to the point where I needed to work with a class that was not timetabled to be taught by me.

I needed to ensure that the students covered the same content as they would do if they were not part of the intervention; this had been one of the concerns voiced by the Head Teacher. For the school and the students' sakes they needed to know that they would be able to access the same assessments and knowledge. My initial plan was to work on a variety of topics within mathematics, however the result of the pilot of materials showed me that there was a significant lack of understanding of decimal place value and fractions and the connections between the two. This is such a significant part of students' understanding in mathematics that ethically I felt obliged to focus on this area.

There were only two times in the year when the Year 7 scheme of work coincided with topics being taught that were suitable for the intervention. At the beginning of the year the scheme of work focused on place value and decimals and in January on fractions. Each of these units lasted two weeks. Neither topic was included in any other part of the scheme of work from the start of the intervention until after the post-test.

5.3 Methods

5.3.1 Research design

Research question: Can the mathematical understanding of students be modified by encouraging students to engage with underlying mathematical structures?

The study adopted a pragmatic research design, involving a pre- and post-test with an intervention group and a comparison group to ascertain the impact of the teaching intervention. Data were collected during and after the intervention and these were mostly qualitative in nature, although there were some quantitative data. Qualitative data were collected to support the nuanced understanding of the learning processes at play during the intervention. The diagram at Figure 5-2 below shows the flow of the first cycle. There was a pre- and post-test of a comparison group, the year before the intervention, then there was a pre-test of the intervention group, an intervention, a delayed post-test followed by interviews of a small selection of students.

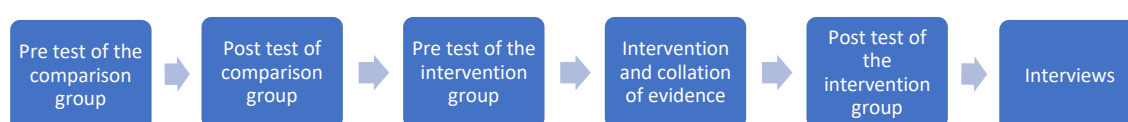


Figure 5-2 Overview of the project

The intervention was designed to follow an action research model and as such required reflection, assessment and adjustment. The first intervention sequence was two weeks long (8 lessons). Analysis of this intervention fed into the design of the second intervention sequence which was also two weeks long.

I undertook this research alone. I did not collaborate with other researchers.

However, I was in a community of PhD students. We met for research days and discussed critically each others' research design. These discussions were useful in ensuring that questions were clear and also helped me to clarify how to proceed at the end of my first intervention. In addition, I discussed the plans with the other teachers in the mathematics departments at both of the schools.

5.3.2 Research tools

As stated in Cohen *et al.* (2011, p. 235) "there is no single prescription for which data collection instruments to use; rather the issue here is of 'fitness of purpose'". In deciding what research tools to use, I considered the practicality of the classroom setting, how the data collection could possibly adversely affect the classroom and what data I would need to be able to analyse the learning of the students in the intervention. The following research tools were adopted.

- Pre and post-test
- Collation of evidence of students' thinking during the lesson
- Transcript of semi-structured interviews
- Reflective journal

5.3.2.1 Pre- and post-test.

Each student sat a pre- and post-test. The questions from these were taken from the CSMS study (Hart, 1981) and were piloted and adapted (see section 6.1). Both the intervention and the comparison groups sat this. All students involved sat the same test (Cohen *et al.*, 2011, p. 493). As the test was not referred back to or marked with the students, the same test was used for the pre-test as for the post-test. The post-test was delayed until several months after the intervention to consider whether any effects persisted. The test had two purposes, one was to identify quantitatively any improvements between the pre- and the post-test of the intervention group over and above the comparison group. The major purpose of the test, though, was to identify misconceptions and ideas for further discussion with the students to ensure that the intervention would address the needs of the students.

5.3.2.2 Collation of evidence of students' thinking during the lessons

To capture evidence of the students' thinking during lessons I collected all written and practical work they produced. In addition, I took photographs of their work and their contributions to class discussions written on the classroom whiteboard. An example of the photographs can be seen at Figure 6-27. The students also had an exercise book, used only for my lessons, where they worked on some of the tasks and where they recorded their own thoughts on any changes to their understanding. Where work was not in the books or on the classroom whiteboard, I took photographs. An example of a photograph of a sorting activity is at Figure 6-31. I ensured no child was identifiable in any photograph. The point of the photographs was to be able to capture ephemeral material in order to be able to analyse it later. This was where their written work was located. Both the books and the photographs were analysed to interrogate the thinking of the students.

5.3.2.3 Reflective Journal

As soon as possible following each lesson, I recorded my thoughts about the lesson in my journal. The journal was recorded on a dictaphone. The journal was a record of my reflections about the lesson. For each lesson I gave a commentary on what I had intended, what I did, and what tasks the students undertook. I recalled conversations with individual children, although over the space of time I did not always recall which student I had spoken to. The journal included a personal record of my reactions to the lesson. In addition, I recorded how I felt the next lesson might need to be adapted to take account of the events of the lesson. By the nature of a lesson, and the fact that I was teaching it was not possible for me to complete notes during the lesson for the journal. For each entry in the journal, I considered the type of interaction and my reflections. For example, I would describe the activity, the responses of students and my reflection as to whether this activity had resulted in the outcomes I had expected. As I was able to access the recordings easily and they were only for my own use, I did not transcribe them. An example of the analysis of my reflective journal is at Appendix 6. The photographs acted as a complementary source of data, enabling me to record details of interactions that I could later combine and triangulate with the reflective journal.

5.3.2.4 Semi-structured interviews

In a lesson a student might answer a question correctly, for example, ordering 8.23 as greater than 8.2. However, unless the student is asked directly it is not clear whether this was a guess; subject to misconceptions (e.g., 23 is bigger than 2, rather than 23 is bigger than 20); the result of an algorithm or based in a secure understanding of decimal place value. Within the lesson, as I was a full participant in the intervention, I was not going to be able to interact with every student.

Interviews were undertaken to give me the opportunity to have these more detailed discussions (Cohen *et al.*, 2011). The purpose of the interviews was to give me an opportunity to investigate further the nuanced understanding and reasoning of the students, something that could only be briefly touched on within a lesson.

I chose semi-structured interviews with questions selected in advance, appropriate for the cycle and current issues, but with the freedom for the students to answer in their own way and for me to ask follow on questions (Cohen *et al.*, 2011). This allowed me room to find out about students' perspectives (Flick, 2023). Semi-structured interviews have a less formal format than structured interviews, which allowed me to divert the interview to follow the lead of the students and to investigate their responses further. To be able to capture all of the discussions, I recorded the interviews, with the permission of the student and their parent and then transcribed the interview. Having transcribed I analysed each interaction. Much of the interviews also included students working, sometimes silently, and so the transcript of the interview had to be analysed alongside the work that they had written. An example of this analysis is at Appendix 5 where you can see the transcript of the interview alongside the work that the students did.

I considered whether it was better to interview the students individually or to interview them in small groups. Although there are advantages to group interviews (Cohen *et al.*, 2011) in terms of a greater variety of answers it is also likely that one student would have dominated and shy students might not have spoken up. In addition, as I was investigating mathematical difficulties, some students might have been reluctant to be honest in front of their peers. It would have been difficult for me to find out about the thinking of the quieter student. Because I really wanted to be able to unpick the difficulties of the students, I decided that individual

interviews would enable me to meet my research aims. Open ended questions are favoured for interviewing children as they are “respondent focused” (Cohen *et al.*, 2011, p. 433) and open ended questions avoid children giving closed answers (Greig & Taylor, 1999) . The semi-structured interview lends itself to open ended questions.

My position as a teacher could have caused difficulties as students might see the interview as a test or might feel the need to find correct answers (Cohen *et al.*, 2011). I was aware that there is a power imbalance in any interview situation (Brinkmann & Kvale, 2018) in terms of deciding on the structure of the interview and in terms of interpreting results. I was also aware that my role accentuated that power imbalance. I needed to consider carefully how to word the questions and how to conduct the interview so as to get authentic replies. All of the questions were based around mathematics and therefore should feel like an extension to a lesson. I was aware that the interview might feel intense to the students. They are used to teachers talking to them one-to-one about lessons for a short time but not moving from one question to the next, without a break. I decided to keep the interviews short, 10-15 minutes at most and reminded them regularly that they were volunteers and could end the interview at any time. I also spent the first few minutes “setting the stage” (Brinkmann & Kvale, 2018, p. 62), explaining to the students what the interview would be like, ensuring that they were comfortable with the process and reminding them that they could stop at any time. In my role as teacher and researcher there was always the possibility of conflict between the quality of research and my ethical duty to the student (Brinkmann & Kvale, 2018). However, I felt ethically bound to ensure that my role as teacher dominated in any situation. For an example see section 6.7.

There are thoughts in interviews and discussions which are internal and are not enunciated (Drake, 2010). Sometimes it is clear that there is something left unspoken and the semi-structured interview allows the researcher to ask further questions. However, there is always the possibility that something important is left unsaid. I asked students if there was anything else they wanted to add at various points through the interviews. In addition, the students will have made meaning during the interview and it might have been a teaching situation for them. The time taken with the students discussing the mathematics one-to-one might have increased their understanding of the mathematics being discussed.

Students were chosen for the interview after an initial analysis of the data. I chose students to be interviewed where I felt I needed to talk to them in more depth to be able to investigate their thinking further. The details of how students were selected for the interviews are in sections 6.5.2 and 7.4.1. The students were interviewed during their mathematics lessons after I had asked permission from the class teacher. Each interview took approximately 15 minutes. The interviews following the first intervention were in May, approximately eight months after the first teaching episode and four months after the second. The time delay was intended to allow for the interviews to be after the delayed post-test. The interviews after the second intervention were in December, approximately two months after the intervention. I intended for a delay between the intervention and the interviews to assess whether overcoming any intuitive responses had any longevity.

The questions I asked in the interviews were.

- What is a fraction?
- Show me $\frac{2}{3}$
- Show me $\frac{2}{3}$ on a number line.

- Show me 0.236 on a decimal.
- Which is bigger 0.236 or 0.24 (also depending on the answer which is bigger 0.225 or 0.21)?
- Show me on the number line.
- Which is bigger $\frac{1}{3}$ or $\frac{1}{4}$?

The first question was intended to discover whether the students could explain the nature of a fraction. Similar questions had been asked at various points through intervention 1. The second and third questions allowed me to discover if the students could represent a fraction or could place it on a number line. The fourth and sixth questions were to assess if the student could represent the magnitude of a decimal. The fifth and seventh questions were to see if the student could compare fractional amounts accurately. For all questions I was also interested in how the students explained their reasoning. The questions themselves appear closed. These were initial prompts that were followed up with further open-ended questions to interrogate the students' thinking.

In the second interviews students were asked to answer a few questions from the pre-test. This was partially to unpick whether certain questions were poorly answered because students could not understand the mathematics or whether they had not understood what was being asked of them.

Once these questions had been answered and discussed, I asked the following questions.

- What is a fraction?
- Which is bigger 0.24 or 0.236?
- A follow up question where necessary, which is bigger 0.225 or 0.21?

- Which is bigger $\frac{1}{4}$ or $\frac{1}{3}$?

The first question is to find out if the students could describe the nature of a fraction. The second and third questions allowed me to discover if the student could compare the magnitude of decimals and describe how they came to the decision. By answering these two questions I would also be able to see if students were likely to hold common misconceptions about the magnitude of decimals. The final question enabled me to find out if students could compare fractions.

5.4 Analysis

The data collected at each stage were analysed as below in Table 5-1. Data were the results of student interactions in response to mathematical stimuli and as Flick states “data analysis is the central step in qualitative research” (Flick, 2014, p. 3). A researcher must decide what the analysis is expected to reveal in order to determine what type of analysis is appropriate (Cohen *et al.*, 2011). Aims of my analysis were “to describe a phenomenon” , to “compar(ing) cases” and “to look for explanations” (Flick, 2014, pp. 5–6) I wanted the data:-

- to show whether the students were able to complete tasks at the end of the intervention that they were not able to complete at the start.
- to generate themes about misconceptions and errors.
- to detail how the students thought about questions when they answered them.
- to explore whether using structure improved the understanding of the students.
- to discover whether difficulties were individual to students or whether there were commonalities.

Given the large amount of data produced in these intervention sessions it was important to initially sort the data into these categories. Much of the data collected fell into the following categories: -

- followed generally accepted mathematics reasoning,
- did not attempt the work,
- displayed one of a known set of misconceptions, or
- displayed a misconception that I had not been aware of.

I then used these to categorise the work. These are “substantive categories” (Maxwell & Chmielecki, 2014, p. 25).

Illustrative cases were used in chapters 6,7 and 8 to present the learning journeys of 3 students for intervention 1 and 2 students for interventions 2 and 3. The explanation about this is at section 6.3 and the illustrative cases are interleaved through the text and are detailed in full at Appendix 9.

From there, examples were selected for further analysis or to describe in the work as either typical of the responses of the group, or as something unusual that was worthy of further investigation. Table 5-1 briefly describes how each set of data was analysed.

Stage	Data	Analysis
Pre-test and post-test	The participants' test papers	The papers were marked with questions being marked as 'correct', 'incorrect' or 'missed'. The results of the tests were analysed, comparing the intervention and the comparison groups. The test papers were analysed. Answers were reviewed to see which of the four categories they fell into.
Intervention 1,2 and 3	Audio file of my daily journal, recorded on dictaphone with my thoughts from each teaching session. This included a brief description about the lesson and where it was altered from the plan. .	The journal was reviewed. I listened to each entry and classified each interaction as 'descriptive' (what happened), 'student interaction' (what the students did or said) or 'reflection'. I wrote a summary for each lesson of the interactions with the students and my reflections. The descriptive entries enabled me to detail the lessons in the sections 6.3,7.3 and 8.1. Student interactions were split into those which were typical of what I would have expected from the literature and those that were unexpected. The reflections helped me shape next lessons and also consider what needed to be altered for the next cycle. I did not transcribe the journal. I used the audio format. I have transcribed one lesson to illustrate. This is at Appendix 6
Intervention 1 and 2	The participants' Decimal Comparison Test (DCT) test papers (see Sections 6.4.5 and 7.3.3)	The students completed the DCT from Steinle. Their test papers were marked. The results of the DCT were analysed as described in Steinle, Stacey and Chambers (2006) to identify whether students held any of the likely misconceptions which can be indicated by the DCT. For the list of students and the likely interpretation see Table 6-9 and Table 7-2. Illustrated cases were used to present the findings (see section 6.3).

Stage	Data	Analysis
Interventions 1,2 and 3	Exercise books and product of the work	All the student work was reviewed. Each piece of work was categorised as above. Each example in section 6.3,7.3 and 8.1 illustrates one of these categories
Interventions 1,2 and 3	Photographs of the work Photographs were taken throughout the lesson of the work on the board as well as the work that the students were producing. Much of this needed to be photographed as there would be no other permanent record.	All the photographs were reviewed. Each photograph was categorised as above. Each example in section 6.3,7.3 and 8.1 illustrates one of these categories
Interviews 1 and 2	Transcript. The interviews were recorded and then transcribed fully.	The analysis of interview transcripts was set out in a grid, with the words spoken by the student and a picture of any work produced, annotated with whether there was an indication of structure in the answer, my thoughts at the time and then any later reflection. For each interview I also stated what the likely misconceptions were at the start and the end of the intervention and what learning might look like. An example of this analysis is at Appendix 5. Illustrated cases were used to present the findings (see section 6.3)

Table 5-1 How data were analysed

Most of the analysis was qualitative. As Cohen *et al.* (2011) note, it is possible to be quickly overwhelmed by the data collected for qualitative analysis. The initial analysis of each piece of data allowed me to reduce the amount of data under consideration by considering whether each piece of data added anything new to the findings to date. As noted in Cohen *et al.*, (2011, p. 220) when discussing education research, “researchers do not know in advance what they will see or look for”. Categorising the data as described above allowed me to know what I was looking for. In addition, where students’ understanding had improved, I was able to analyse whether they were using the structures provided or had acquired an instrumental understanding. For an explanation of the different types of understanding see section 7.1.1. Where the students’ work showed errors then I wanted to be able to analyse the error in terms of the error classification in section 7.1.5.2. As this takes a detailed conversation with the student it was only possible in the interviews.

Because I had a class of students, I would not have been able to record every interaction in the classroom. This is one reason why I selected an interview as a research tool. I wanted to record this so that I could be “faithful to the exact words used” (Cohen *et al.*, 2011, p. 539). Using recordings allowed me, for example, to distinguish between vague, unformed mantras (e.g., decimals go the wrong way) and the more polished, potentially schooled mantras (e.g., to multiply by 10 add a zero). I could not necessarily be certain as to which were unformed and which were schooled. However, when spoken the schooled mantras have a rhythm and a clarity. They are also repeated by different students from different schools. A defining feature is also that there are circumstances where they will work. For example, to multiply by 10 add a zero, works when we want to multiply 5 by 10, but not when we want to multiply 0.5 by 10. The unformed mantras tend to be

hesitant, and are often grammatically incorrect. Although I did not realise at the start of the project that this is something I would need to be able to distinguish, it has become central to my discussion on intuitions.

5.5 Validity and Reliability

I am aware that as 1) a participant in the study and 2) as a person of authority within the school I am instrumental in shaping my study. Taking the viewpoint of social constructivism, it is inevitable that I, as a participant, will shape the outcomes of the project. My positionality is considered in the reflexivity section (section 5.6) and the positioning section (section 10.4).

Qualitative research is often viewed with suspicion and labelled ‘unreliable’ and ‘not scientifically sound’. Scepticisms include its perceived inability to be reproduced, a lack of statistical generalisability of findings, and doubts as to whether a different researcher would produce the same account (Buckner, 2005, p. 60)

For quantitative research with a positivist approach to be reliable it needs to be shown that the same research, done by other researchers would give similar results. This is neither possible, nor desirable from a constructivist viewpoint where I, as the teacher, shaped the research and my interactions with the students will have shaped the outcomes of the lessons and answers in interviews. There is still a need for reliability in qualitative research, but it would have a different interpretation. Barbour (2014, pp. 497–498) acknowledges that “overarching criteria in qualitative data analysis remain elusive.” Writers suggest other measures more suitable to qualitative research are; “consistency” (Noble & Smith, 2015, p. 34), “credibility” “neutrality”, “trustworthiness” and “dependability” (Cohen *et al.*, 2011, p. 201).

Validity refers to whether or not the method measures what it is supposed to measure. Validity will have a different meaning whether quantitative or qualitative

research is being considered. For qualitative research validity includes; “descriptive validity” and “interpretive validity” (Cohen *et al.*, 2011, p. 181).

“Research aims at presenting reality, not reproducing it” (Flick, 2023, p. 492).

Descriptive validity describes how factually accurate the account is. This is a measure of the “truth” (Blumenfeld-Jones, 1995) of the research. It also has a moral aspect in that the researcher is trustworthy and basing all of the research on fact, not making up anything or distorting it.

Interpretive validity is similar to “fidelity” (Blumenfeld-Jones, 1995)

In this distinction I take truth to be "what happened in a situation" (the truth of the matter) and fidelity to be "what it means to the teller of the tale" (fidelity to what happened for that person). Truth treats a situation as an object while fidelity is subjective. (Blumenfeld-Jones, 1995, p. 26).

My research was undertaken in a systematic manner. The interventions were planned, then the work analysed. All data were initially analysed and categorised and then analysed in further depth. The further analysis is described in the sections 6, 0 and 8. I interpreted all data with an open mind not prejudging what I was expecting to find. Examples of initial and further analyses are in the appendices (See Appendix,3,4,5 and 6).

Interviews and interactions with students are subjective, but this allows a “sensitive understanding” of the mathematical world of the students (Brinkmann & Kvale, 2018, p. 62).

Aspects of reliability and validity appear to rely on the honesty and integrity of the researcher. It is the measure of whether the researcher is reporting and interpreting the data in a credible, trustworthy and objective manner. As my research is qualitative and viewed from a constructivist viewpoint, I am aware that my work is not necessarily reproducible. It might even be the case that reviewing

my evidence and data another researcher might have reached different conclusions, but that both sets of work, if conducted in a credible, trustworthy way, could be valid. I reported all work as accurately as I could and, as much as is possible, I aimed to ensure that all interpretation was as well-justified as possible, using the literature, the categories that developed through the research cycles, by comparing various sources of data and from my own mathematical and teaching experience. I also ensured that I tested alternative interpretations. For example, when in the interview 2 (section 7.4.2) I asked Charles whether 0.236 or 0.24 was larger. Although he said 0.24, I did not accept that his reasoning was necessarily correct and so I asked a further question to rule out the possibility that Charles held a shorter-is-larger misconception. It is not possible for any research to be 100% valid (Cohen *et al.*, 2011), it is the aim of researchers to maximise the validity of their research. In this validity can be considered to be “trustworthiness” (Flick, 2023, p. 493)

I had designed the research tools so that I was able to triangulate data (Flick, 2023). Triangulation is a way of “demonstrating concurrent validity” (Cohen *et al.*, 2011, p. 195) . Relying on a single type of data might lead to bias, however, by triangulating the data the analysis is more robust. For example, the photographs illustrate and reinforce the data in the journal. The written records within the interviews alongside the recording and transcript give a fuller picture of the thinking of the students than either the work or the transcript would have given me alone. The fact that the results of the post-test agree with the findings of the interview is a powerful indicator in the validity of the results.

5.6 Reflexivity

Reflexivity is the process where I, as researcher, acknowledge and account for the fact that I shaped the research; both during the research and any outcomes

(Berger, 2015). As a teacher I have worked for many years in a reflective profession and as such I am used to reflecting on a day at school and thinking what I could have done differently. Reflexivity in research takes this further. It is recognising that I impacted on the learning but also on the research. It is recognising where I played the part of teacher and where I played the part of researcher. It is how I dealt with the competing roles as researcher and teacher.

As such, the idea of reflexivity challenges the view of knowledge production as independent of the researcher producing it and of knowledge as objective (Berger, 2015, p. 220)

As a researcher I needed to ensure that I looked both at the data I collected but also at my actions and my interpretations and how my interactions affected the data and the interpretation. It was important for me to consider at all points the way in which I was shaping the study. By the nature of the research, I was fully involved in the study, and I was shaping the study throughout. I needed to ensure, though, that my shaping of the study was desirable. It is worth noting that it is not possible or even desirable to be completely objective.

However, within a research environment that is also a political environment, the researcher's neutrality is often not desirable and is always unachievable (Drake, 2010, p. 86)

In lessons, I needed to be teacher first, and researcher second. When I was reviewing data, planning the intervention and analysing evidence, I needed to be researcher first. To ensure that the interpretations of the data were thorough I needed to ensure that I was honest and objective when reviewing data.

All through the research, I have detailed my own involvement and have tried to evaluate where my interactions might have been less than ideal.

reflexivity is a researcher's conscious and deliberate effort to be attuned to one's own reactions to respondents and to the way in which the research account is constructed, (Berger, 2015, p. 221)

Berger (2015) discusses how the researcher, as an insider, is able to understand the subjects in the study and that it might be that the subjects are more open.

the worldview and background of the researcher affects the way in which he or she constructs the world, uses language, poses questions, and chooses the lens for filtering the information gathered from participants and making meaning of it, and thus may shape the findings and conclusions of the study (Berger, 2015, p. 220)

Because I was a teacher in the school, I had some common understandings and experiences with the students. However, as a teacher in the school I did not have a shared experience and there was necessarily a power imbalance. In the school I was *in loco parentis* and needed to be extremely aware of this situation at all times. This is always true when teaching, but I needed particularly to consider the imbalance when doing the interviews. For example, I needed to ensure that the students understood that the interview was optional. It is not often that as a teacher in school students are free to say 'no' to teachers and I needed them to understand that pulling out of the interview at any stage was perfectly acceptable.

More crucial, though, to the rigour and robustness of the research was the difference between the students' and my mathematical background. There were two relevant differences in backgrounds. Firstly, as a teacher I would have been considered the expert in the room. Secondly my school experience of mathematics had been very different from theirs. I had understood fractional amounts when it was taught in primary school and had never felt that mathematics was difficult. I had, though, been aware at school that others did find mathematics difficult, and I had experienced difficulties in some mathematical ideas when I was at university and so I was able to appreciate the fact that the students did find the ideas and concepts difficult.

One suggestion from Berger (2015) is to write up interviews, with what was said, what it might mean and then to go back, after a few weeks, and reflect on what was said with a new lens afforded by time. My interview analysis (see Appendix 5) contained the statements, the work, then my thoughts at the time of transcribing and finally my reflections which was an interpretation added after the passage of time.

The concept of reflexivity is often explored in terms of social positioning. This approach asks how the social location of the researcher and the researched influence the investigation process and outcomes. Another way in which reflexivity has been examined is the ways in which the researchers' emotional responses to respondents can shape the formers' interpretations of respondent accounts (Buckner, 2005, p. 61).

One consideration is that as a teacher there is an emotional element if a lesson goes well or badly. If, at the end of a lesson, I felt that the students had learned what I taught and had enjoyed the lesson, I would consider that the lesson had 'gone well'. If the students did not appear to learn or enjoy the lesson, then I might consider that the lesson had 'gone badly'. Both these feelings are accompanied by a significant emotional response. As a researcher, I needed to ensure that I did not allow that emotional response to cloud the objectivity of my recall of the lesson or of my analysis.

As a teacher it is important to maintain the distance and not to be emotionally involved with the work. In this sense, my research was an extension of my everyday work. Drake, having undertaken educational research in her own workspace warns about the issues, but also points out that this research gives the researcher privileged access. She suggests that keeping and using diaries 'to stimulate reflexivity whenever possible is encouraged as an important aspect of 'self-triangulation' (Drake, 2010, p. 85).

Insider researchers often choose their project as a result of several years of experience of working with the issues. Thus, they often have assumptions and ideas about what they expect to find and, on the basis of professional experience, have a theoretical stance before beginning the project.(Drake, 2010, p. 98)

This quote from Drake resonates with me. My experience of teaching and one of my projects with my MSc led me to believe that teaching using structure was worth investigating. In my capacity as a researcher, I discovered that structure alone was not going to be enough, and that intuitions and misconceptions were more persistent than I had anticipated. My teaching experience, therefore, gave me my initial idea. One limitation of reflexivity is that being aware of one's possible biases does not stop them from happening (Cohen *et al.*, 2011). However, my role as researcher allowed for me to adapt my beliefs in a way that I might not have considered. I chose to use photographs to record ephemeral moments. The photographs allow me to record what was happening at a given moment. A photograph records what is in the view of the lens accurately. There is, however, subjectivity in the interpretation of the photograph and the decision of what to photograph (Drake, 2010). I am confident that my reflexive stance supported both my approach to data analysis and indeed the fidelity of my findings.

The outcome of a cycle was not intended to be a verification of a hypothesis but questions to be pursued in the next cycle. Each cycle was an opportunity for me to learn more about the mathematical thinking of the students and how I could alter my teaching to help them improve their understanding. I had no intention to produce a generalisable positivist statement; rather I intended to expand knowledge about potential teaching design and learning in a limited conceptual area.

5.7 Cycles

Each cycle was designed to focus upon closely exploring questions raised following analysis of the previous cycle. The design of each cycle was informed by the results of the previous cycle alongside further references to the literature. This is shown diagrammatically in Figure 5-3.

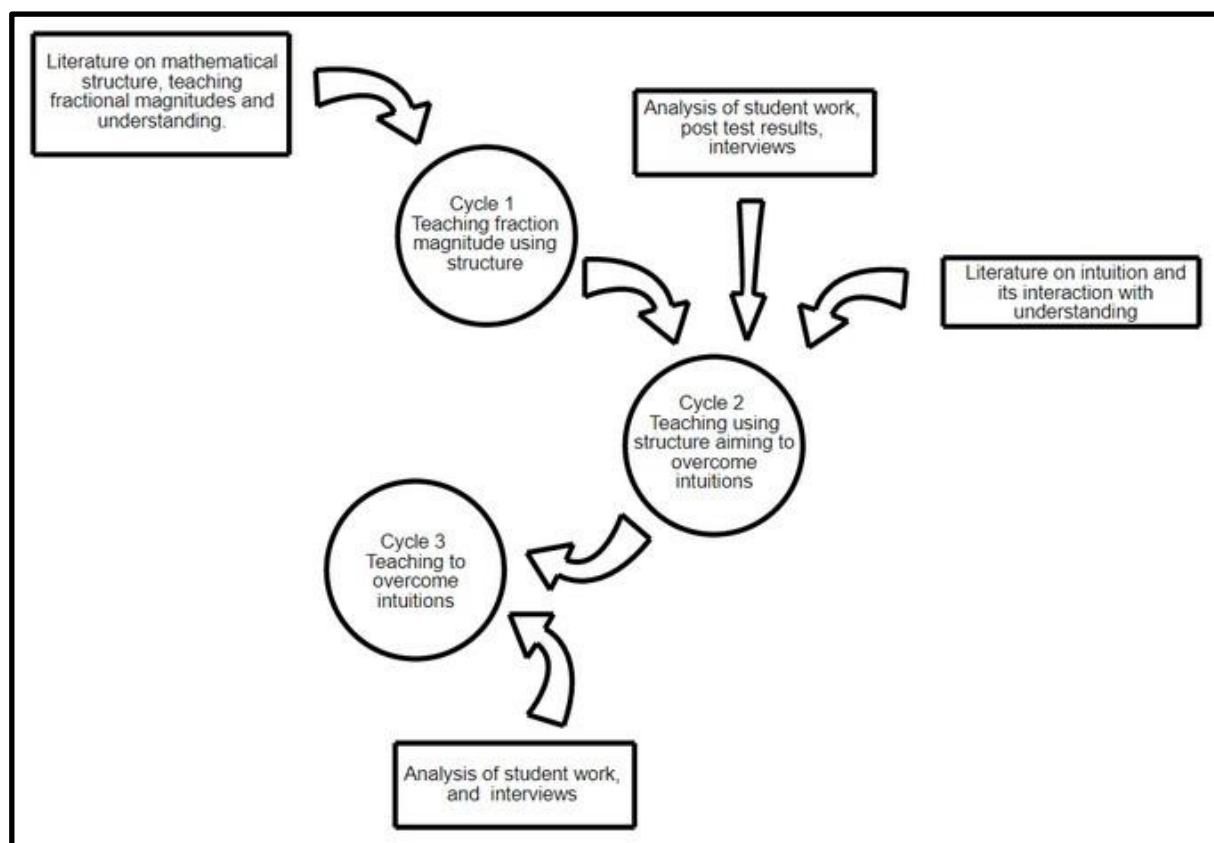


Figure 5-3 - Cyclical nature of the study

The first cycle focused on how to improve teaching of a particular concept. My approach was informed by the literature and by my experience as a teacher. The second cycle was informed by my findings from the first cycle and my approach became more focused on the learners' interpretation of the concepts and their errors.

As depicted in Figure 5-3, the purposes of the three cycles were: -

- Cycle 1 - teaching fraction magnitude using structure,

- Cycle 2 - teaching using structure aiming to overcome intuitions, and
- Cycle 3 - teaching to overcome intuitions.

Figure 5-4 shows the timescale of the project, the elements of each cycle, how each cycle informed the next and where COVID-19 disrupted the planned work.

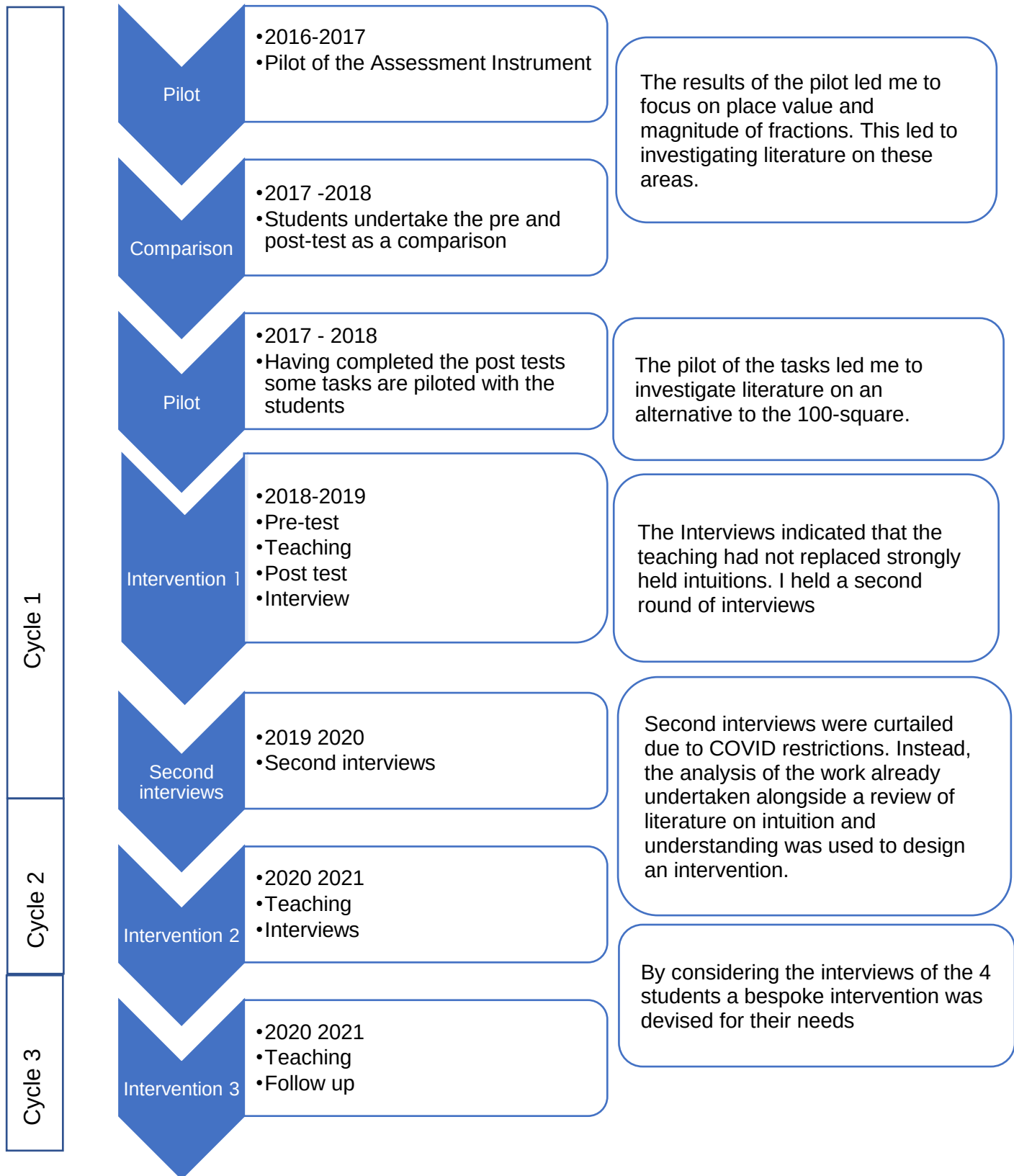


Figure 5-4 cycles

This chapter has provided the methodology and methods for my study and has examined how the data that were collected were analysed. It also considers the validity and robustness of data and any conclusions drawn. Chapters 6,7 and 8 describe the 3 cycles, detail examples of the data collected and my interpretation of the data. They also describe how I used the results of each cycle to shape the design of the subsequent cycle

Chapter 6

6 Cycle 1

6.1 Design of the Assessment Instrument for the pre-test and post-test

The initial plan for the study was a mixed methods intervention with several classes looking at different topics in Year 7 mathematics. Topics included place value, ordering decimals, representing fractions, and generalising simple arithmetic with algebra. The decision to intervene with Year 7 and the choice of these topics, in particular, was made because the selected topics were considered to be fundamental prerequisites of the KS3 and KS4 mathematics courses. Most of the concepts are introduced in KS2, however my experience and that of others (Steinle & Stacey, 2004) indicated to me that many concepts are not securely understood by students when they arrive in secondary school and indeed, many errors persist through secondary school. In the Third International Mathematics and Science Study (TIMMS, 1999) 13-year-old students were asked to find the smallest of five decimals (Question B10). The international average of students giving the correct answer was 46%. Steinle (2004b) details several other studies where difficulties with decimals are common and persist into adulthood.

These difficulties as discussed in section 2.5 include: -

- Ordering decimals, particularly those of different lengths called “ragged decimals” by Steinle (Roche, 2005, p. 13; Steinle, 2004b, p. 473).
- Knowing that there is no unique successor to a decimal.
- Knowing that there is always a decimal between two unequal decimals.

The topics under consideration were limited to understanding place value, ordering and representing fractional amounts and generalising arithmetic with algebra. The purpose of the inclusion of the algebra was to ask students to engage with the structures behind the arithmetic that they will have been taught at primary school, thus adding in another layer of structure.

Given the initial decision to consider both quantitative and qualitative data, I designed an assessment instrument to be undertaken both pre and post the intervention, this was piloted by Year 7 classes in 2016-17. In 2017-18, a group of students (the comparison) group sat the pre-test and post-test, several months apart but were not subject to any intervention. In 2018-19 a different group of students took the pre-test and post-test and were part of the intervention sessions.

	September	January	April	May-June
2016-2017 Year 7 Pilot	Pilot of the pre-test	Pilot of the pre-test		
2017-2018 Year 7 Comparison group	Pre-test		Post-test	Some students piloted tasks
2018-2019 Year 7 Intervention group	Pre-test Intervention	Intervention	Post test	Interviews
2019 – 2020 Year 8 Intervention		Further Interviews (disrupted by COVID)		
2020 – 2021 Year 7	Intervention Interviews	Further Intervention		

Table 6-1 Timeline of intervention

The assessment instrument used in the pre-test was intended to discover which of a number of misconceptions were held by students. The questions used were taken from the CSMS tests on place value and decimals, fractions, multiplicative reasoning and algebra (Brown *et al.*, 1984a, 1984b, 1984c). These were long tests

and part of a full suite of tests used to test the understanding of secondary aged students in several areas of mathematics and, as such, not suitable in their entirety for use in this project. I selected specific questions from each focussing on place value, ordering decimals, representing fractions and generalising simple arithmetic with algebra.

I needed a test that would be quicker than the CSMS test to be able to be administered within one lesson (50 minutes). I selected 11 questions (some multi-part) from questions on place value, ordering decimals, representing fractions and generalising simple arithmetic with algebra (Brown *et al.*, 1984a, 1984b, 1984c). The questions are set below in Table 6-2 alongside the intended area of assessment. As discussed later in section 6.1.6, I decided to remove any focus on algebra from the study and so analysis of algebra questions has been omitted from the following.

Question			Area to assess
Q1a	Add one tenth	4.254	Assessing how students deal with having more than 9 in a given column.
Q1b	Add one tenth	2.9	
Q1d	Take away 100	583	
Q1e	Take away 100	30000	
Q2a	Add ten	3597	
Q2b	Add ten	0.15	
Q2c	Add one hundred	21534	
Q2d	Add one hundred	19930	

Questions		Area to assess
Q3	Add one to 06399	Assessing place value and what happens when a column exceeds 9
Q4	Is there a difference between 4.9 and 4.90	Assessing ordering integers and decimals
Q5a	Which is bigger 0.75 or 0.8	
Q5b (i)	Draw a ring around the bigger number 20100 or 20095	
Q5b (ii)	Draw a ring around the bigger number 7.55 or 7.5	
Q5b (iii)	Draw a ring around the bigger number 4.06 or 4.5	
Q6 a	Write down any number between 4000 and 5000	Assesses whether students can find a decimal between two unequal decimals
Q6 b	Write down any number between 4100 and 4200	
Q6c	Write down any number between 0.4 and 0.5	
Q6d	Write down any number between 0.41 and 0.42	
Q6e	How many different numbers are there between 0.41 and 0.42	Assessing whether students know that decimals do not have a unique successor.
Q7a (i)	Write down the largest of these; $n+1$, $n+4$, $n-3$, n , $n-7$	
Q7a (ii)	Write down the smallest of these; $n+1$, $n+4$, $n-3$, n , $n-7$	
Q7b	$2n > n+2$. Is this always, sometimes, or never true	
Q8a	What can you say about a if $a+5=8$	
Q8b	What can you say about b if $b+2$ is equal to $2b$	

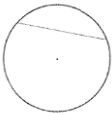
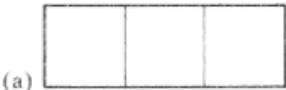
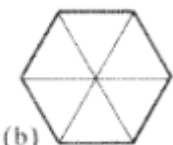
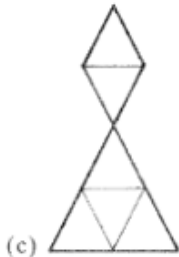
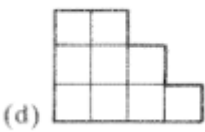
Questions		Area to assess
Q9a	$A+B+C=C+A+B$ Always, never, sometimes, when	
Q9b	$L+M+N=L+P+N$ Always, never, sometimes, when	
Q10	Has this circle been split into 2 halves? 	Assessing whether students know that representing a fraction requires the parts to be equal.
Q11a	Shade in two thirds of each of these shapes 	Assessing whether students recognise a fraction of a shape which is subdivided into a multiple of the denominator
Q11b	Shade in two thirds of each of the shapes 	
Q11c	Shade in two thirds of each of the shapes 	
Q11d	Shade in two thirds of each of the shapes 	

Table 6-2 Questions for the assessment instrument

6.1.1 Pilot of the Assessment Instrument

The purposes of the pilot (Cohen *et al.*, 2011, p. 492) include finding out about “clarity of the items”, “time taken” and “omissions, redundant and irrelevant items”. Although my initial intention was to carry out one pilot test of the materials, the analysis of the results showed that a second pilot, following revisions, was desirable. In the description below I will refer to these as the first and second pilot.

6.1.2 First Pilot

Because the intervention was intended for Year 7 students, I wanted to pilot the assessment tool with a Year 7 class. In school B, there were seven Year 7 classes, each set by prior attainment. I selected set 2 to pilot the assessment because I did not want to test a set with the highest or lowest attaining students. This was because I wanted to ensure a spread of prior attainment, which was more likely in the sets 2 to 6. Any set from 2 to 6 would have suited the pilot, but timetabling and staffing constraints meant that set 2 was the easiest class to use for the pilot.

Each student was given the option to withdraw at any point and they were assured anonymity. Ethical implications of the pilot and the main study are considered further in section 5.2. Twenty-nine students were present during the test. One student had a one-to-one Learning Support Assistant (LSA) to support his behaviour and literacy. She read to him and kept him on task but did not assist in any other way. No other adult intervened during the test. The class teacher, reviewing the content of the test, was surprised by the questions, stating that all the students would be getting everything right.

On completion of the test by the students, I reviewed all the responses. Of the 12 questions, only one part of one question received a correct response from all students.

What can you say about a if $a + 5 = 8$

Once the pilot had been completed, I reviewed the effectiveness of the assessment. The time taken by students to complete the test was reasonable: 20 minutes. Most students completed the test and most answered or attempted all questions. One of the questions on the pilot asked whether the students found the instructions clear, and no student stated that they had been confused by the language or the instructions.

Considering the first pilot, I analysed the results by how many students gave correct responses, but also by looking at the range of incorrect answers to see where errors were being made. The incorrect answers tended to be similar and so, in the revision of the pilot, I decided to use multiple choice for some of the questions. The aim was to make processing of the assessment instrument easier and quicker for the main study. I used the incorrect answers from the first pilot to develop the choices in the multiple-choice questions. I selected the most commonly selected wrong answers as possible answers on the multiple choice.

From the answers to the first pilot, it was possible to see a pattern of responses and thus be able to analyse the answers and question what this meant about students' understanding. For example, analysing question one: -

Add one tenth

(e) 4.254 →

Although the correct answer is 4.354, the answers given were:

Answer	Frequency
4.244	1
4.254	1
4.264	11
4.354	4
42.52	1
42.54	6

Table 6-3 - answers to question 1e

There are therefore three more common answers:

- 4.354, the correct answer,
- 42.54 where the question has been multiplied by 10 and
- 4.264 where the place value for one-tenth appears to have been misidentified.

It was difficult to understand why students had multiplied by 10 but it is a more common response than the correct answer. It was not possible to talk to the students about this answer because, to ensure anonymity, the students did not name their answers. If the same answers were repeated in the main study, then conversations with the individual students would be possible as their scripts were named and such conversations should expose whether this is a misconception or,

for example, students misreading the question. It is possible that, although the students thought that the instructions were clear, they were unknowingly confused.

6.1.3 Second Pilot

The purpose of the second pilot was to see if data would be better collected by using multiple choice questions. The results of the first pilot were used to select the possible answers in the second pilot. For example, question 1e became.

Add one tenth

- 4.254

A 4.244 B 4.264 C 4.354 D 42.54 E None of these

Where B, C and D represented the most popular answers in Table 6-3 above.

There were three more answers which I could have selected, 4.244, 42.52 and 4.254. I decided not to use 42.52 as I could not see a misconception that could lead to this answer, 4.254 was unchanged from the question. 4.244 seemed to be a relatively reasonable answer for whatever the misconception was that was leading many to select 4.264.

For the second pilot, I selected a different Year 7 class. I could not use set 2 again because the questions were largely the same. Again, I wanted to select a class from set 3 to set 6. I selected set 5, again, mostly due to timetabling and staffing constraints. The students in set 5 had a lower prior attainment than set 2. As there was no intention to compare the results of the first and the second pilot it did not matter that the students did not have similar academic profiles. There were 20 students present on the day of the second pilot, one who was supported one-to-one by an LSA. The student is autistic and had one-to-one support to help her access lessons and with literacy. The LSA was not helping with the mathematics.

Again, all students were allowed to stop the pilot at any point and do alternative work.

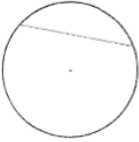
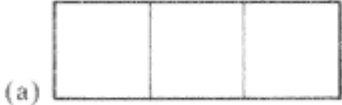
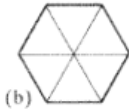
A summary of the results of both the pilots is at Table 6-4. On reviewing the results, some errors appeared less frequently in the second pilot than the first. This may have been, at least in part, due to the introduction of the multiple choice. Students faced with possible answers would be forced to discard their own erroneous answer if it is not included in the list. This might be akin to recognising a correct spelling rather than being able to recall it.

The first question is a case in point. Only 14% wrote the correct answer in the first pilot but in the second pilot, faced with multiple choice question, 25% selected the correct answer. Although there is no intention to compare the results of the first and the second pilot, it is surprising that the second pilot would have this question answered better when the students have a lower prior attainment. The rest of the test, much of which is the same as the first pilot and therefore includes no multiple choice, shows that the second pilot was less well answered than the first. This bears out the possibility that the multiple-choice answers for question 1 guided the students towards a correct answer, potentially supporting appropriate thinking. If multiple choice questions are provided, then the student always has the option to guess. It is not clear therefore whether a given answer is a deliberate choice from a calculation or just a way of completing a task quickly. A multiple-choice test would have made the processing of results easier for me. On the other hand, the richness of answers appeared to be better from the first pilot. This is a small-scale project, so I decided that the extra processing time from not using multiple choice is worth the extra richness of the data collected.

6.1.4 Analysis of results

Table 6-4 below summarises the results of each of the pilots. A discussion on the implementation of the pilot and a further analysis of the answers to individual questions follows.

Question			Pilot 1 % correct	Pilot 2 % correct
Q1a	Add one tenth	4.254	14	25
Q1b	Add one tenth	2.9	34	30
Q1d	Take away 100	583	86	55
Q1e	Take away 100	30000	72	40
Q2a	Add ten	3597	79	50
Q2b	Add ten	0.15	59	30
Q2c	Add one hundred	21534	93	85
Q2d	Add one hundred	19930	90	80
Q3	Add one to 06399		93	75
Q4	Is there a difference between 4.9 and 4.90		90	55
Q5a	Which is bigger 0.75 or 0.8		93	70
Q5b (i)	Draw a ring around the bigger number 20100 or 20095		97	85
Q5b (ii)	Draw a ring around the bigger number 7.55 or 7.5		86	80
Q5b (iii)	Draw a ring around the bigger number 4.06 or 4.5		93	75
Q6 a	Write down any number between 4000 and 5000		100	90
Q6b	4100 and 4200		97	90

Question		Pilot 1 % correct	Pilot 2 % correct
Q6c	0.4 and 0.5	83	70
Q6d	0.41 and 0.42	76	45
Q6e	How many different numbers are there between 0.41 and 0.42	10	12
Q7a (i)	Write down the largest of these; $n+1$, $n+4$, $n-3$, n , $n-7$	76	50
Q7a (ii)	Write down the smallest of these; $n+1$, $n+4$, $n-3$, n , $n-7$	79	60
Q7b	$2n > n+2$. Is this always, sometimes, or never true	34	35
Q8a	What can you say about a if $a+5=8$	100	60
Q8b	What can you say about b if $b+2$ is equal to $2b$	24	10
Q9a	$A+B+C=C+A+B$ Always, never, sometimes, when	62	40
Q9b	$L+M+N=L+P+N$ Always, never, sometimes, when	17	20
Q10	Is this circle split into 2 halves 	76	95
Q11a	Shade in two thirds of each of the shapes 	97	85
Q11b	Shade in two thirds of each of the shapes 	69	25

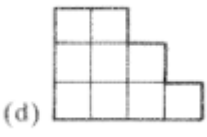
Question		Pilot 1 % correct	Pilot 2 % correct
Q11c	Shade in two thirds of each of the shapes 	62	30
Q11d	Shade in two thirds of each of the shapes 	69	35

Table 6-4 Summary of Pilot results question by question

Looking at the results, 90% of students in the first pilot successfully added 100 to 19930 whereas only 14% correctly answered one tenth to 4.254. This potentially indicates an increased competency with whole numbers than with decimals, although it could also indicate a lack of understanding the question.

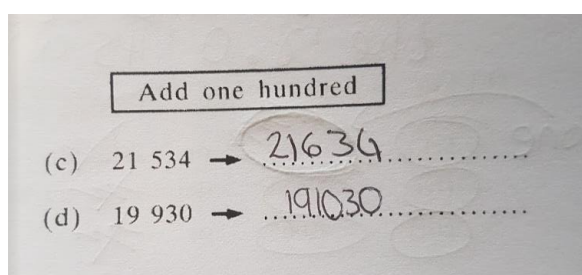


Figure 6-1 student answer to add one hundred.

Considering Figure 6-1, it can be seen that this student has managed to add 100 correctly to 21 534. This is a simple addition with no need to consider carrying because the addition does not increase the digit in any column past nine. On the other hand, where they have added 100 to 19 930, it can be seen that the correct column has been identified but rather than recognising that adding one to 900 will have a knock-on effect in the thousands column the student has placed 10 in the hundreds column, perhaps indicating a lack of understanding of place value.

6.1.5 Pilot of the tasks

Students in the comparison group undertook the pre- and post-test in the academic year 2017-2018. Once both tests were complete, I worked with classes and students to pilot tasks that I considered using for the teaching sessions.

I asked a whole class to pilot my human number line task (see 1.1.1 for details of this task). I piloted some other tasks using six volunteers from one of their mathematics lessons. In this lesson I asked the students to use various representations of decimals and fractions and talked to them about which they found most accessible and why. I gave the students a sheet of paper with 100 squares on it. The intention was that each 100-square would represent one. I told the student each square was one and asked them to show me a half. Two groups folded the whole piece of paper. The concept of a half was apparent, but it was not clear that the students could answer ‘half of what?’ This is similar to a finding in Wong (2013), that children mix the concept of $\frac{1}{2}$ as a measurement and $\frac{1}{2}$ as a magnitude. I considered whether or not my instructions had been clear. I reiterated the instructions carefully and students folded one square. Although the context is different, this being an area model, this interaction is reminiscent of the findings in Wong (2013), when asked to place $\frac{1}{2}$ on a number line, participants placed a mark for $\frac{1}{2}$ halfway along the line whatever the end points.

Once the idea of showing me a half was sorted and each student was showing me half of the square, I asked them to show me a quarter.

Me: show me a quarter.

(Student cuts out a square of 25 small squares)

Me: how do you know it's a quarter?

Student: I halved it and halved it again.

Me: how do you know it's a quarter?

Student :

Me: Is this a quarter? (I folded a square into four unequal portions)

Student: no they all need to be the same size

This indicated to me that the student knew that the quarters needed to be equal. I realised later that I had missed the opportunity to check their understanding of the meaning of equal. I noted that in a future exchange at this point I would show quarters that were not congruent shapes.

Roche (2005) suggests that colouring in a 100-square to represent decimals or fractions allows the student to consider only the numerator and therefore reinforces whole number thinking. This is akin to the criticism in Watanabe (2002) that the denominator should partition rather than be counted. My experience resonated with this finding. The pilot led me to consider that use of the 100-square as a decimal representation might reinforce the idea of whole number thinking. For example, using the 100-square to represent 0.25, 25 blocks out of 100 are coloured. Another criticism of this representation is that it is restricted to two decimal places. Having reflected on the use of the 100-square during the pilot I will not use it in the intervention, instead, I will use the decimat (Roche, 2005, 2010).

6.1.6 Summary from the Pilot

Considering the results of the pilot, and my role at school changing from Head of Department to being on the Senior Leadership Team, it became clear that I would not be able to cover all of the intended subject areas. The pilot showed that even considering children with reasonably high prior attainment, many had significant difficulties in understanding magnitudes of fractional amounts. In my experience, the issues around fraction magnitudes can persist through secondary school. I

decided at this stage to restrict the intervention to fraction magnitude and place value, leaving out any focus on algebra and generalised arithmetic.

My experience tells me that students who are placed into set 1 or 2, are usually able to find fraction magnitudes or will quickly do so with the teaching in secondary school, whilst the students placed in the set with the lowest prior attainment will often be unable to use fraction magnitudes even at aged 16. In 2018/19 there were three sets. Because I was no longer going to be able to work with more than one class, I chose to work with set 3.

Given the choice of the full pre-test and post-test and the multiple-choice version I decided to keep the full test. It would mean that I had more analysis to do but would not restrict the richness of data that I would be able to collect and analyse.

The 100-square is used in the Maths4Life task (Department of Education, 2007) that I intended to use during the intervention. My experience of piloting the tasks resonated with the warning in Roche (2010) that using the 100-square (modelling an area in the Maths 4 Life task) can potentially reinforce whole number thinking. I decided to introduce the Decimat (Roche, 2005) and replace the use of the 100-square in any of the tasks I used.

6.2 The Comparison Year

In the school year 2017/18, all the Year 7 students were asked to take the pre-test in September and the post-test in April, this was planned to be at the same time as the student in the intervention year would take the tests. Both the pre-test and the post-test were identical to the first test piloted at school B. The scheme of learning across both school years was consistent and so the students would be taught the same content between the tests. In the comparison year there were 77 students. Only students present in school on the day that the test was taken took the test.

The result of the testing is in Table 6-5 below. A full student by student break down is at Appendix 3. At the time of the test there were questions included on algebra (these were questions 7,8 and 9). Because the scope of the project changed to exclude algebra the results here are reviewed without these questions. This leaves 24 questions being analysed. The average improvement is 2.4 questions out of the 24 asked for the 52 students who took both tests.

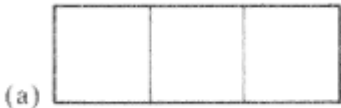
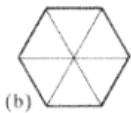
	Number of students	Average pre-test correct answers	Average Post-test correct answers
Students who sat neither test	5	NA	NA
Students who sat the post-test but not the pre-test	8	NA	14.5
Students who sat the pre-test but not the post-test	12	12.5	NA
Students who sat both	52	14.5	16.9
Overall	77	14.2	16.3

Table 6-5 - results of the pre and post-test (comparison group)

6.2.1 Comparison pre and post test

The overall results of the comparison pre- and post-test excluding the algebra questions are below at Table 6-6.

Question			Pre-test% correct	Post-test % correct
Q1e	Add one tenth	4.254	23	44
Q1f	Add one tenth	2.9	36	44
Q1g	Take away 100	583	88	78
Q1h	Take away 100	30 000	52	52
Question			Pre-test% correct	Post-test % correct
Q2a	Add ten	3 597	67	81

Q2b	Add ten 0.15	44	40
Q2c	Add one hundred 21 534	81	83
Q2d	Add one hundred 19 930	64	77
Q3	Add one to 06399	72	90
Q4	Is there a difference between 4.9 and 4.90	55	80
Q5a	Which is bigger 0.75 or 0.8	62	89
Q5b (i)	Draw a ring around the bigger number 20 100 or 20 095	89	88
Q5b (ii)	Draw a ring around the bigger number 7.55 or 7.5	64	76
Q5b (iii)	Draw a ring around the bigger number 4.06 or 4.5	75	85
Q6 a	Write down any number between 4 000 and 5 000	86	100
Q6b	4 100 and 4 200	86	100
Q6c	0.4 and 0.5	67	83
Q6d	0.41 and 0.42	52	77
Q6e	How many different numbers are there between 0.41 and 0.42	8	8
Q11a	Shade in two thirds of each of the shapes 	72	95
Q11b	Shade in two thirds of each of the shapes 	41	70

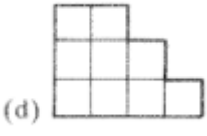
Question		Pre-test% correct	Post-test % correct
Q11c	Shade in two thirds of each of the shapes 	33	64
Q11d	Shade in two thirds of each of the shapes 	35	58

Table 6-6 Question by question comparison group

6.2.1.1 Place Value

Questions 1-3 concentrated on place value and being able to manipulate how changes in one column affect other columns (e.g. adding 1 to an integer number ending in a 9). There were many difficulties with the first two questions; 1e and 1f (add one tenth to 4.254 and 2.9). These were answered correctly by 23% and 36% respectively in the pre-test and 44% each in the post test. As in the pilot some students appeared to multiply by 10 (see Figure 6-2). Some used a different strategy for e and f: add $\frac{1}{100}$ for 1e and 1 for 1f. Although the intention of these questions is to see if students can manipulate place value, some resorted to formal addition algorithms. One student wrote an aide memoire showing the place value columns. Adding 10 caused issues (see Figure 6-3), particularly when added to a decimal. Whilst most added 10 successfully to 3597, only 50% added 10 to 0.15. It appears possible that might have appeared to the students as a trick question.



Figure 6-2 Students' work - adding one tenth.

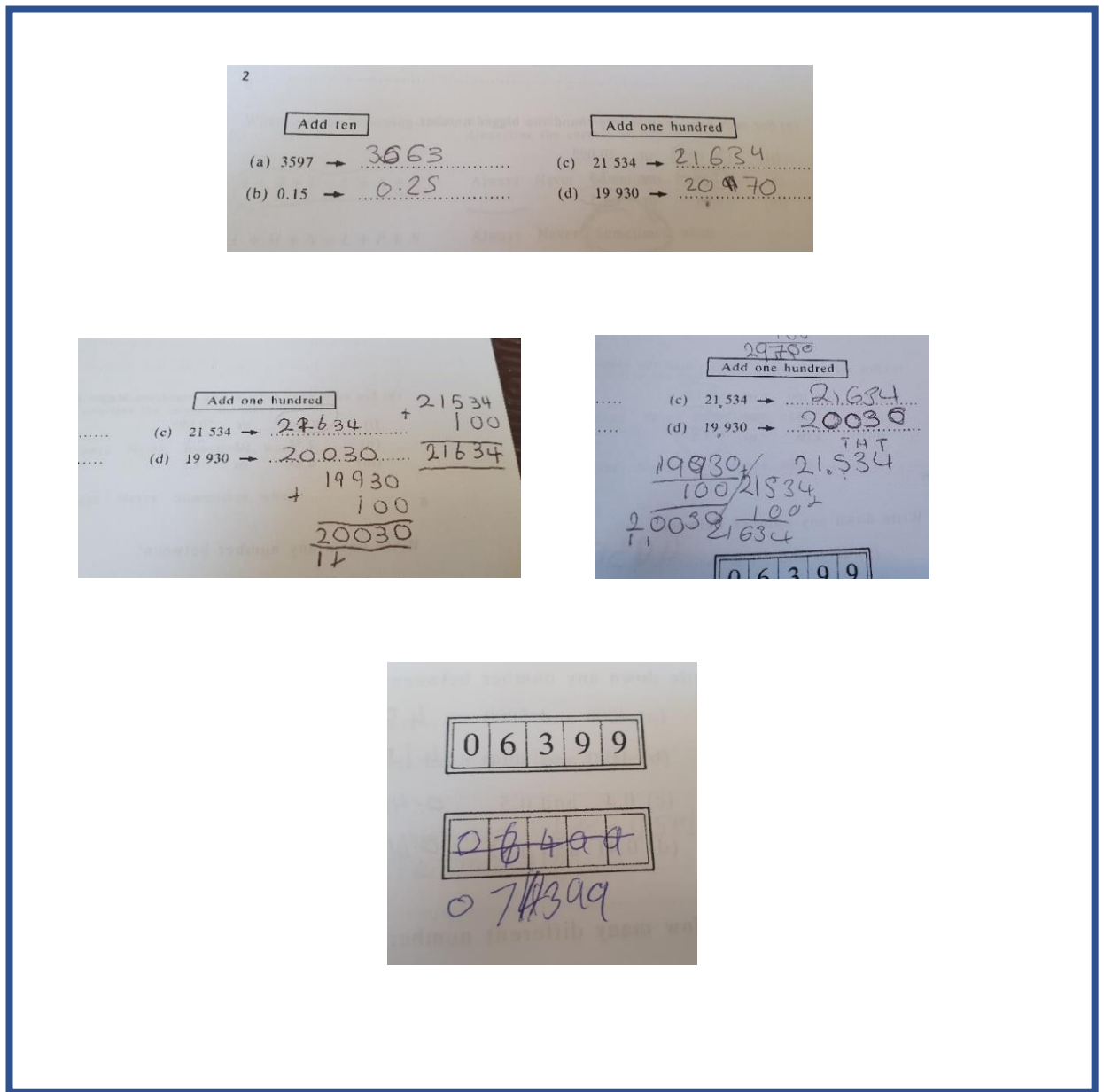


Figure 6-3 - Adding 10 and 100

6.2.1.2 Decimals

Although questions 4 to 6 (see Table 6-6) were also place value questions they were more about the magnitude of numbers. Question 6 (see Figure 6-4) appears to become increasingly harder for students (Table 6-7). Most students (86% pre-test 100% post-test) were able to find a number in between 4000 and 5000. Once asked to find numbers in between two decimals the students were more likely to make an error.

6

Write down any number between:

(a) 4000 and 5000

(b) 4100 and 4200

(c) 0.4 and 0.5

(d) 0.41 and 0.42

(e) **How many** different numbers **could** you write down which lie between 0.41 and 0.42?

Figure 6-4 Question 6

	Pre-test	Post-test
A number between 4000 and 5000	86%	100%
A number between 0.4 and 0.5	67%	83%
A number between 0.41 and 0.42	52%	77%

Table 6-7 correct answers to question 6

Some answers listed 0.04 as lying between 0.4 and 0.5: others used a second decimal place (0.41.5). There seemed to be a bias in some answers towards two digit decimals, with many answers having a question mark instead of an answer at this point

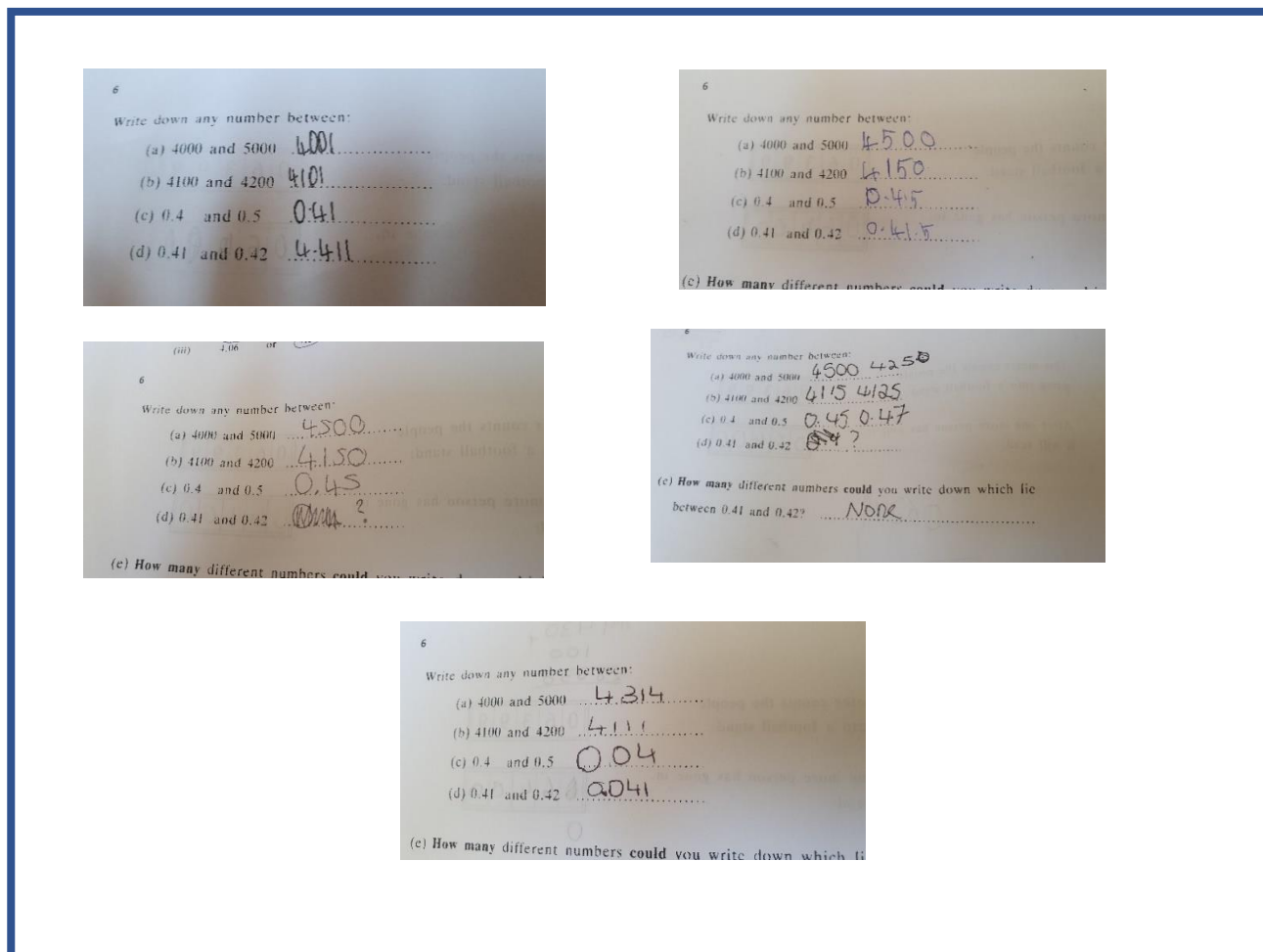


Figure 6-5 finding numbers between

Question 5 asks students to select the bigger of two numbers (see Figure 6-6).

Interestingly, more students were able to select the bigger of 4.5 or 4.06 rather than 7.55 and 7.5, even though the latter question would be selected by those with a longer-is-larger misconception as well as those who have an accurate method for finding the bigger of two decimals (see Table 6-8).

	Pre-test	Post-test
5bi 20 100 20 095	89%	88%
5bii 7.55 7.5	64%	76%
5biii 4.06 4.5	75%	85%

Table 6-8 correct answers to question 5

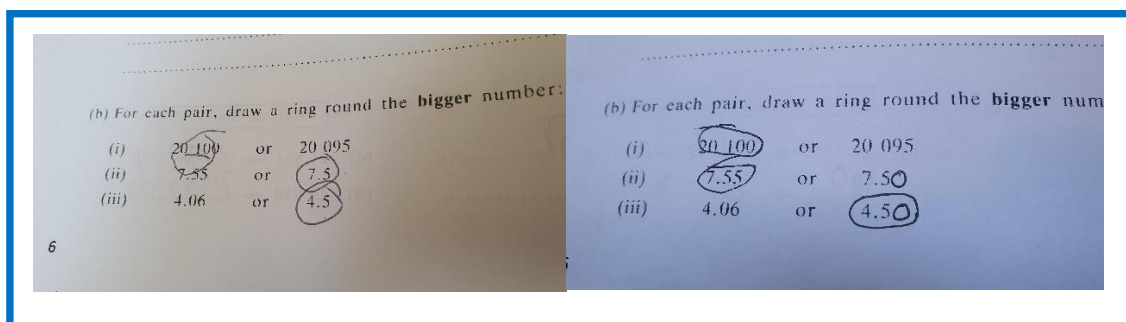


Figure 6-6 examples of answers to question 5

6.2.1.3 Fractions

Figure 6-7 shows a selection of post-test answers to question 11. Answer 1 shows a child selecting $\frac{2}{3}$ as one part less than the whole. Answer 2 is interesting, because although the student has not shaded in $\frac{2}{3}$ it could be argued that by shading they have represented $\frac{2}{3}$. Answer 3 shows work where a student has coloured in 2 parts with no reference to the whole. Answer 4 shows where the student has converted $\frac{2}{3}$ into equivalent fractions to shade in the correct number of parts.

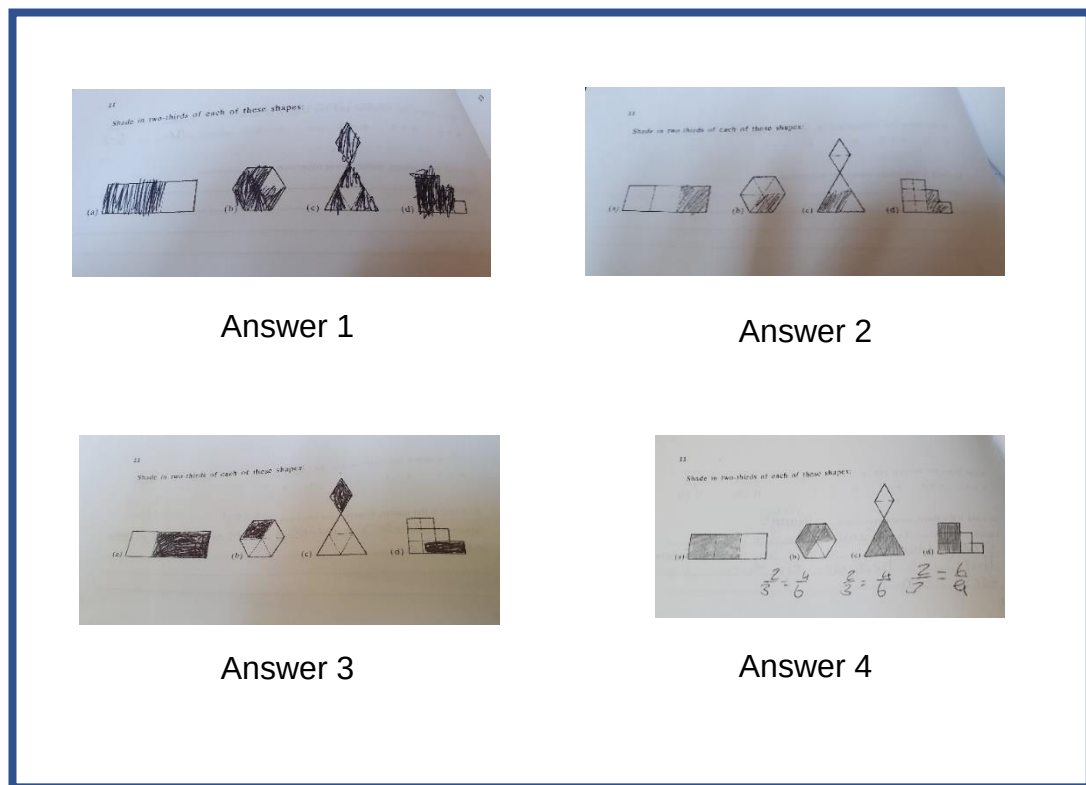


Figure 6-7 fraction answers

6.2.1.4 Indication of intuitive / automated responses

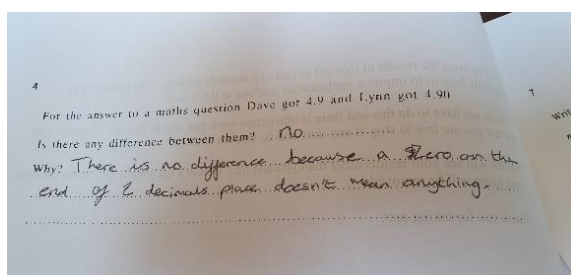


Figure 6-8

In Figure 6-8, when asked if there was a difference between 4.9 and 4.90, the student has written 'There is no difference because a zero on the end of a decimal place doesn't mean anything'. Although little can be read into this, without talking to the student further, it appears to have the nature of an automated response: a phrase that has been learnt and used frequently.

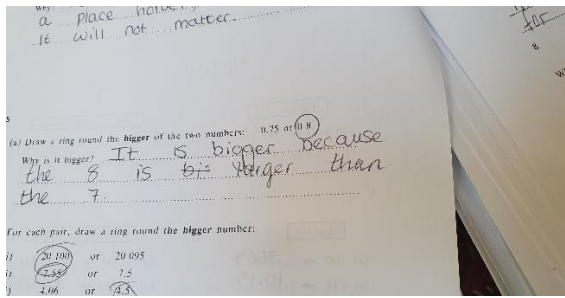


Figure 6-9

Here, in Figure 6-9, the student, asked which is bigger 0.75 or 0.8 has replied 0.8 because 'It is bigger because the 8 is bigger than the 7'. The student appears to have at least a partial algorithm to assess the magnitude and order decimals. It could be that that is a full algorithm that will work for any number of decimal places, but here only one decimal place can be seen. The student appears to be using the full left-to-right comparison algorithm.

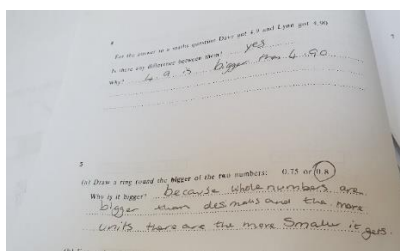


Figure 6-10

Figure 6-10 appears to show another automated response, with decimals 'the more units there are the smaller it gets'. This appears to be shorter-is-larger misconception.

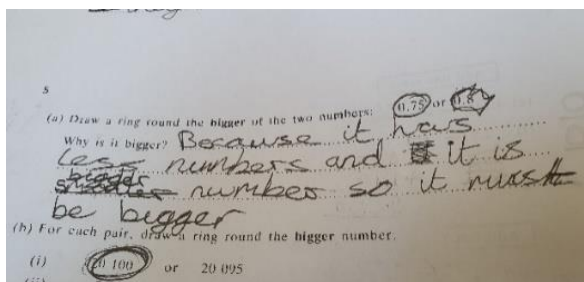


Figure 6-11

Here, in Figure 6-11, the student has written 'Because it has less numbers and it is bigger numbers so it must be bigger'. It is interesting to see what is going on here, as originally the student had written 'Because it has less numbers and it is smaller numbers so it must be bigger'. Originally the student had stated that 0.8 was bigger but had changed that to 0.75. The student does not appear to have a fixed algorithm or method, but maybe has mantras and intuitions, which are perhaps conflicting.

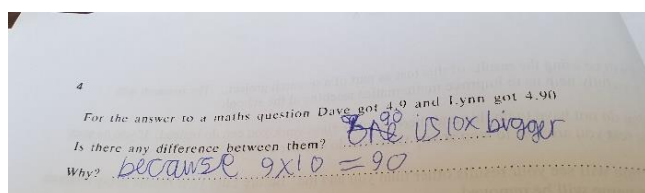


Figure 6-12 multiply by 10 add a 0

Here (Figure 6-12) asked if there is a difference between 4.9 and 4.90 the student has said that '(4.)90 is 10 x bigger because $9 \times 10 = 90$ '. This appears to be the result of the automated 'to multiply by 10 add a zero'.

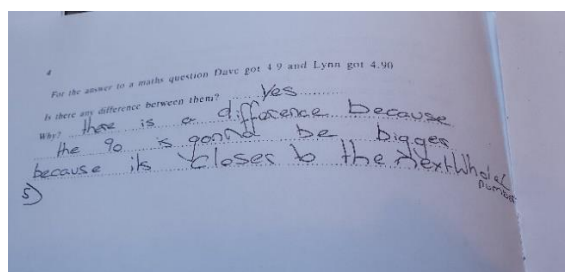


Figure 6-13 difference between 4.9 and 4.90

Asked whether there is a difference between 4.9 and 4.90 this student (Figure 6-13) has said 'there is a difference because the 90 is gonna (sic) be bigger because its (sic) closer to the next whole number'.

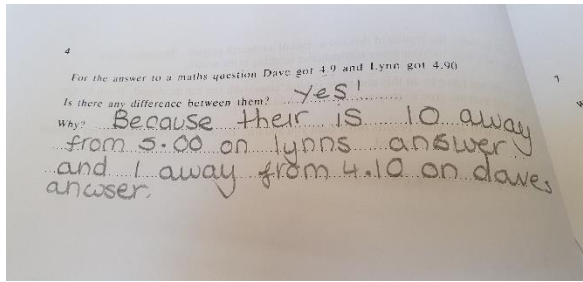


Figure 6-14 4.9 and 4.90

This student (Figure 6-14) shows a lack of understanding in their place value. Again, answering whether there is a difference between 4.9 and 4.90 they answer yes, ‘Because their is 10 away from 5.00 on Lynns answer and 1 away from 4.10 on daves anwser (sic)’.

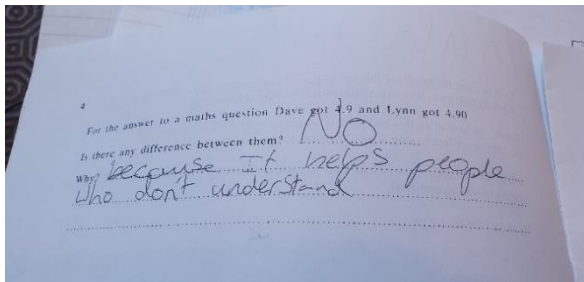


Figure 6-15 - 4.9 and 4.90

This student asked the same question (Figure 6-15) has said ‘No’, ‘Because it helps people who don’t understand’. This could be a student who has been taught that to order decimals you can add a zero to the end to ensure that there are the same number of decimal points in the numbers.

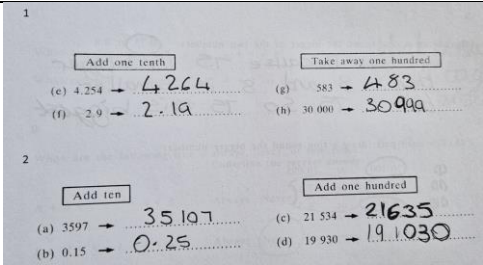
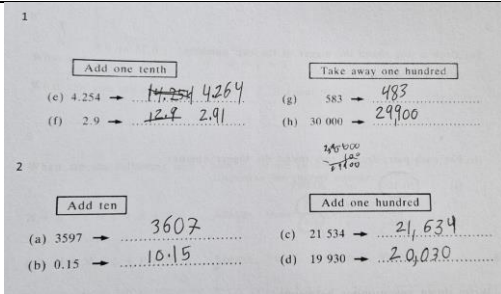
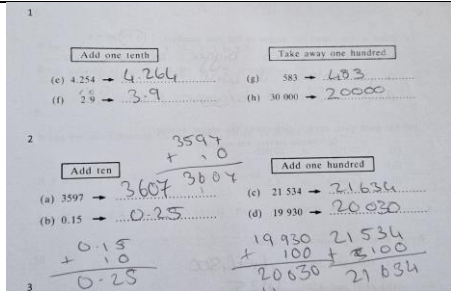
6.3 Illustrative Cases

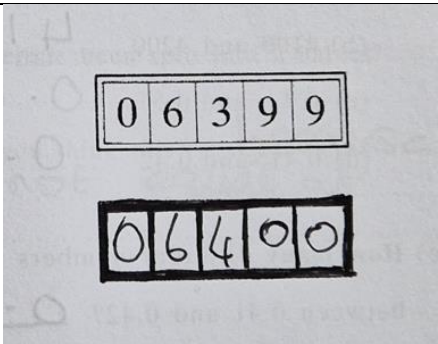
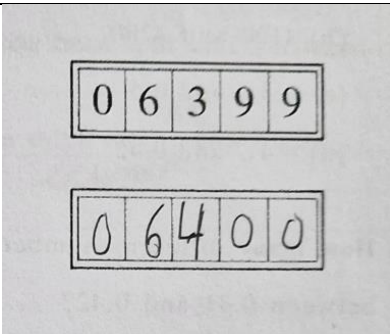
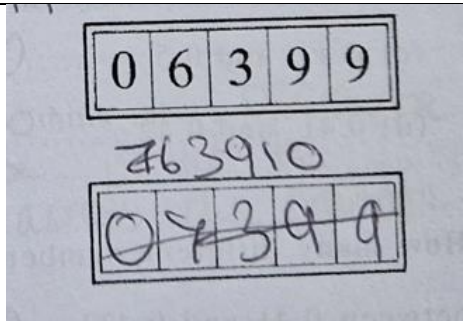
All of year 7 were invited to sit the pretest. Analysis of the pre-test and post-test results are at section 6.5. To illustrate the journey of students through the interventions and their changes in understanding I have detailed illustrative cases for three students in intervention 1 and two students in intervention 2. The three students are Tasha, India and Rachel. I selected these students from students

who had been in both teaching episodes in the intervention and were also included in the interviews. Each displayed a misconception at the start of the intervention, were able to work correctly with decimals in the lessons, but displayed misconceptions in the post-test or interview. The text draws from evidence from other students as well.

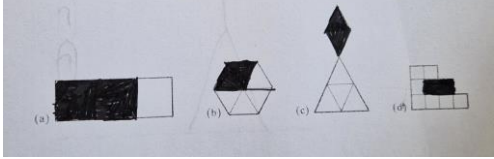
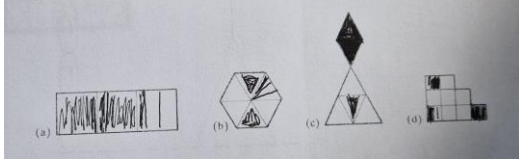
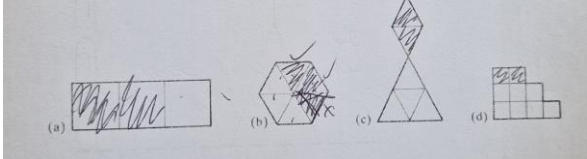
The illustrative cases are presented at each stage of the intervention . They are labelled as “illustrative case” (see Illustrative Case 6-1, Illustrative Case 6-2, Illustrative Case 6-3, Illustrative Case 6-4, Illustrative Case 6-5, Illustrative Case 6-6, Illustrative Case 6-7, Illustrative Case 6-8, Illustrative Case 7-1, Illustrative Case 7-2, Illustrative case 8-1, and Illustrative Case 8-2). The entire illustrative case can be seen in Appendix 9

Illustrative Case 6-1 lists the students, their KS2 scores and question by question their answers to the pre-test. The illustrative case will be expanded as relevant points in the text. The entire illustration is also at Appendix 9.

Student	Tasha	India	Rachel
	Key Stage 2 scaled score (80 to 120 – national average 105)		
KS2 maths	92	96	96
	Pre-test		
Pretest	8 (out of 24)	12 (out of 24)	13 (out of 24)
Q1/2			
	Tasha displays an insecure application of place value.	India displays a more secure application of place value. However, there are still mistakes when adding one tenth	Rachel has used algorithms to calculate the answers rather than applying the properties of place value. Even using the algorithms there are errors in applying place value. Adding 10 to 0.15 is an example of this

Student	Tasha	India	Rachel
			 Rachel seems unsure how to tackle this at all. It appears that she has placed a 10 in the units column.
	5 (a) Draw a ring round the bigger of the two numbers: 0.75 or 0.8 Why is it bigger? Because 75 is bigger than 8 and 8 is smaller than 75 so 75 is biggest This potentially displays a longer is larger misconception with a whole number bias	5 (a) Draw a ring round the bigger of the two numbers: 0.75 or 0.8 ? Why is it bigger? India does not appear willing to start the question.	5 (a) Draw a ring round the bigger of the two numbers: 0.75 or 0.8 Why is it bigger? It is bigger because in a percentage it will 75% and 8% and 75% is bigger than 8% Rachel has a strategy here to convert to percentages, something she repeats in lesson. Unfortunately, she converts 0.8 to 8% and therefore her answer displays a longer is larger misconception.

Student	Tasha	India	Rachel
	(b) For each pair, draw a ring round the bigger number: (i) $\textcircled{20\ 100}$ or 20 095 $\textcircled{7.55}$ or 7.5 $\textcircled{4.06}$ or 4.5 Tasha appears to have a longer is larger misconception where 4.06 is larger than 4.5 potentially due to the greater number of digits in the first.	(b) For each pair, draw a ring round the bigger number: (i) $\textcircled{20\ 100}$ or 20 095 (ii) 7.55 or $\textcircled{7.5}$ (iii) 4.06 or $\textcircled{4.5}$ India appears to display a shorter is larger misconception with both of 7.5 and 4.5 being selected as larger.	(i) $\textcircled{20\ 100}$ or 20 095 (ii) $\textcircled{7.55}$ or 7.5 (iii) $\textcircled{4.06}$ or 4.5 Rachel appears to have a longer is larger misconception where 4.06 is larger than 4.5 potentially due to the greater number of digits in the first.
	Write down any number between: (a) 4000 and 5000 <u>4137</u> (b) 4100 and 4200 <u>4125</u> (c) 0.4 and 0.5 <u>0.04</u> (d) 0.41 and 0.42 <u>0.42</u> It is not clear why c is wrong and d is arguably correct.	Write down any number between: (a) 4000 and 5000 (b) 4100 and 4200 (c) 0.4 and 0.5 (d) 0.41 and 0.42	Write down any number between: (a) 4000 and 5000 <u>404500</u> (b) 4100 and 4200 <u>4125</u> (c) 0.4 and 0.5 <u>0.45</u> (d) 0.41 and 0.42 <u>0.42</u> Rachel has made a good attempt here. If the question had been more specific and ensured that the answer for d could not be 0.41 or 0.42, it might have been more informative.
	10 (a) Has this circle been split into 2 halves? <u>NO</u> (b) Why do you think this? <u>because it is not equal at all</u> Tasha can express the concept of a half	10 (a) Has this circle been split into 2 halves? <u>NO</u> (b) Why do you think this? <u>because it is not equal</u> India can express the concept of a half	10 (a) Has this circle been split into 2 halves? <u>NO</u> (b) Why do you think this? <u>because other wise it will be straight across the middle</u> Rachel can express the concept of a half

Student	Tasha	India	Rachel
	11 Shade in two-thirds of each of these shapes:  Although A is correct, Tasha has coloured two for each shape, not considering the denominator	11 Shade in two-thirds of each of these shapes:  It is not clear what India intends here.	Shade in two-thirds of each of these shapes:  Although A is correct, Rachel has coloured two for each shape, not considering the denominator

Illustrative Case 6-1

6.4 Intervention 1

6.4.1 Design of the tasks

I selected or designed tasks that would work with a particular instantiation or multiple instantiations and would encourage students to utilise the essential and necessary properties to help them appreciate what stayed the same and what would be different.

We suggest that any sensible approach to teaching combines work on understanding concepts with work on mastering procedures, and combines tasks designed to stimulate learners to express their own thinking using technical terms with tasks designed to highlight the use of important routines. Keeping the notion of mathematical structure in mind, together with seeking the structural underpinnings of any proposed tasks, provides ways for students to experience these important elements. (Mason *et al.*, 2009, p. 26).

It is important to note that it is not enough just to select such tasks. It can be that the students do not interpret the results in the way that the teacher intends but also with our current model the instantiations that the teacher anticipates might not be the instantiations that the students experience.

The purpose of the tasks and the tasks instantiations being addresses are included in Table 6-10 and Table 6-13 for the first intervention.

6.4.2 The Intervention lessons

The intervention comprised two series of lessons with a Year 7 class, most of whom had been assessed at Key Stage 2 as having been below the expected standard. Their Key Stage 2 mathematics test scores were between 84 and 95. The national average score is 103. The range of possible scores is from 80 to 120. Details about the students are at Table 6-9.

Student	September intervention	January intervention	Interview	Gender	HMF	FSM	SEN	EAL	Age	Reading Age 2018	Spelling Age 2018	KS2 English scaled score	KS2 Mathematic scaled score	DCT primary classification	DCT secondary classification	Pretest correct	Pretest incorrect	Pretest not answered	posttest correct	posttest incorrect	posttest not answered
Tayla	Withdrawn	Withdrawn		F	No	No	Yes	No	11.37	6.08	5.9	93	86			3	21	0	7	15	2
Cameron	Y	Different set		M	Yes	No	Yes	No	11.65	10.11	9	95	95	shorter-larger	reciprocal / negative	8	11	5	12	5	7
Lucy	Y	Y		F	No	No	Yes	No	11.43	9.06	10.1	87	95	longer-larger	whole number	6	7	11	4	6	14
Joseph	Y	Y		M	No	No	No	No	12.31	9.06	11.4	95	94			17	5	2	18	4	2
James	Y	Y	Y	M	No	No	No	No	11.97	13.02	8.1	102	92	shorter-larger	reciprocal / negative	11	9	4	13	9	2
Anton	Different set	Y		M	Yes	No	Yes	No	12.34	7.05	7.5	93	97			7	8	9	9	14	1
Liz	Y	Y		F	Yes	No	No	No	12.06	7	8.8	90	90			4	18	2	5	19	0
Amber	Y	Y		F	No	No	Yes	No	11.68	9.01	10.5	96	86	shorter-larger	reciprocal / negative	5	9	10			
Ruth	Y	Y	Y	F	Yes	No	No	Yes	11.41	8.04	7.1	87	83	longer-larger	whole number	4	18	2	0	3	21
Fergus	Y	Different set		M	Yes	Yes	No	No	11.37	12.1	11.1	111	110	apparent - expert		23	1	0	21	3	0
Katie	Y	Left		F	No	No	Yes	No	11.48	9.01	12.1	89	89	longer-larger		13	11	0			
Billy	Y	Different set		M	Yes	No	No	No	11.67	11.06	12.1	104	102	apparent - expert		21	3	0	19	4	1
Tasha	Y	Y	Y	F	No	No	No	No	11.58	10.11	12.4	93	92	longer-larger	whole number	8	16	0	11	13	0
Nuri	Y	Left		M	Yes	No	No	No	11.99	10.11	13.7	104	110	apparent - expert		21	3	0			
Chantelle	Y	Y	Withdrew	F	Yes	No	No	No	11.71	12.02	13.7	102	96	longer-larger	whole number	10	13	1	8	12	4
India	Y	Y	Y	F	No	No	Yes	No	11.40	8.1	8.4	97	96	longer-larger		12	6	6	13	11	0
Claire	Y	Y		F	Yes	Yes	No	No	11.52	9.06	10.05	100	94			9	11	4			
Ciaran	Y	Withdrawn		M	No	No	Yes	No	11.75	8.07	6.8	90	93	longer-larger		7	15	2	9	8	7
Louise	Y	Y		F	No	Yes	No	No	11.87	8.04	12.8	96	89			10	8	6	8	15	1
Rachel	Y	Y	Y	F	No	No	No	No	12.20	10.11	13.7	99	96	longer-larger	whole number	13	10	1	11	12	1

Table 6-9 the intervention group

Where some of the students have higher KS2 SATS scores than 95 it is because their information was not available when the classes were initially put into sets. In this case, other information, such as primary school teacher's comments were used to select a set. Where nothing was known about the incoming student, they would be placed in any set with the view of moving them, if applicable, when more was known. Given the nature of our student population and how frequently students move into and out of the school this happens relatively frequently. In this eventuality, the students were more likely to be placed in set 2 or 3. The aim of the first sequence of lessons was to improve the students' understanding of decimal place value, whilst the second was focused more on understanding fractions.

The content of the lessons; ordering decimals and fractions and engaging with place value are all aims of the KS1 / KS2 national curriculum. All the students should have encountered these concepts in Year 3,

Pupils connect tenths to place value, decimal measures and to division by 10. They begin to understand unit and non-unit fractions as numbers on the number line, and deduce relations between them, such as size and equivalence. They should go beyond the $[0, 1]$ interval, including relating this to measure. Pupils understand the relation between unit fractions as operators (fractions of), and division by integers. They continue to recognise fractions in the context of parts of a whole, numbers, measurements, a shape, and unit fractions as a division of a quantity. (Department of Education, 2013b, p. 21)

And in Year 4

They begin to extend their knowledge of the number system to include the decimal numbers and fractions that they have met so far. (Department of Education, 2013b, p. 24)

Pupils learn decimal notation and the language associated with it, including in the context of measurements. They make comparisons and order decimal amounts and quantities that are expressed to the same number of decimal places. They should be able to represent numbers with one or two decimal places in several ways, such as on number lines (Department of Education, 2013b, p. 27).

By Year 6

compare and order fractions, including fractions > 1 (Department of Education, 2013b, p. 40).

Ordering decimals is only mentioned once and this states that by year 4, pupils

order decimal amounts and quantities that are expressed to the same number of decimal places in several ways, such as on number lines (Department of Education, 2013b, p. 27).

My experience tells me that many students cannot demonstrate these understandings by the start of Year 7, and in fact many do not have a secure understanding by Year 11. The pre-test showed that the students in the intervention made many errors in this area.

Many approaches to teaching decimals in school mathematics are carefully constructed so that students only ever encounter the same number of decimal places as described above. Steinle (2004b) describes that carefully tailoring teaching so that only the same number of decimal places is considered compounds the misconception of whole number thinking. She states: -

Students need to be exposed to ragged decimals to detect their misconceptions and to help them to realize that there is something that they do not know (Steinle, 2004b, pp. 473–474).

As described in section 2.7, Steinle and Stacey (2004) found that young children have a tendency towards a longer-is-larger misconception but by Year 8 this is often replaced by the more persistent shorter-is-larger misconception. My experience tells me that many students I have taught still have a longer-is-larger misconception and this was borne out by the fact that many students in the pre-test displayed this as a likely misconception (see question 5 in section 6.2.1.4 where students were asked to select between 0.75 and 0.8). None of the questions in the pre-test address the possibility of the shorter-is-larger misconception, this however was addressed by asking each student in the

intervention to complete the Decimal Comparison test (DCT) (Steinle & Stacey, 2004).

The aim of the intervention was to improve understanding and address misconceptions by engaging the students with tasks and activities that made them draw comparisons between fractions, decimals, and the result of division sums – using the fact that they are all instantiations of the real numbers. To understand the structure of the continuous number system, I felt that the instantiation of the number line was important. This pictorial representation allows students to visualise smaller, larger and in between as a proxy for the real numbers. Each instantiation was related back to the number line (Siegler *et al.*, 2011).

At the start of the intervention the students undertook the DCT (Steinle & Stacey, 2004). The test identifies possible misconceptions held. I intended to use the test to allow me to identify the misconceptions that were likely to be held by the students in the intervention. By doing this, I was able to tailor the intervention to the address the particular misconceptions held. The results of the DCT can be seen in Table 6-9. This showed that, according to the Steinle and Stacey's classification it was likely that eight students held a longer-is-larger misconception and three held a shorter-is-larger misconception, perhaps more typical, according to Steinle and Stacey of a primary school profile.

Different pedagogical approaches were used for different tasks e.g. discussion (Alexander, 2008; Sfard, 2010; Swan, 2006b), collaboration (Swan, 2006a) and some explanation (Westwood, 1996).

6.4.3 Tasks

A progression through the tasks was as follows.

	Task	Description	Purpose	Instantiation
Lesson 1	Decimal Comparison Test	Students select the smaller and larger of 30 pairs of decimals (Steinle & Stacey, 2004)	To assess whether students are likely to hold one of a number of common misconceptions	Notation
Lesson 1	Human Number Line	Students were asked to place themselves in a line then one student was selected and given a reference point (e.g., 0). Other students were asked to say their number	To see whether there was an automatic assumption that the gaps were integer gaps. To see whether the students had the idea of sequences and could deal with gaps other than 1	Number line
Lesson 2	Folding strips	Students were given metre long strips and asked to fold them into equal pieces, measure them, and work out 1 divided by the number of pieces	Each time the paper was folded the students would encounter a fraction, a decimal, a measure, and a division sum. Discussion around the activity was intended to help the students make the link between decimals, fractions, division, sharing and visualising the number line	Number line Division Shapes Fractions Measure Decimals
Lesson 3 and 4	Decimat (Roche, 2005)	The students were introduced to the decimat; a pictorial way of representing decimals. (see(Roche, 2005))	To be able to represent a decimal by tenths, hundredths, and thousandths – to be able to compare decimals up to three decimal places	Notation Decimat

	Task	Description	Purpose	Instantiation
Lesson 5	Number line	Placing decimals	To be able to place decimals on a number line (Siegler <i>et al.</i> , 2011)	Notation Number line
	Task	Description	Purpose	Instantiation
Lesson 6	Decimals Card Sort from Maths4Life (Department of Education, 2007)	Comparing and ordering decimals, via a number line	The task asks students to order decimals. As part of this they also match the decimal to the representation on the number line.	Number line Notation Decimats

Table 6-10 – Tasks within the first intervention

The tasks were selected to ensure that students engaged with finding and estimating the magnitude of fractional amounts. In most tasks and all lessons, the students worked with magnitudes of amounts on the number line following the advice from Siegler *et al.* (2011) and others (Gabriel *et al.*, 2012; Stacey, Helme, Steinle, *et al.*, 2001; Wong, 2013). The first round of two weeks (six lessons) was intended to ensure students could compare two numbers accurately. At the end of each lesson, I reflected on the lesson in my journal, which I recorded on a dictaphone. This reflection, alongside the work of the students, enabled me to adjust the next lesson in response to what I had learned from the previous lesson. Each task is discussed in more detail later in this section.

6.4.4 The Students

Because it is common practice at the school to move students round between sets the constitution of the class changed between September 2018 and January 2019. The details of the students are below in Table 6-11 and a summary of the number of the student in each group is at Table 6-12.

Throughout chapters 6,7 and 8 I have reported conversations with students.

These are not transcriptions, but are derived from journals which were recorded soon after the lesson. The conversations are, therefore, not word-for-word but instead indicate what was being said.

Student	Sep Group	Jan Group	Interview	Comment
Tayla	N	N		Could not recite numbers from 1 – 10. Left the group to work with the class teacher
Cameron	Y	N		Moved out of the class in December
Lucy	Y	Y		
Joseph	Y	Y		
James	Y	Y	Y	
Anton	N	Y		Moved into the class in December

Liz	Y	Y		
Amber	Y	Y		
Ruth	Y	Y	Y	
Fergus	Y	N		Moved out of the class in December
Katie	Y	N		Left the school
Billy	Y	N		Moved out of the class in December
Nuri	Y	N		Left the school
Tasha	Y	Y	Y	
Chantelle	Y	Y	N	initially agreed to be interviewed, but she declined to take part on the day.
India	Y	Y	Y	
Claire	Y	Y		
Ciaran	Y	N		By January joined Tayla to work with the class teacher
Louise	Y	Y		
Rachel	Y	Y	Y	

Student	Sep Group	Jan Group	Interview	Comment
Tayla	N	N		Could not recite numbers from 1 – 10. Left the group to work with the class teacher
Cameron	Y	N		Moved out of the class in December
Lucy	Y	Y		
Joseph	Y	Y		
James	Y	Y	Y	
Anton	N	Y		Moved into the class in December
Liz	Y	Y		
Amber	Y	Y		
Ruth	Y	Y	Y	
Fergus	Y	N		Moved out of the class in December
Katie	Y	N		Left the school
Billy	Y	N		Moved out of the class in December
Nuri	Y	N		Left the school
Tasha	Y	Y	Y	
Chantelle	Y	Y	N	initially agreed to be interviewed, but she declined to take part on the day.
India	Y	Y	Y	
Claire	Y	Y		
Ciaran	Y	N		By January joined Tayla to work with the class teacher
Louise	Y	Y		
Rachel	Y	Y	Y	

Table 6-11 - the students

Total Students involved in either intervention	20
September	18 (19 started but Tayla moved to work one to one with the class teacher during the intervention)
January	13
Both	12
Selected for interview	6
Interviewed	5

Table 6-12 summary of the students

6.4.5 Decimal Comparison Test (DCT)

The first task that the students undertook was the Decimal Comparison Test (DCT) using the DCT2 from Steinle and Stacey (2004). The purpose of the test is to give an indication of the likely misconceptions following a short set of 30 questions. For each question the student is required to indicate which is the larger of two decimal representations. A copy of the DCT is below at Figure 6-16

Name: _____ Class: _____ Date: _____

For each pair of decimal numbers circle the one which is LARGER.

0.4	0.457	1.06	1.053	4.4502	4.45
0.86	1.3	4.08	4.7	17.353	17.35
0.3	0.4	3.72	3.073	8.245	8.24563
1.85	1.84	2.621	2.0687986	3.2618	3.26
3.71	3.76	8.052573	8.514	3.741	3.746
4.8	4.63	5.62	5.736	0.35	0.42
0.5	0.36	0.5	0.75	2.186	2.954
0.75	0.8	0.426	0.3	0.872	0.813
0.37	0.216	2.516	2.8325	0.038	0.04
3.92	3.4813	7.942	7.63	0.006	0.53

Figure 6-16 DCT (Steinle, 2004a, p. Appendix 1)

Using the DCT allows a classification for each student. A primary classification is longer-is-larger, shorter-is-larger, apparent expert. The test can also be inconclusive. If there are patterns within the answers, then there is a possible secondary classification e.g., whole number thinking or negative/reciprocal thinking. The possible secondary classifications are described by Steinle (2004b). These are referred to in section 2.5.3. The guide to the marking scheme is in Appendix 7.

The results of the DCT for students in the intervention are above at Table 6-9. This includes their demographic data, reading and spelling ages, as well as KS2

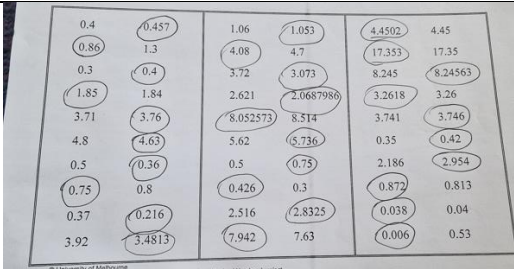
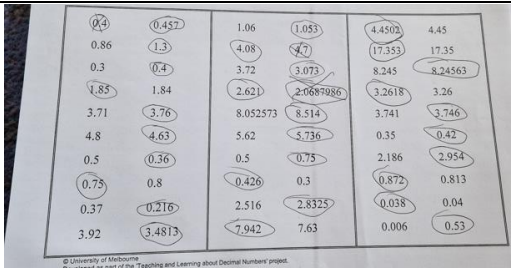
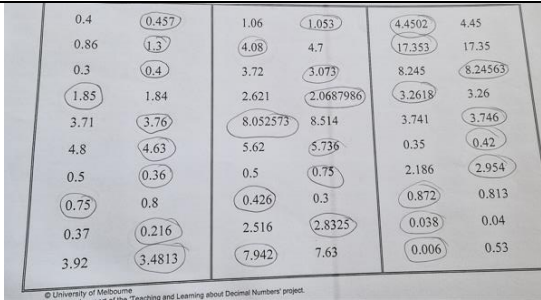
results, DCT results and the number of questions, correct, incorrect and not answered on the pre-test and post-test.

Of the 18 students who were in the intervention group in September, 14 took the DCT. Others were absent on the day of the DCT. Using Steinle and Stacey's (2004) analysis of DCT results, three of the intervention group were apparent experts, three possibly held the shorter-is-larger misconception and eight possibly held the longer-is-larger misconception. The three who were identified as apparent expert were no longer in the group in the January intervention, because after testing and working in the class they were moved into a different set. Steinle and Stacey (2004) refer to participants who achieve a certain level of accuracy as apparent experts rather than experts. This is because from the written test it is only possible to test procedural competency. It might be that students really do understand which decimal is bigger or it could be that they have acquired an algorithm allowing procedural competence. It could be that this algorithm is incomplete and would not work under a different set of circumstances. It would seem key that for someone to be labelled as expert they would be competent with comparing any two decimals.

Steinle (2004a) offers that for younger students the more likely misconception is the longer-is-larger but that older students can replace this with a shorter-is-larger misconception. The results show both present in the students in the intervention. Those with a shorter-is-larger misconception had a reciprocal / negative number subclassification. The students classed as longer-is-larger did not all have a subclassification.

The results of the DCT meant that future tasks were designed to focus on the value of each decimal place and how to assess the magnitude of a decimal.

Illustrative Case 6-2 (below) details the DCT test shows that Tasha, India and Rachel were all likely to hold a longer-is-larger misconception

Student	Tasha	India	Rachel
	First Intervention – September Tasks		
			
DCT	Longer is larger misconception (whole number bias)	Longer is larger misconception	Longer is larger misconception (whole number bias)
Analysis of pre-test and DCT	At the beginning of the intervention, Tasha shows a longer is larger misconception. She does not display a secure understanding of place value or fractions	At the beginning of the intervention, India shows a longer is larger misconception. She does not display a secure understanding of place value or fractions	At the beginning of the intervention, Rachel shows a longer is larger misconception. She does not display a secure understanding of place value or fractions. Rachel appears to have some algorithms which she applies to help her answer these questions.

Illustrative Case 6-2

6.4.6 Human number line

The human number line can be seen as “People Maths” (Bloomfield & Vertes, 2005); an approach where the students themselves are used as a resource or a manipulative. I have used this approach in many parts of my teaching. For example, when teaching a group of students about loci, I would invite a student to stand 1m from a point on the floor. Gradually I would invite more and more students to find another point that satisfied this condition. The students would talk to each other and talk to me, and they would be able to discuss the fact that they had formed a circle. Another advantage of this activity is the fact that I can continue to refer to it with the class later when I was looking at it from a more abstract perspective.

The aim of the human number line task was to be a relevant icebreaker. I wanted to introduce the intervention with a ‘People Maths’ activity because my experience tells me that it would promote discussions about decimals; not only between the students and me but also between the students themselves. In addition, as I had not taught this class before it would allow me to see how the students worked together and would demonstrate to the students my ways of working. I wanted them to know that I valued discussion and participation over giving correct answers and that I would not always be asking questions that had only one right answer. Although the students in the class had varied mathematical experiences, my experience tells me that students who end up being placed in set 3, with lower than average KS2 results have rarely found much success in their mathematical career up to this point and so I wanted them to feel comfortable and safe in my classroom.

For the human number line, I asked students to stand in a line across the room. I started by selecting a student and saying ‘you are number 0’ then asking other

students their number. Both when I piloted this task, and in the intervention, the students counted in ones, although I gave them no guidance on this, and no student asked. All the answers showed the students counting in integers. At the beginning of the activity the student I selected to be '0' was standing at the end of the line, with all the other students to their right. Students only needed to consider positive integers. Later, I labelled a student towards the middle of the line as '0', meaning that students to their 'right' labelled themselves with positive numbers but those to the 'left' had to label themselves with negative numbers.

One student, Fergus, labelled himself as negative 6. I asked him why and he explained that 0 was here and so counting down he would be negative 6. I asked for any other opinions and one student said.

Student: No miss, you said that numbers were increasing in this direction and so he would be 6 not negative 6.

Fergus: Oh, I didn't hear that.

This exemplifies the interactions with Fergus. His answers were frequently showing a different aspect of the question from others. This might have been, at least in part, because Fergus was finding the concepts under discussion easier than others were. He was a student who had been placed into the set without us having access to his KS2 SATS scores. Fergus moved to set 1 when the mathematics department rearranged the classes, so was not in the January group.

I extended the task by saying one student was 0 and the student 5 to the right was 10, thus requiring the students in between to count in 2s. I set numbers so that students had to count in 2s, 3s and 4s and into the negative numbers.

Next, I extended the task by labelling two students as 0 and 1. When I asked each child what their number was, they labelled themselves with very few errors. Tayla, however, did not appear to understand the task. Students around her were

whispering the answers to her. I was concerned that Tayla was not gaining much from the intervention.

The human number line showed me that the students were familiar with decimals with one decimal place and were able to count in these quantities. The issue with this particular task is that once one student has identified their number others around can work out their place by pattern matching, so it does not assess competence. It did however, foster some very supportive conversations, with students helping each other and checking their answers. I took from this that the group had a supportive nature that would work well with pairs tasks that I had planned.

Although students were able to visualise the number line, I recognised from the DCT that they were less competent comparing decimals of different lengths. I have not yet tested how the students deal with 0.11. If students hold a whole number thinking misconception, they might consider that 0.11 is larger than 0.9. This needed to be considered in future lessons.

6.4.7 Folding Strips

For the next lesson I started the folding strips task. The aim of this task is to create a connection between fractions, decimal, division and measure by requiring students to move between different instantiations; from fraction to decimal to division whilst maintaining the continuous nature of number. Students, in pairs, were given 1-metre-long strips of printing roll. They were also given a tape measure. They took the strip, folded it, recorded how many pieces the strip was folded into and what fraction each piece was, how many centimetres and metres that was and also, on the calculator divided 1 by the number of pieces. I demonstrated the first fold, with the students copying my actions and recording

Two pairs' results are in Figure 6-17

[illegible]

There were difficulties that had not happened when I had piloted the tasks. Measurement errors occurred. After two folds it became difficult for the students to measure with a degree of accuracy that would have been suitable for the task. In addition, the students started recording by the pattern they could see rather than working out each part. On the left-hand work in Figure 6-17 the third entry at $\frac{1}{8}th$ of the strip, a measurement of 0.14m is quite reasonable, but $1 \div 8$ would not give an answer of 0.14, even when rounded. From this, the students are working with the relationship between decimals and measure, if they had completed the work as required, they would have a measurement of 0.14m but 1 divided by the number of strips would have been 0.125. The students would not have been able to see the relationship. This frustration of using an experiment to find structure but imprecise measurements making the answer less than obvious is discussed in Mason *et al*, (2009). As well as inevitable measurement errors, some pairs mixed up the number of pieces with the number of folds and some appeared to have difficulty with converting between cm and m. A tape measure with both metres and

centimetres on the same side would have been good to use. I had provided the students with standard metric sewing measuring tapes, these had the centimetres marked in the centre of the block rather than at the point (see Figure 6-18 measuring). Although the measuring tape is perfectly acceptable as a sewing measuring instrument, it has not been pedagogically designed and so was not as good for the task as it could have been. Although this had not presented problems in the pilot, the students in the intervention had difficulty reading the lengths.



Figure 6-18 measuring tape.

Other than reiterating the connection between fractions and decimals, this task did not contribute as much to the students' understanding as I had intended it to. I decided that when I introduced decimats I would need to start more slowly than I had planned.

6.4.8 Decimats

The aim of introducing the decimat was to introduce the students to an instantiation, which was a visual representation of a decimal which linked the decimal with its related proper fraction, and also emphasised the “ten-ness” (Roche, 2010) of our number system. Although the decimat is limited to 3 decimal places, the reasoning can be extended to explain how four and further decimal places could be represented. The visualisation supports the idea that each subsequent decimal place is 10 times smaller than the last.

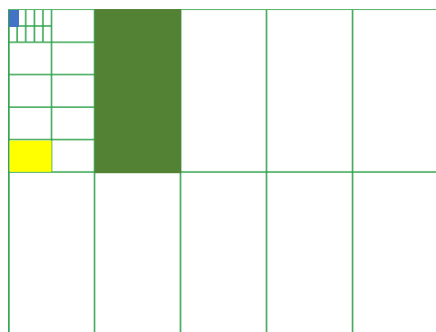


Figure 6-19 Decimat example

In Figure 6-19 the whole decimat represents 1, the 10 large rectangles each represent 0.1, the 10 medium rectangles each represents 0.01 and the 10 small rectangles each represent 0.001. Thus, every decimal between 0 and 1 with up to three decimal places can be represented. For example, the decimal represented by the coloured rectangles in Figure 6-19 is 0.111.

I introduced the decimat slowly. I gave the students a piece of paper and, with no guidance, asked them to show me a half. Some wrote $\frac{1}{2}$, some folded the paper in two, this is a similar response to one seen by Wong (2013) where students confused the magnitude of the fraction with a measure. It is similar to the confusion in the piloting of the tasks. It could have been due to this or due to my questioning. However, because of the pilot I was expecting the response and was able to use it to move the students towards what I wanted them to do. One folded the paper in two and then wrote $\frac{1}{2}$. I clarified that I wanted the students to show me a half by folding and compared this back to the folding paper strips task. I asked the students to show me $\frac{1}{4}$ and $\frac{1}{8}$. These were all quite successfully done, however, these fractions had all been practised in the folding strips task. One interesting outcome was that when I asked students to show me $\frac{1}{3}$ some of them folded the paper in half three times (hence showing me $\frac{1}{8}$). In this the students have a potential cognitive conflict (Zazkis & Chernoff, 2008). They believe they are

showing me $\frac{1}{3}$, but they have a shape folded into 8 pieces. I asked the student about what they were showing me.

Me: what fraction is this?

Student: $\frac{1}{3}$?

Me: How do you know that?

Student: I folded it three times.

Me: Unfold it and count the pieces.

Student: (unfolding and then counting) Oh It's $\frac{1}{8}$

Me: How do you know that?

Student: It is in 8 pieces.

The cognitive conflict did not appear to be activated for the student until I asked him how many pieces he had and therefore what fraction he was showing me. When I saw the two realities I went back to the structure and asked a question to activate their cognitive conflict. The students though, did not see the conflict, nor see the need to go back to the structure to sort out the problem.

I asked the students to show me $\frac{1}{10}$. The students did not automatically come up with an answer for this. Nuri, though, split his paper into 5 and then folded it lengthwise to make 10 parts. I asked Nuri to come to the front and draw his answer on the board, asking the class to comment on what they thought about this. The consensus was that the shape was split into 10 equal parts and so, each part was $\frac{1}{10}$. Nuri's drawing was in effect the simplified decimat in Figure 6-20. This simplified decimat was my starting point for introducing the decimat to the class.



Figure 6-20 simplified decimat

I asked students to draw this shape on their whiteboards and then show me $\frac{1}{10}$, $\frac{2}{10}$, etc. Asking students to show whiteboards at the same time allowed me to check that all of them were participating, and all were managing to show me a correct answer. I went to a number line on the board. Students filled in $\frac{1}{10}$ $\frac{2}{10}$ etc underneath the line and 0.1, 0.2 etc on the top of the line. It was noted, for example, that $\frac{1}{10} = 0.1$.

Next, I asked the students to show me 0.1, 0.2 etc. on their whiteboards. Again, I asked enough questions until I was comfortable that the students were all able to show me any tenth on the decimat.

The only exception was Tayla. She was the student who had been unable to engage with the human number line. She seemed unaware of what to do with the simplified decimat representation. It may have been due to the stresses of arriving at secondary school, but Tayla was exhibiting significant learning needs within the school. She had become electively mute, had regressed to a baby voice when she did talk and was unable to recite the numbers 1 to 10. She came from primary school with a report that assessed her mathematical age as 5 years old, even so, having only taught in mainstream secondary schools, I had never taught a child

who was unable to recite the numbers from one to ten. It was at this point that, in conjunction with the class teacher, I decided that it would be better for Tayla to have the rest of the intervention lessons being given one-to-one support with the class teacher and to leave the intervention.

Once I was satisfied that the students were able to represent a decimal with one decimal place on the decimat, I took one of the rectangles in the decimat and split it into 10. I asked how large each one of the smaller rectangles were. One student suggested $\frac{1}{100}$. I asked whether other students agreed, most said they did. I asked how many times a small rectangle fitted into each tenth and therefore into the large rectangle.

I asked students to show me $\frac{1}{100}$, $\frac{2}{100}$ etc on their whiteboards. Having equated $\frac{1}{100}$ with 0.01, I asked students to show me 0.01, 0.02.

I then asked about 0.21. I missed the opportunity to discuss $\frac{21}{100}$ and went straight to 0.21 being 0.2 and 0.01. I had a version of the decimat on the board and asked the students to fill in 0.21 then 0.23 on their whiteboard.

I displayed the full decimat and discussed that the smallest rectangles were $\frac{1}{1000}$ or 0.001. I asked a student to come to the front and represent 0.125. I gave each student a blank sheet of decimats (See Figure 6-30). On their sheets, I asked them to represent some decimals with three decimal places. One important aspect when considering the magnitude was ensuring that students were able to explain how they knew one decimal was larger than another. I asked them to use the decimat to explain why 0.3 was larger than 0.25 and all the students drew both on their sheets and were able to tell me that the representation of 0.3 had more of the decimat shaded than 0.25 did.

I chose to use the decimat rather than the 100-square. In the 100-square you can represent 0.25 as 2 columns and 5 squares similar in thinking to the decimat, The advantage of the 100-square is that this can be seen easily as $\frac{25}{100}$ and, depending on how you colour in the 25 squares, $\frac{1}{4}$. The advantage of the decimat though is that it is easy to see the difference between $\frac{1}{10}$, $\frac{1}{100}$ and $\frac{1}{1000}$. Because the decimat uses three decimal places it restricts the possibility of money thinking where decimal amounts are just integer amounts out of 100.

I found this lesson offered a potentially powerful way to visualise the decimals. 0.11 can be shown easily to be smaller than 0.9. Students quickly were able to represent any decimal to three decimal places. Although the area and relative size of each block was discussed at length there was still the possibility that students have learned how to colour in the decimat by rote, so that colouring in 0.235 would be two large rectangles, three medium ones and five small ones. If students did learn how to complete the decimat by rote then I would imagine that if asked in the future they could mix up the roles of each size of rectangle and might well shade in, for example, 5 large rectangles, three medium ones and two small ones.

6.4.9 Extending the Number Line to more than one decimal place.

The aim of this task is for students to be able to position decimal numbers of more than one decimal place on the number line. To do this they will need to appreciate that, for example, $0.7 < 0.72 < 0.8$. This will build on from the decimat task where students were able to visually represent the fact that 0.01 was 10 times smaller than 0.1. The purpose is to build a connection between the decimat, place value in a written number and the position of the number on the number line, when the decimal has two or more decimal places.

When I first introduced the number line, I referred back to the human number line and students split the number line in tenths, and then added in the decimals. I would ask students to come to the board and add a number to the number line on the board. They could check with classmates, or they could correct incorrect numbers.

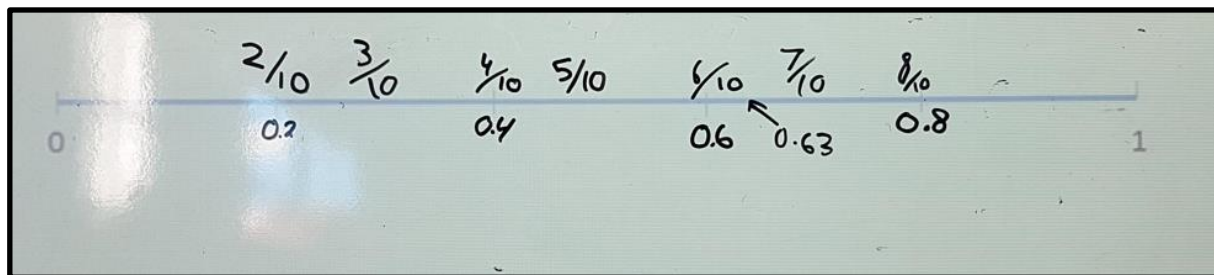


Figure 6-21 number line

As part of this exercise students placed 2 and 3 digit decimals onto a 0 to 1 number line. At first the number line was subdivided by pips to help the students work out where 0.1, 0.2 etc were. This was revisited each lesson, once introduced. Gradually I reduced the scaffolding in the task by removing the pips so that the task was more about estimating a distance on the continuous number line. Siegler *et al.* (2013, p. 15) found that students at a younger age were more competent judging amounts on a continuous number line.

I made sure that once I had introduced the number line fully, every justification included reference to the number line. I asked students to come up and fill in any number on the number line. One came up with 0.3, another 0.4 and then Fergus correctly placed 0.216 (see Figure 6-22).

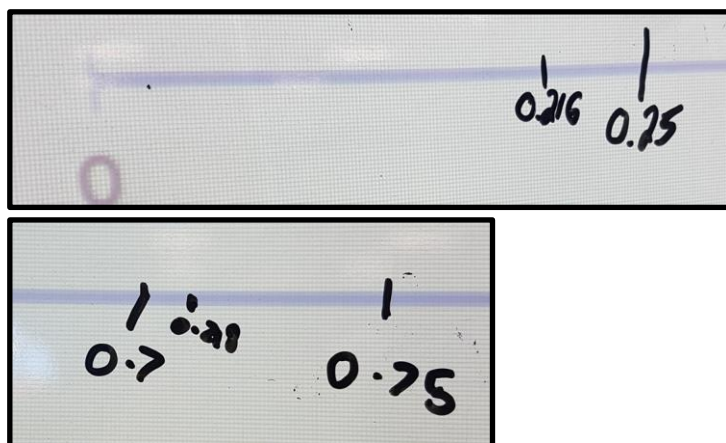


Figure 6-22 students placing numbers on a number line.

Students, with help from each other, were able to place decimals on the number line. One student, James, came to the board. From the position of his hand with the pen it seemed likely that he was going to place 0.25 to the right of 0.9, but as I watched he appeared to check himself, there was nowhere past 0.9. He brought the pen down and appeared to spend some time thinking before coming back towards 0 and placing 0.25 in its correct position. However, it is perhaps not surprising that students feel as if the decimal part of a number is a separate integer entity. They do face this as a possibility in everyday life. For example, the chapters in this document are numbered in such a way that 7.12 follows 7.9, 7.10 and 7.11.

Now that students were putting decimals on the number line and were representing numbers on the decimal, the next thing I wanted them to do was to order decimals.

6.4.10 Ordering decimals

Students were asked to order 5.248, 5.4 and 5.63 and to say whether $\frac{1}{2}$ or $\frac{1}{6}$ was bigger. They were also asked to give reasons for their choices. This was done as a starter activity in their exercise books with no guidance.

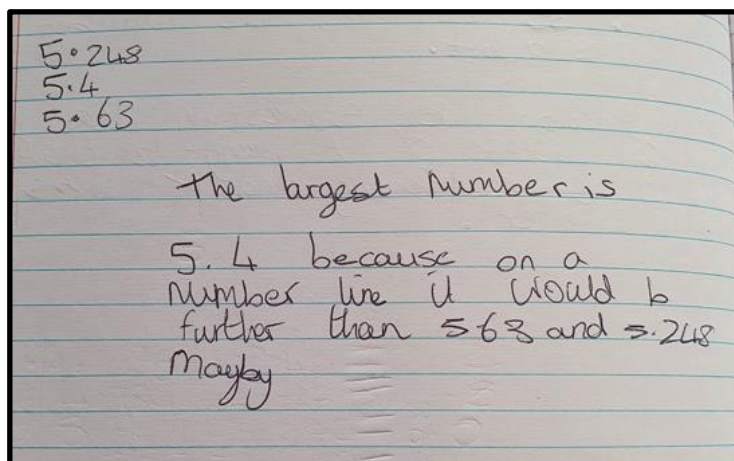


Figure 6-23 student work

The student in Figure 6-23 appears to have tried to visualise the decimal on the number line, referring to one number being 'further along', but is not certain of what they are doing. The inclusion of the word 'mayby' (sic) indicates that the student also was not sure that they had done what they meant to.

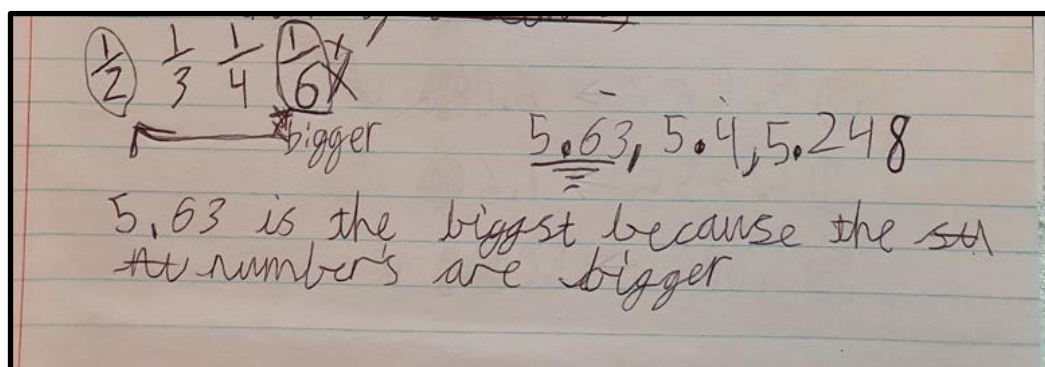


Figure 6-24 student work

This student (Figure 6-24) appears to have initially used the whole number thinking to say that $\frac{1}{2}$ is smaller than $\frac{1}{6}$, but then has changed their mind. When asked to order the three decimals they have picked the right answer, though from the description it is not entirely sure that their reason is correct.

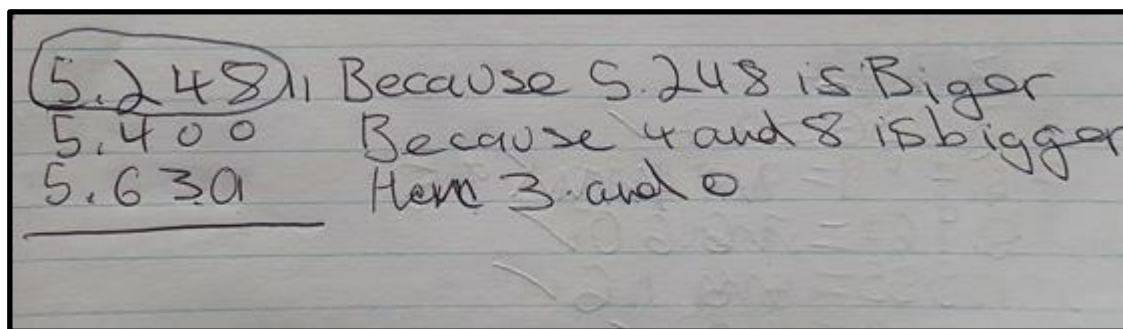


Figure 6-25 student work

This student (Figure 6-25) appears to have an algorithm for checking the magnitude of decimals. She has made all three decimals the same length. This is the annexe zero method from Steinle(2004a) but then when checking for the magnitude has applied the rule to the second and third decimal place – but not the first.

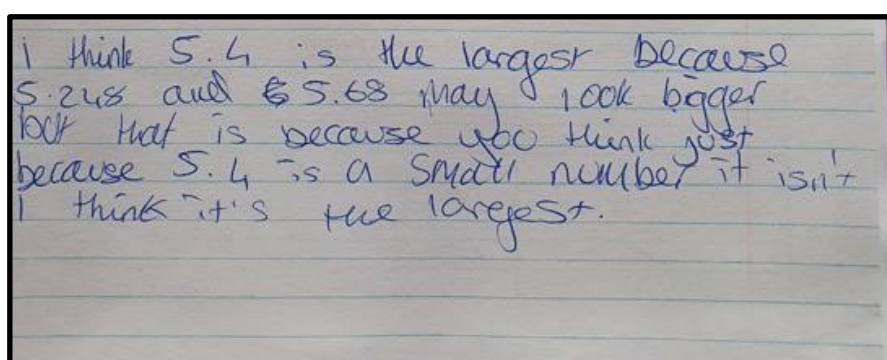


Figure 6-26 student work

When selecting the largest decimal, the description in Figure 6-26 indicates that the student might be showing a shorter-is-larger misconception.

The whole class discussion following this resulted in the diagram in Figure 6-27 with the students placing the decimals on the number line. The consensus of the results is shown in the figure.

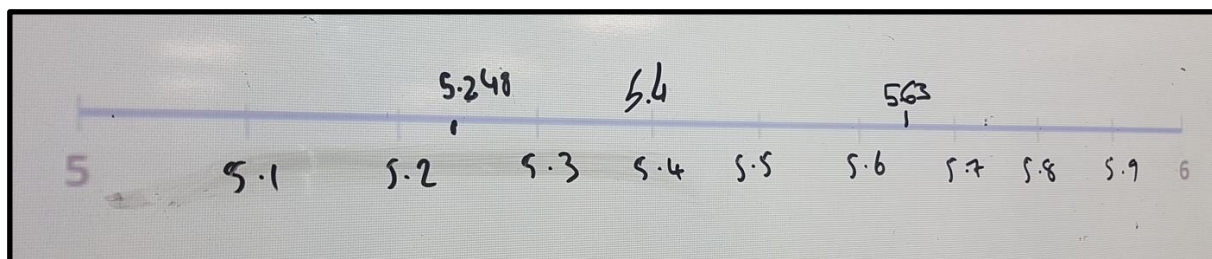


Figure 6-27 results of a class discussion

As part of the discussion following this task, I selected 19 numbers – and each child took a turn to come to the board and place their number. They were able to talk through their discussions and check with the rest of the class. Each child was able to place their number either independently or with the support of their peers. The results are seen in Figure 6-28.

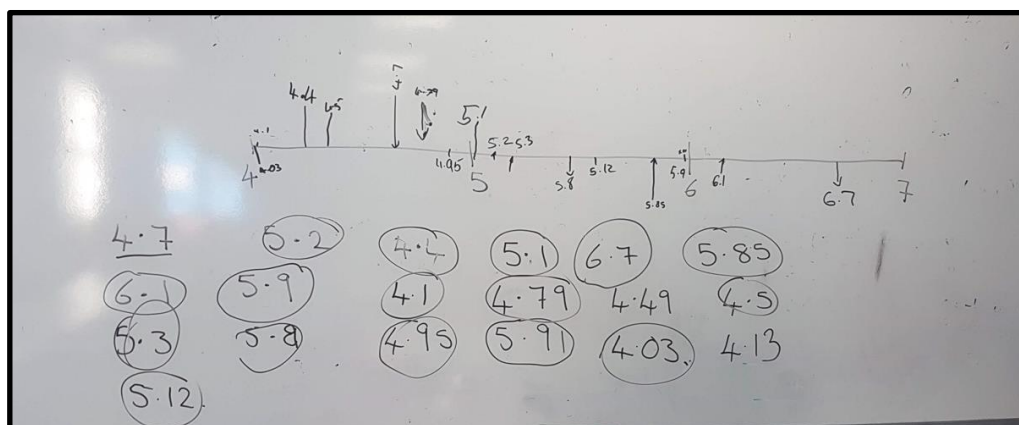


Figure 6-28 placing numbers on a number line.

Following the work on decimals and number lines I used the task ‘Comparing decimals of different length: Part 1’ from the Thinking Through Mathematics: Maths4life project (Department of Education, 2007).

The aim of this task was to require students to use each of the instantiations and move between them. Students would order the set of numbers comparing sizes using the different instantiations (numbers, decimals and number lines). An extended version of this task is included in the January tasks (see section 6.4.20).

In the task, as published, students are given each of the three sheets (see Figure 6-29) which they cut into cards to sort and match them – then order them. The work is done in pairs and much of the learning is intended to arise from the conversations between the students and also between the students and the teachers. There are opportunities for cognitive conflict. Some of these are discussed below.

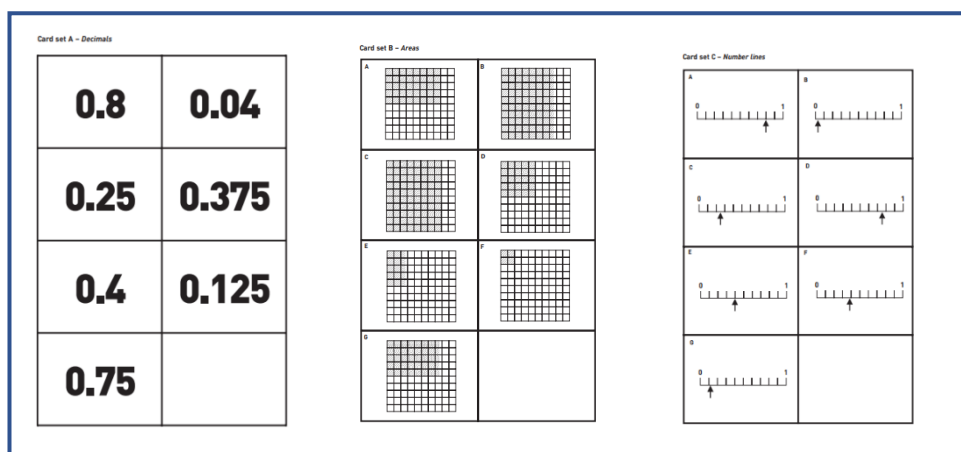


Figure 6-29 comparing decimals of different lengths.

However, during the piloting of tasks I had found that students were using the 100-square in a way that could be reinforcing whole number thinking so I had decided not to use the instantiation of the 100-square representation of decimals. Instead, I just gave the students the decimal cards and the number line cards. I also gave them a blank sheet of decimat, to help them (see Figure 6-30).

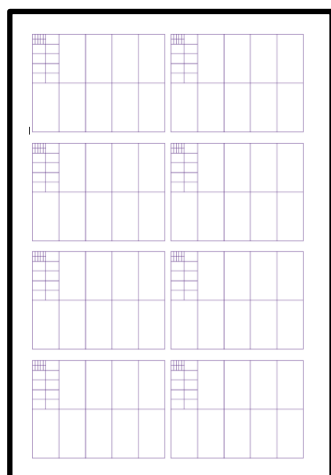


Figure 6-30 blank decimat sheet

I allowed students to work in their own way. However, if any pair were stuck, I suggested that they matched the number line with the number by way of drawing the decimal.

Once the cards were matched, different students used different strategies for ordering. As I was circulating, I wanted to know how the students were intending to order. I asked one pair:

Me: How do you order them just using the number lines?

Student: By looking for the lines that are further over here (pointing to the right of the number line).

The student was able explain that the numbers further towards 1 were greater.

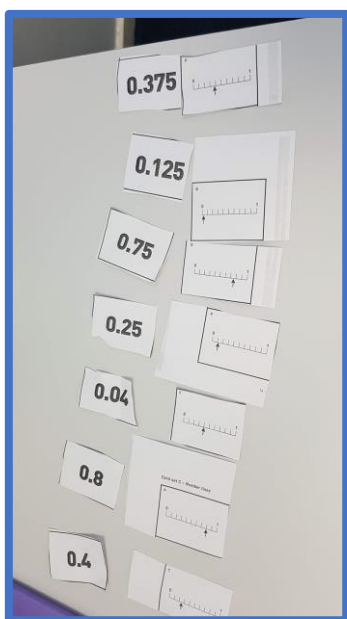


Figure 6-31 student work

In Figure 6-31 the 0.4 card has been matched with 0.25 on the number line: i.e., $\frac{1}{4}$.

This was a common misconception seen in GCSE examiners' feedback (see section 1.3).

In a first viewing of the work, most pairs matched some of numbers with the number line. For example, 0.8 was usually correct. One very common error was

mixing 0.4 and 0.04 (see Figure 6-32), potentially a reflection of the issues surrounding the purpose of 0 in a decimal (see (Steinle & Stacey, 2001)). I asked students about these matches and at least one student thought that 0.4 and 0.04 were the same. For these students, I asked them what they would have done with the decimal. In answering that question, they were able to swap them by themselves, sometimes without ever verbalising their response to my question. It appeared to be a memory jog. If they had not used the number line to order, I asked them to check via the number line. Ordering by looking at the position on the number line appeared to be easier than considering the digits.

One pair (including Rachel) had a working algorithm to order decimals. They explained to me how they made all the decimals the same length, in terms of the number of digits, and then ordered them. Although this is a working algorithm, annexing zero has the disadvantage that it can encourage whole number thinking (Steinle & Stacey, 2004). Also, if the algorithm is not accompanied by an understanding of place value, then it can be incorrectly applied.

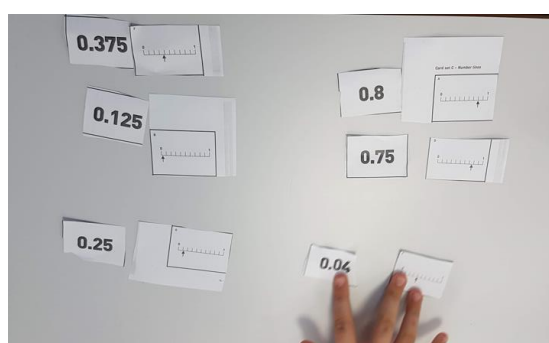


Figure 6-32 student work

As I was circulating I saw a pair who had paired 0.04 with a diagram representing 0.4. This can be seen in Figure 6-32. I asked the pair to reconsider and one realised that 0.04 was incorrect.



Figure 6-33 student work

Figure 6-33 shows one pair of students using the decimats to help them.

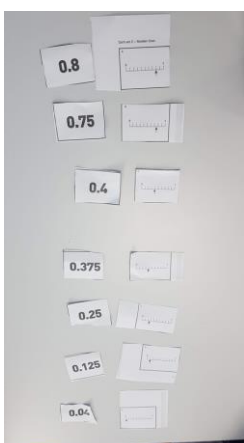
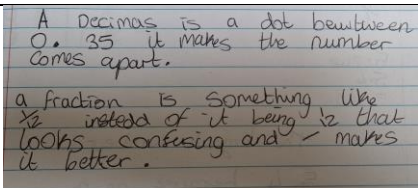
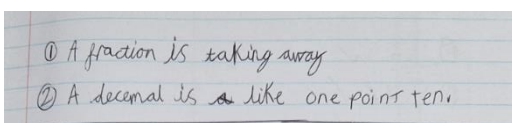
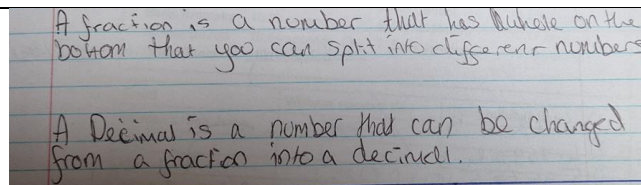
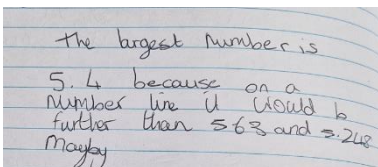
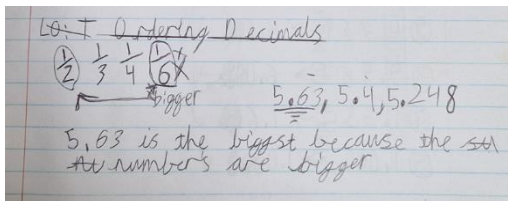
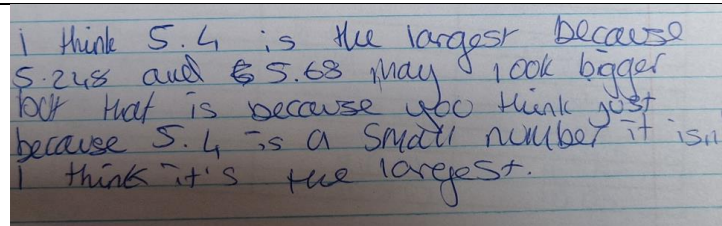


Figure 6-34 student work

By the end of the task all the pairs of students had put the number lines in order. The task had been completed by the students using different techniques. One pair already had an algorithm to utilise, and they did this without an error, others had ordered by considering the number line, some by considering the decimal, some by the numbers. Most, though, used a combination of strategies.

6.4.11 Illustrative Cases

Below at Illustrative Case 6-3,

	Tasks		
What is a decimal / fraction			
	Tasha is does not clearly explain what a fraction or decimal are.	India does not explain what a fraction is but gives an example of a decimal	Rachel appears to have some understanding of what a fraction and a decimal is but does not express it clearly
Ordering 5.4, 5.63 and 5.248			
	Tasha displays a shorter-is-larger misconception here	India correctly identifies the largest number.	Rachel appears to select 5.4 as the largest number because it feels it should be the smallest. It appears that her 'gut' feeling is that longer-is-

			larger but she is choosing the opposite to her 'gut feeling.

Illustrative Case 6-3

6.4.12 January tasks

When I initially planned the intervention, I had intended to focus on understanding decimals in September and understanding fractions in January. This was in line with what the students would have been learning with their teacher and I had agreed with the school that I would not stray too far from the scheme of work. Both sets of tasks rely on the relationship between fractions and decimals and so, although the January tasks lead to comparing fractions and the September tasks towards comparing decimals, there is a large overlap between the approaches and the outcomes. The January tasks can really be viewed as an extension of the September tasks.

In January the tasks I used were:

Lesson	Task	Description	Purpose	Instantiations compared
Lesson 1	Fraction Comparison Test	Students select the larger of 24 pairs of fractions	To assess whether students are likely to hold one of a number of common misconceptions	Notation
Lesson 1	Number line	Placing fractions on a number line	Moving between the instantiation of the fraction and the number line.	Notation Number line
Lesson 1	Folding paper	Students fold paper sheets into 2,4 and 8 to visualise the fractions. They then play a game to be the first to cover their board	To visualise fractions as part of a whole and to recognise the equivalence of various combinations of fractions	Notation Shapes
Lesson 2/3	Fractions task	Card sort where students match up different instantiations of fractions	Students are working with fractions in the form of a continuous number line, a discrete number line, a shaded shape and in unsimplified forms	Shapes Notation Number line
Lesson 4	Distractors	Students are asked to shade in $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}$ of a given set of shapes. Some of the shapes already have distractors which can confuse the students	To visualise and represent fractions	Shapes Notation
Lesson 5	Ordering Decimals and Fractions	The N1 task from the standards unit	Students match decimals, fractions and some visual representations – then order them	Shapes, notations, number lines (in 10ths), number lines in fractions

Table 6-13 January tasks

6.4.13 Number Line

I started each lesson with a number line. This was familiar from the September intervention, as was asking students to come to the front to add in a number. As in the September lessons, I started with the number line subdivided by pips. The subdivision would depend on what I was expecting so when talking about tenths the pips would guide students to $\frac{1}{10}$, $\frac{2}{10}$ etc. As in September, I removed the scaffolding of the pips and students were faced with a continuous number line from 0 to 1. The initial attempts at using the line are in Figure 6-35

Each student was given the opportunity to add to the line. The first student picked $\frac{1}{10}$. Others then built up from there.

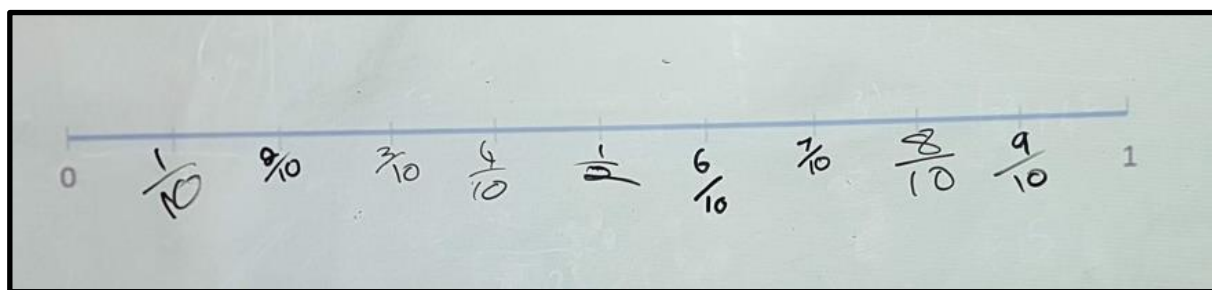


Figure 6-35 fraction number line

One student, Anton, came to the board when $\frac{1}{10}$, $\frac{2}{10}$ and $\frac{3}{10}$ were on the board.

Anton had moved into the intervention group and so had not been in the group during the September intervention. He came to the board and correctly placed $\frac{1}{2}$.

Another student then wrote up $\frac{4}{10}$. The next student, Lucy, came to the board. She seemed unable to put anything on the board and even with encouragement was not writing anything. She appeared to have some level of cognitive conflict. She was happy that the first few numbers went up in tenths but now there was a half there she was not so confident. She counted up in tenths and managed to place $\frac{6}{10}$ correctly.

Once all the numbers were placed, I discussed with the class whether Anton's answer was correct. One student said that it was wrong and that it should be $\frac{5}{10}$.

Another felt that $\frac{1}{2}$ was correct, but agreed with the logic of $\frac{5}{10}$, he appeared quite conflicted. The conversation, guided by me, led to a discussion open to all the students that compared the number line to the one that had been used in September with decimals and that $\frac{1}{2}$, $\frac{5}{10}$ and 0.5 were equivalent.

I asked the class whether there were any other numbers that could fit at the $\frac{5}{10}$. At first there were not any answers, but then someone said $\frac{50}{100}$ and then another said 50%. The final line is at Figure 6-36.

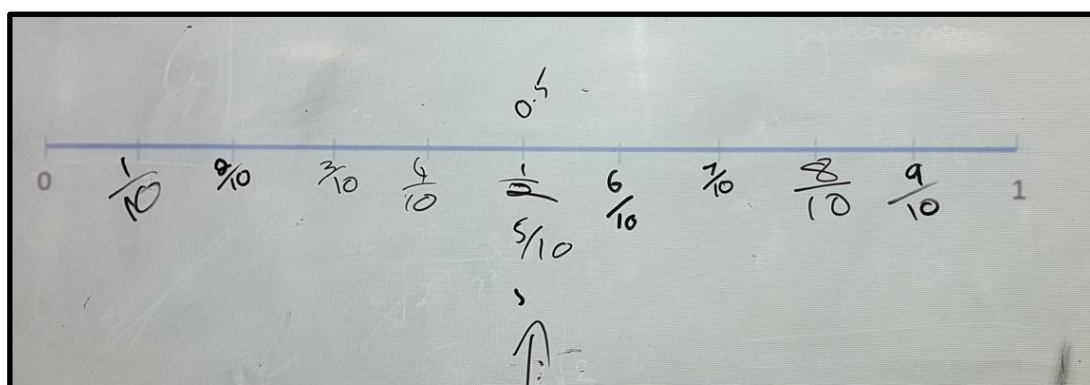


Figure 6-36 number line

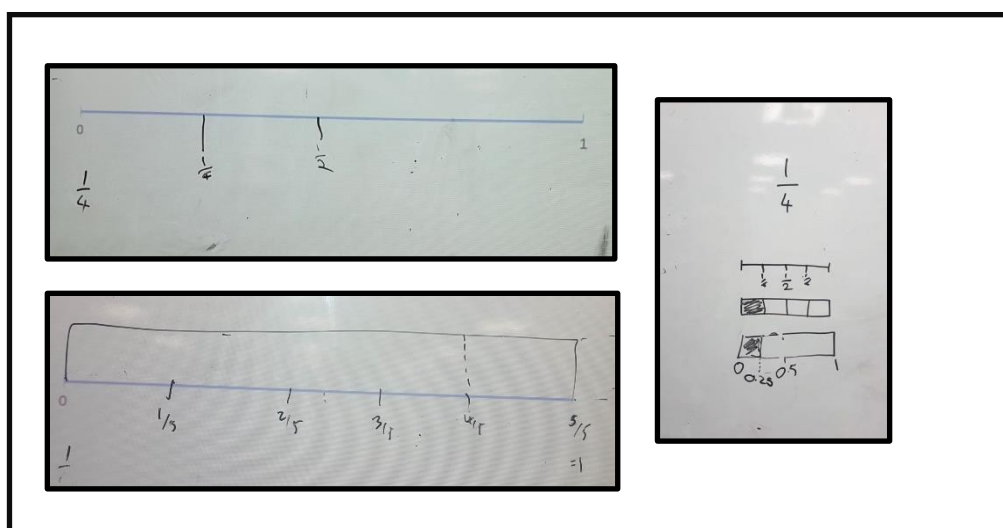


Figure 6-37 number line and shape

Trying to place $\frac{1}{4}$ on the line, this student has used $\frac{1}{2}$ as a reference point. Another student asked to place $\frac{1}{5}$ on the number line seemed unable to start. The bottom left-hand picture on Figure 6-37 illustrates how moving from one instantiation to another aided the student. The student was initially asked to place $\frac{1}{5}$ on the number line. She seemed reluctant to start, despite encouragement. The instantiation that she was trying to engage with was the continuous number line. By creating a rectangle around the number line, she was now engaging with splitting a shape into fifths. She was able to do this and marked off each of the fifths on the line. The right-hand picture shows a student's work moving from a fraction to a decimal utilising the continuous number line and splitting shapes.

6.4.14 Fraction Comparison Task

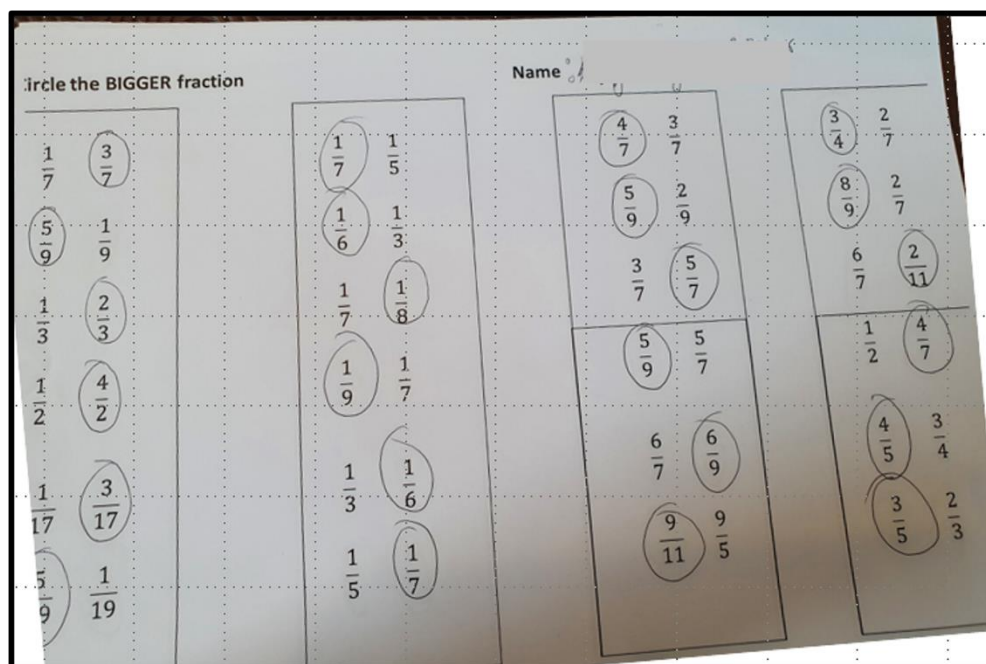


Figure 6-38 Fraction Comparison Test completed by a student

This test was based on the idea of the DCT by Steinle and Stacey (2004). I created 24 questions which, section by section, addressed the likely misconceptions. The first six questions (section A) compare a unit fraction with a non-unit fraction with the same denominator. The second six questions (section B)

compare unit fractions with different denominators. The next three (section C) compare two non-unit fractions with the same denominator, the next three (section D) compare non unit fractions with the same numerator. The final 6 questions (section E) compare dissimilar fractions.

Students were given this task to do with no help or support and with no feedback straight after. My aim was to assess a level of competency and identify possible misconceptions. If students randomly assigned answers, then approximately a 50% success rate would be expected. Given my understanding of the literature, and what I knew of the class, my expectations were that section A would be accurately completed and that section B would show up those with a whole number misconception. Section C would be accurately completed, Section D would show similar results to section B and that section E would show a random distribution of correct answers. The results, by student, are in Table 6-14.

Student	Section A	Section B	Section C	Section D	Section E	Total
Maximum	6	6	3	3	6	24
Lucy	6	0	3	0	4	13
Joseph	5	6	3	3	3	20
James	6	0	3	0	3	12
Liz	6	0	3	0	4	13
Amber	2	2	0	1	2	7
Ruth	0	0	3	0	3	6
Chantelle	0	0	3	0	6	9
India	6	0	3	0	4	13
Claire	5	0	3	0	3	11
Louise	6	0	3	0	4	13
Rachel	0	6	0	3	3	12
Anton	0	6	0	3	3	12
Tasha	6	0	3	0	3	12
Average Mark	3.7	1.5	2.3	0.8	3.5	11.8
% Correct	62%	26%	77%	26%	58%	49%

Table 6-14 Fraction Comparison Test results

Joseph was the only student who appeared to be able to order all the fractions in sections A to D.

Lucy, James, Liz, India, Claire, Louise and Tasha, followed the pattern of my expectations, performing well when comparing fractions with the same denominator but poorly when the numerators were the same and the denominators different. This appears to show a whole number misconception where $\frac{1}{3} < \frac{1}{4}$ because $3 < 4$. I found Rachel's results and Anton's interesting. It seems possible that they have selected the smaller rather than the larger of each fraction pair. This appears more likely than a student being aware that $\frac{1}{3} > \frac{1}{4}$ but unaware that $\frac{2}{3} > \frac{1}{3}$. Unfortunately, I did not have the opportunity to talk to the students about their answers.

The results indicated to me that I was going to need to ensure that the tasks and discussions emphasized the meanings of the component parts of a fraction.

6.4.15 Folding Paper

The purpose of this task is to have a visual representation of a fraction and to have a visual representation to support equivalent fractions. It allows two instantiations (the notation and the area of a shape model) to interact.

This task requires students to fold paper into $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. Each student started with four pieces of coloured paper. On one sheet they wrote the number 1. On the next



Figure 6-39 folding paper game.

sheet I asked the students to fold it into two. When describing what to do I was careful to ask them to 'fold the piece of paper into 2, sharing one piece of paper equally'. I asked what fraction each piece was and one student said 'a half', when asked how they knew they said 'because there are two equal parts'. The students folded, then cut this piece of paper into 2 and wrote a half on each piece. The students were asked to place the two halves, so that it covered the whole piece of paper. This was used to illustrate that $\frac{1}{2} + \frac{1}{2} = 1$. On the next sheet the students folded the sheet into 4, cut it out and wrote $\frac{1}{4}$ on each part. On the final sheet the students folded the sheet into 8, cut it out and wrote $\frac{1}{8}$ on each piece. At each stage of folding and cutting there was a discussion to explain why, for example, folding into 4 equal pieces is $\frac{1}{4}$. Once the pieces were cut out, students were asked to show that (for example) 2 quarters made one half. Then they were asked to find ways of making $\frac{1}{2}$ or making 1.

Because the students had the shapes with the fraction written on them, they were supported by being able to interact with two instantiations. In Figure 6-40 the student can cover the area of the $\frac{1}{4}$ with two $\frac{1}{8}$ shapes, to see that $\frac{2}{8} = \frac{1}{4}$.

Once the students had demonstrated that they could find equivalences using the folded paper they played a game in pairs. The purpose of the game was to practise finding equivalents. For the game, each child has their selection of pieces. When they roll a 1,2 or 3, they can place an $\frac{1}{8}$ piece onto their whole piece. When they roll a 4 or 5, they can place a $\frac{1}{4}$ piece and when they roll a 6, they can place a $\frac{1}{2}$ piece. They can only place a piece if there is enough room on the whole piece. The first in each pair to cover their piece of paper wins.

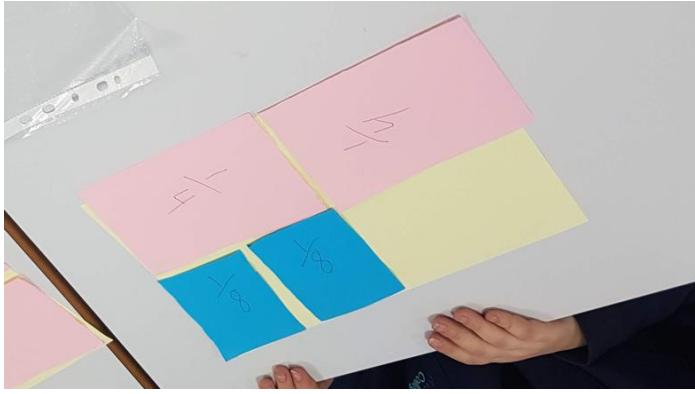


Figure 6-40 working with folded fractions.

Sometimes when playing the game, the students found out that they needed to rearrange the pieces they had already placed to be able to place their next piece.

There were supportive student discussions during this activity. Students were all able to find different ways of making, for example, 1 or $\frac{1}{2}$ or $\frac{3}{4}$ with the fraction pieces.

6.4.16 Fractions Task Part one

Now that the students had been working with fractions of different sizes and equivalent fractions, the next task was to match fractions across different instantiations. Students were given the sheet in Figure 6-41.

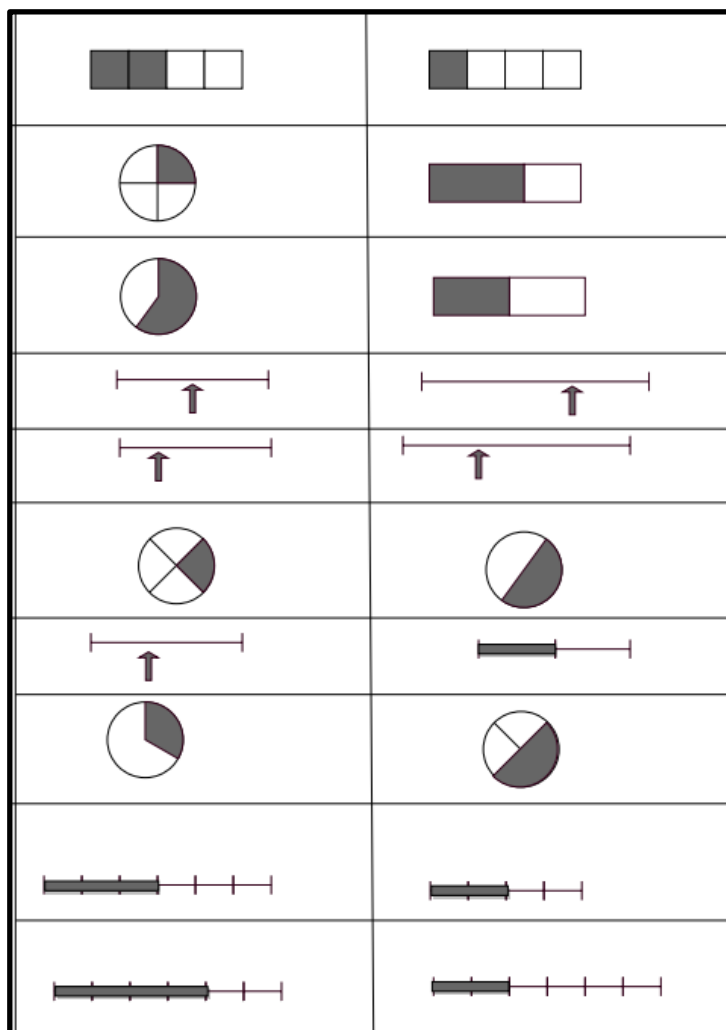


Figure 6-41 fractions task

They chopped them into cards, and, in pairs, they sorted them onto $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{2}{3}$, or $\frac{2}{4}$.

The inclusion of $\frac{2}{4}$ was deliberate to encourage discussion about equivalences.

Each set of cards contained circles, rectangles and number lines. Each shape could either be discrete or continuous.

All the different instantiations are shown in Figure 6-42. For a full list of the examples given to the students see Figure 6-41.

Instantiation	Example
Discrete Circle	
Discrete Rectangle	
Discrete number line	
Continuous	
Continuous rectangle	
Continuous number line	

Figure 6-42 fraction instantiations

Although there was always something to match, it was not the case that each fraction had exactly six matching cards, some of the cards required the students to consider finding equivalent fractions.

A possible error that I was expecting was that student would consider that Figure 6-43 represents $\frac{1}{3}$ because there are three sections, and one is shaded.



Figure 6-43 Possible error number 1

In the discrete example (Figure 6-44) It is perhaps straightforward to see that this is $\frac{1}{4}$, because there are four equal sections, where only one is shaded.



Figure 6-44 discrete quarter

However, it is harder to gauge the fraction here in Figure 6-45 where there are no guidelines to show how the shape has been split.



Figure 6-45 continuous third

As students worked, they grouped together the fractions that were the same size.

All of the fractions were one of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$ or $\frac{2}{4}$. I circulated around the class, talking to the pairs as they worked.

They could pick out the halves, particularly of the shapes: circles and rectangles. The quarters were also selected, when discrete. The same was not true of the continuous examples; these were being left unsorted. It was not that the students did not know how to measure; more that they did not realise that this was something they could or should measure. It was almost as if the narrative of quarter being one out of four meant that they were expecting to see shape split into 4 there were expecting fractions to somehow be discrete. This is the criticism by Watanabe (2002) that fractions are often considered in a part-whole model where both the numerator and the denominator are counted.

I talked to the groups who had left the continuous pictures to the side about how many times it would fit in. They could see that the shape was not $\frac{1}{2}$ but were unable to see how many times it should fit in other than that. I suggested that they used the polythene to cover the fraction in question and turned it round to see how many times the shape fitted into the whole (Figure 6-46).



Figure 6-46 using a guide to estimate a fraction.

When they saw this, the students were able to say that the shape was $\frac{1}{3}$. One of the pictures had three blocks coloured in out of six and this caused problems. I discussed with the students, comparing the situation to a chocolate bar, and they thought it was a half. Most of the students had left the pictures with the arrows (the continuous number line) to the side. The students found this harder than the other pictures. I asked a student how they might approach this problem:

Me : What could you use to help you?

Student: A ruler

The student then went and got a ruler and measured the line and the shaded part. This enabled them to work out the fraction.

Examples of students sorting the fractions is at Figure 6-47.

The students had all stuck down their work but it was the end of the lesson and so this work could not be reviewed with them immediately. I decided to complete this in the next lesson.

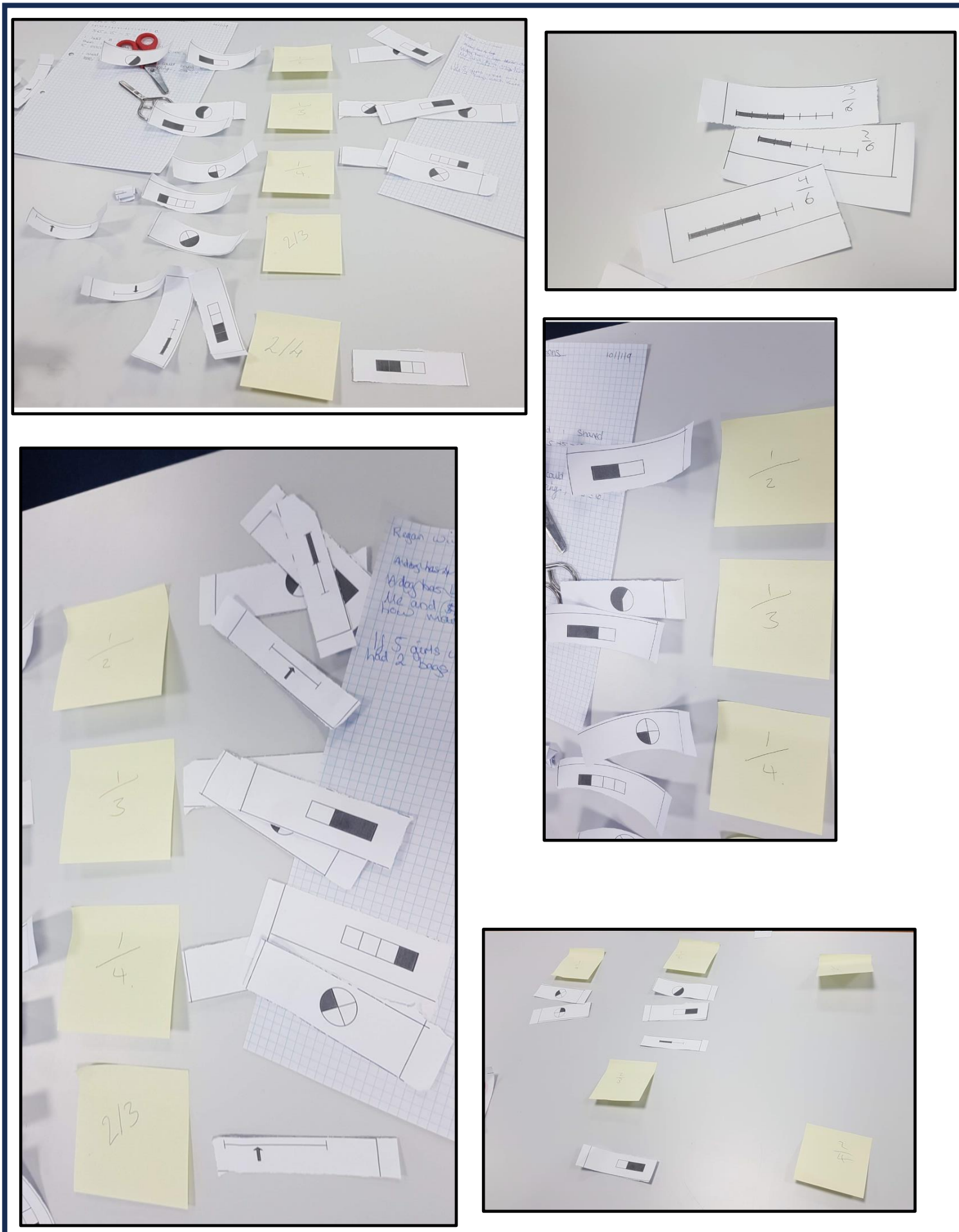


Figure 6-47 Students sorting the fractions task

6.4.17 Draw a Half

This lesson was intended to continue from where I left off with the fractions task.

As a starter the next day, I asked the students to draw an image of a $\frac{1}{2}$ and invited some to share their ideas. This was only intended to be a short interaction, but it grew into a more significant interaction which is worth reviewing.

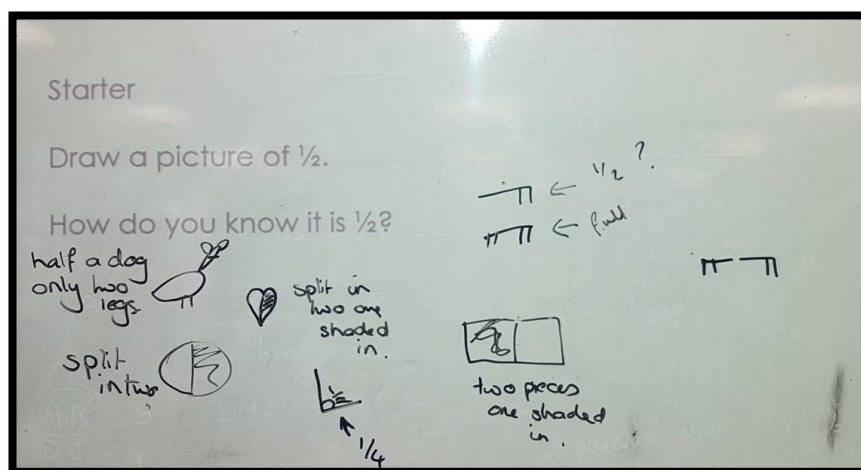


Figure 6-48 picture of a half

The students each came up to the board and shared their ideas. From a slightly gory half a dog, (i.e. a dog with two legs), to a circle split in two, a heart split in two, a rectangle split in two. The majority of the pictures showed a shape split into two equal parts. Anton, however, drew half a table as well as half a dog.

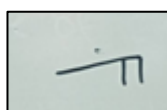


Figure 6-49 half a table

This was intended to be a short starter and I had anticipated that each student would have drawn a circle or rectangle split into two equal parts. Most of the answers were like this but the table and the dog needed to be addressed. Rather than addressing them directly I chose to create a side task. I drew five shapes on the board (A to E in Figure 6-50).

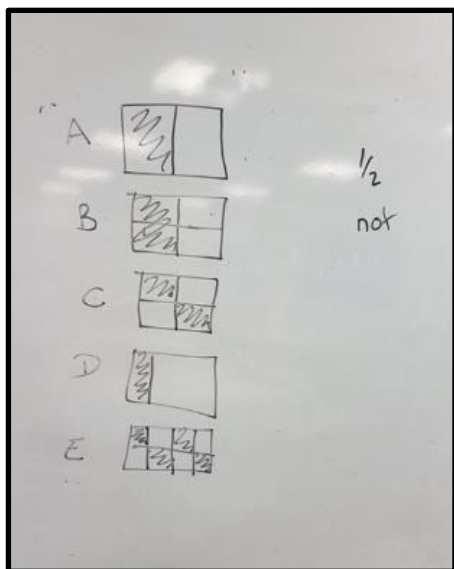


Figure 6-50 $\frac{1}{2}$ or not?

I went through each question in turn and asked the students to stand if it was a half and to remain seated if not a half. For A, all the students stood. For B, half stood, and half sat. I went round each child seated to find out what they thought the fraction was. They all thought it was $\frac{2}{4}$. This led to a discussion between the class, moderated by me, confirming the equivalence. Five students said C was $\frac{1}{2}$, two said it was not, five were not sure. A discussion followed with C. All recognised that D was not $\frac{1}{2}$ because although the shape was divided into two parts; the two parts were not equal. By E the students were beginning to recognise that there were two equal shares and so E did represent a half.

The conclusion was that to be a half, the shape needed to be in two equal parts. Having concluded this conversation, I went back to the halves that the students had drawn.

I drew the class's attention back to the drawings of the dog and the table that Anton had drawn.

Me: Is that half a dog?

Anton: No. It's a full dog with two legs missing.

Looking at Anton's half a table I wanted to know if only legs were missing or if it was just a very long table. I asked him if that was half a table.

Anton: yes

Me: How do you know?

Anton: because it has two legs?

Me: but does that make it half a table?

Anton drew the same length table but this time with four legs.

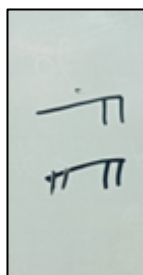


Figure 6-51 half a table?

I opened up the question to the class and Joseph came to the board and drew the table split into two halves, explaining that Anton's half table was not half as the two parts were not equal.

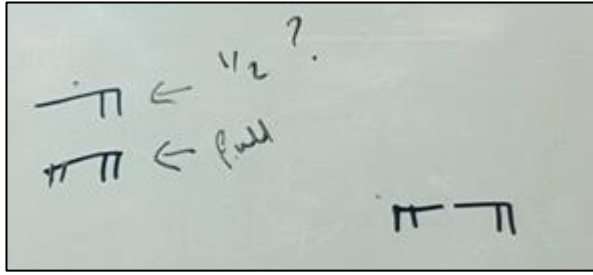


Figure 6-52 Anton and Joseph's work on half a table

6.4.18 Fractions task Part 2

Having completed the discussion on what constitutes a half I wanted to review the fractions task from the day before. Students referenced their work, and I displayed the questions on the board. Students came up to the board, in turn, to write up their answers.

Figure 6-53 shows the answers given by students as they were reviewed by the class. The right hand side shows some of the answers being amended after discussions

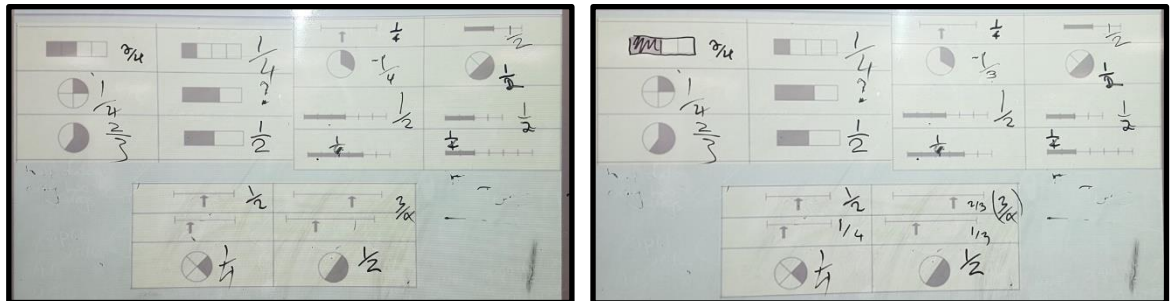


Figure 6-53 initial student answers (left) and revised (right)

6.4.19 Fractions with distractors

For this task students were given two copies of the question below in Figure 6-54.

They were asked to shade in $\frac{1}{2}$ on each shape in one copy then $\frac{1}{3}$ on each shape on the other. For someone who is secure in their understanding this task is straight forward. The extra information inside the shapes though can cause issues; these are the distractors.

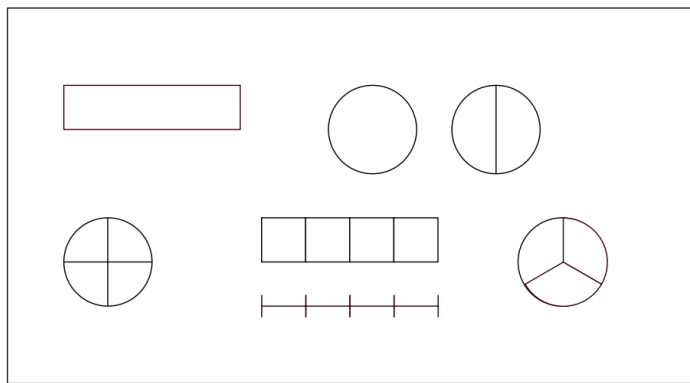


Figure 6-54 fractions with distractors

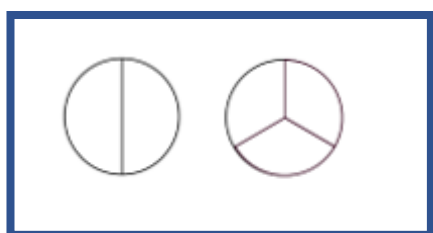


Figure 6-55 circles from the fractions with distractors

Asking students to shade in a half of each of the circles in Figure 6-55 is essentially the same question. This is an example of where a teacher might consider a situation as one instantiation, but a student might consider it as two. Some students shaded both circles in immediately, but many were reluctant to even start shading the circle on the right. They found this hard, they could see that colouring in one third was not right, but neither was colouring in two thirds. This is the use of a “perceptual distractor” (Swan, 1983, p. 16).

Me: What would you do if this were a pizza

Student: Ah yes. I would divide that last piece into two.

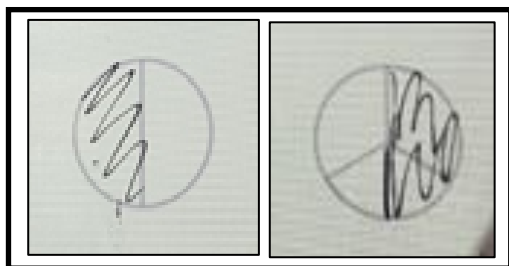


Figure 6-56 colouring in one half

It seems as if this is more of a mathematics question problem than a mathematics problem. In everyday life, if faced with a pizza split in three, they would easily share it between two by taking a slice each and cutting up the third slice. Students seem to be thinking that their answers had to be using only the information in the question. This does not appear to be a mathematical issue, but more about what learners perceive to be an acceptable answer.

I displayed the questions on the board twice; once for a half and once for third. I asked each child to come up and show one of their halves or one of their thirds on the board. All successfully filled in the halves. Two of the thirds were incorrect and this was adjusted by other members of the class helping.

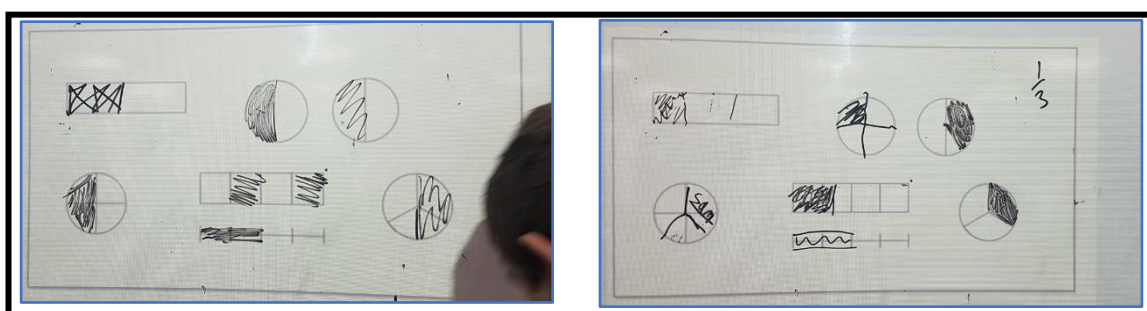


Figure 6-57 fractions with distractors

I asked which is bigger $\frac{1}{7}$ or $\frac{1}{9}$. Almost all thought $\frac{1}{9}$ was bigger. I asked them which would give them more of a chocolate bar one divided into 7 pieces or one divided into 9 pieces. I was trying to encourage the link between fractions and division (sharing) and also trying to consider the purpose of the numerator and denominator in evaluating the magnitude of a fraction.

6.4.20 Ordering Fractions and Decimals task

I wanted to end the intervention with a task that required students to order fractions making use of any, or all of, the different instantiations that had been

covered. I decided to use the N1 task from Improving Learning in Mathematics (Swan, 2005). Here, there are six sets of cards set A, decimals; set B, fractions; set C an area representation of the fraction; set D a scale based on the denominator or the fraction; set F, a decimal scale and set E a 100-square. Sets A, E and F were the same as the cards used in the September task from Thinking Through Mathematics (Maths 4 Life) (Department of Education, 2007). In September I had replaced the 100-square with a blank decimat.

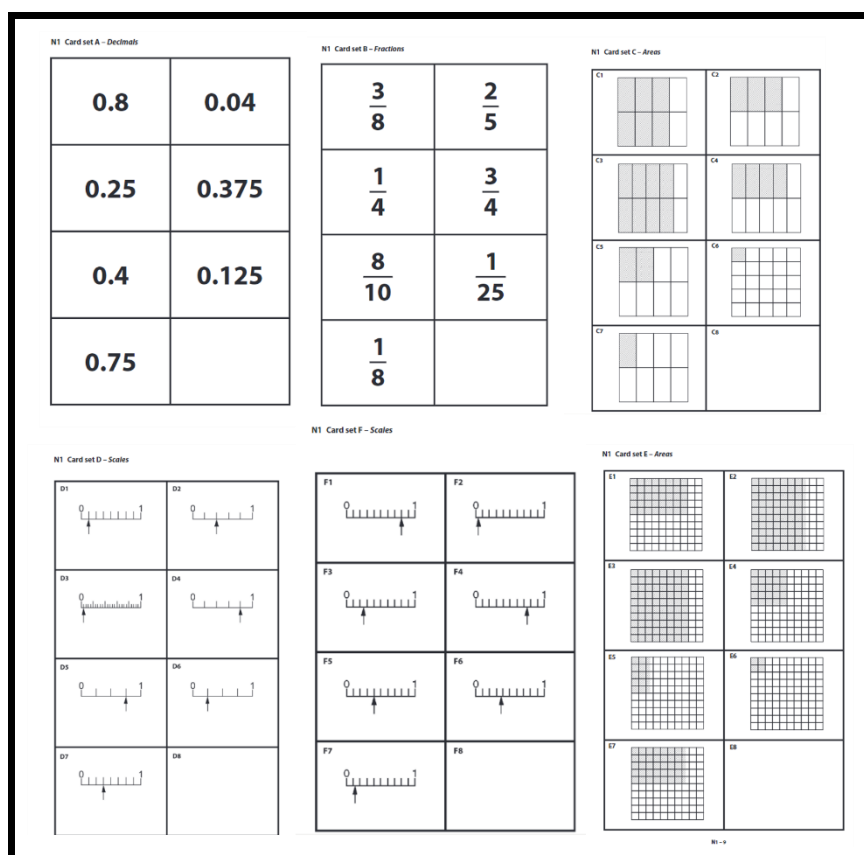


Figure 6-58 N1 Ordering Fractions and Decimals SU task.

Students, in pairs, matched each of the sets of cards. Once they were matched, they were put in order. Completed work can be seen at Figure 6-59.

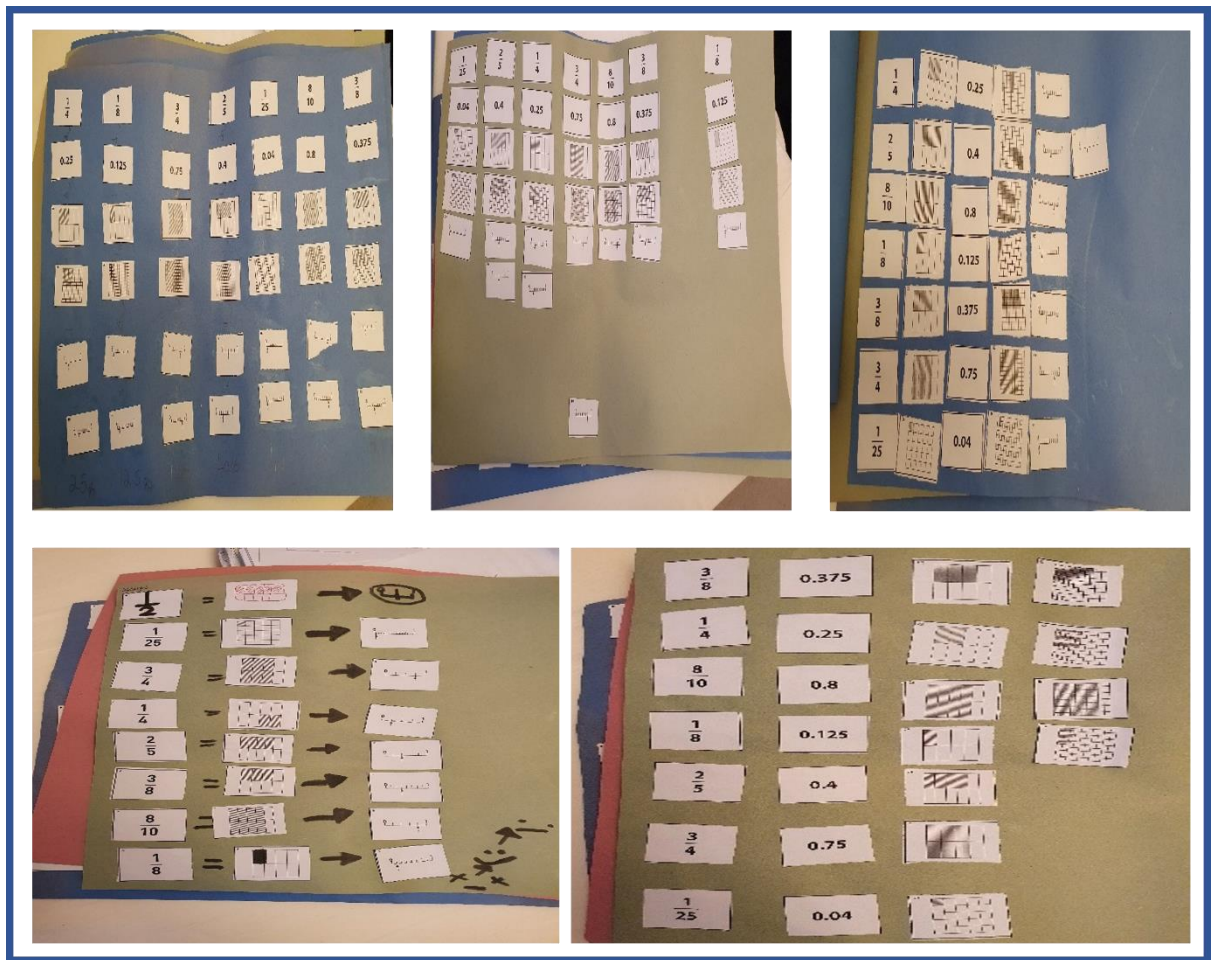


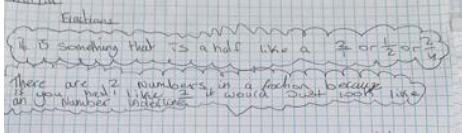
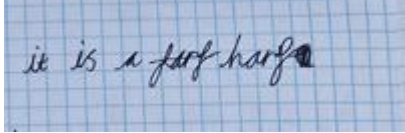
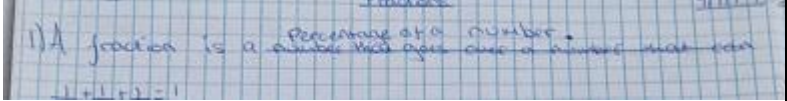
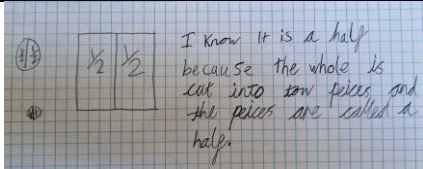
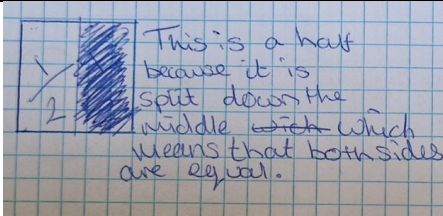
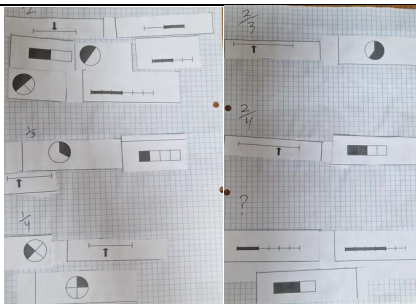
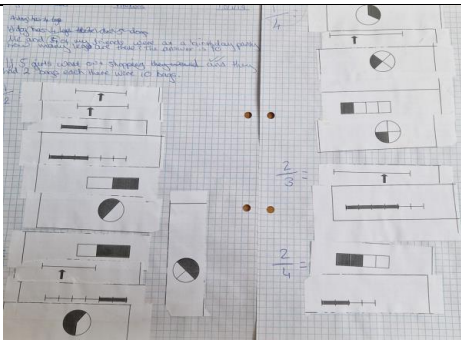
Figure 6-59 ordering fractions and decimals students' work.

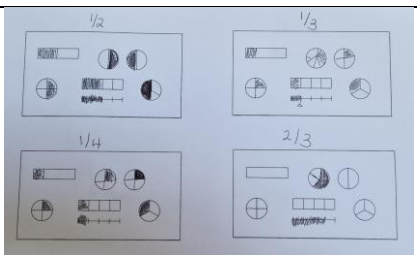
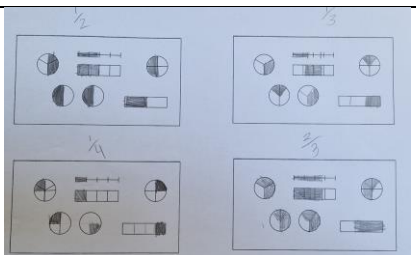
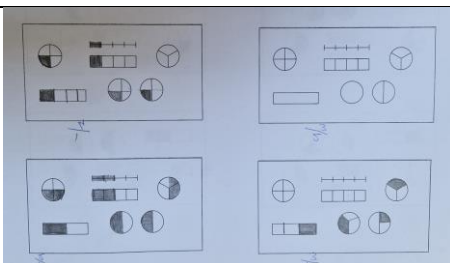
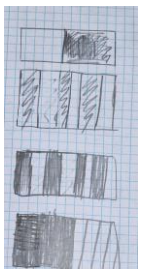
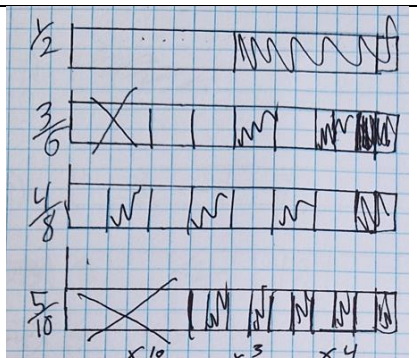
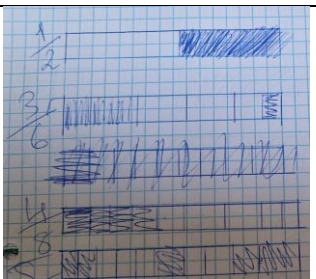
My reflections on the last day were that although some students were more competent than others all had been able to order the fractions using the extra instantiations, for example the number lines and the visual representations. Considering this months later, I think that the students had a better concept of what a fraction was but needed more opportunities to compare and order fractions

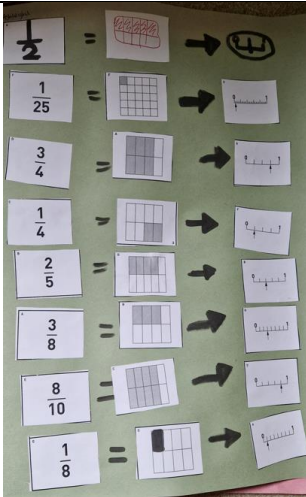
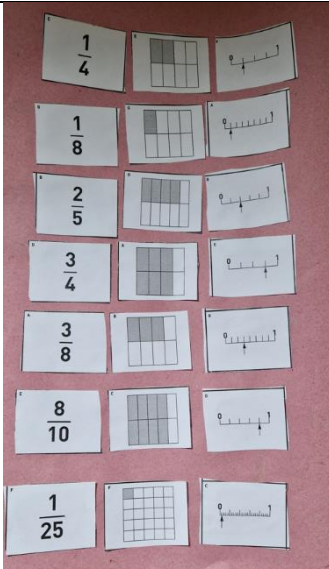
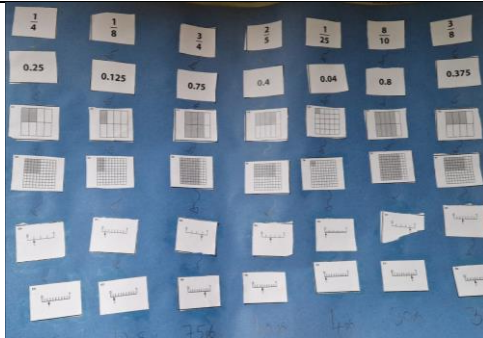
6.4.21 Illustrative Cases

Illustrative Case 6-4 shows that at the start of the intervention it likely that all three students had a whole number bias. Through the lessons the students have all managed to represent fractions, match them to decimals and the number line and thereby order them.

	Tasha	India	Rachel
	First Intervention – January tasks		
FCT section n 6.4.1 5			
FCT result s	6,0,3,0,3 Able to compare fractions with the same denominator but not where the denominators are different. This suggests a whole number bias ($3 < 4$ implies $\frac{1}{3} < \frac{1}{4}$)	6,0,3,0,4 Able to compare fractions with the same denominator but not where the denominators are different. This suggests a whole number bias ($3 < 4$ implies $\frac{1}{3} < \frac{1}{4}$)	0,6,0,3,3 It seems likely that Rachel has found the smaller in each pair. If this is the case her results are the same as Tasha and India. If not, then she is able to see that $\frac{1}{5} > \frac{1}{3}$ but not that $\frac{2}{3} > \frac{1}{3}$

	Tasha	India	Rachel
9 1 What is a fraction			
Draw a half			
		India explains the concept of a half but does not mention equal parts.	Rachel explains the concept of a half and does mention equal parts.
			
		India has identified fractions in their continuous, discrete, shape and number line forms. She has not identified three of them.	Rachel has identified fractions in their continuous, discrete, shape and number line forms. She has not identified three of them.

	Tasha	India	Rachel
Fractions with distractors			
	Tasha has sketched representations of fractions with distractors	India has sketched representations of fractions with distractors	Rachel has sketched representations of fractions with distractors
Equivalent fractions			
	Asked to draw fractions equivalent to a half, Tasha has found some equivalent fractions	Asked to draw fractions equivalent to a half, India has found some equivalent fractions	Asked to draw fractions equivalent to a half, Rachel has found some equivalent fractions

	Tasha	India	Rachel
Matching Fractions Task			
	Tasha has matched fractions to decimals to a number line	India has matched fractions to decimals to a number line	Rachel has matched fractions to decimals to a number line

Illustrative Case 6-4

6.5 Delayed Post test

6.5.1 Results

The Post test was administered in April; some three months after the intervention.

The results of the pre-test and the post test for all students who took either are in Table 6-15

	Number of students	Average KS2 scaled score	Pre-test correct. (Average out of 24)	Post-test correct. (Average out of 24)
Students from the comparison year	52	101.7	14.5	16.9
Set 3 students from the comparison year	14	93.8	7.8	11.7
All students – intervention year	33	95.9	12.3	13.4
Neither group – intervention	17	99.4	14.2	16.1
One group – intervention	4	96	14.8	15.3
Both groups - intervention	12	91.2	8.8	8.9

Table 6-15 - results of the pre and post-test for students in the intervention and comparison who took both pre- and post-test.

The initial plan was to compare the students in the intervention group with the whole year group from the comparison year. In the comparison year, the students improved on average by 2.4 more correct questions in the post-test from the pre-test. In the end all of the cohort of the intervention year did the pre- and the post-test. With students moving in and out of groups students were split into three groups, those who had been in the September and January interventions (both), those who had been in either the September or the January groups (one) or who had not been in either of the intervention groups (Neither). A measure of an effect of the intervention is in Table 6-15. Out of the intervention cohort, 33 students sat both the pre- and the post-test. The corresponding number for the comparison

group was 52. The results in the table exclude questions 7,8 and 9 which dealt with algebra, which was not included in the intervention. Although there was a difference in the comparison year, none of the different groups shows a material difference in the intervention year. In addition, the group sizes are very small. Because the group sizes are small there was not enough data to perform significance testing.

It is worth noting that the KS2 results for the two cohorts are very different. The comparison year had an average KS2 result of 101.7, this is near to national average. This is significant because the ideas I tested in the pre-test and post-test have all been taught at primary school and so a student who is 'secondary ready' would be expected to have performed well on both the KS2 mathematics test and the pre-test. The year of the intervention the students did not do so well at KS2 – the average mark is 95.9, with the intervention group having an average score of 91.9. The details of the individual students, including their KS2 scores is at Table 6-9

It is perhaps more interesting to consider which questions students could or could not answer pre- and post- intervention. The analysis for this is below in Table 6-16 and these questions were followed up in the interviews.

Question			Pre-test% correct	Post-test % correct
Q1e	Add one tenth	4.254	25	35
Q1f	Add one tenth	2.9	38	52
Q1g	Take away 100	583	85	75
Q1h	Take away 100	30 000	53	55
Q2a	Add ten	3 597	67	85
Q2b	Add ten	0.15	47	54
Q2c	Add one hundred	21 534	79	86
Q2d	Add one hundred	19 930	71	81
Q3	Add one to 06399		80	91
Q4	Is there a difference between 4.9 and 4.90		63	81
Q5a	Which is bigger 0.75 or 0.8		68	85

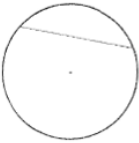
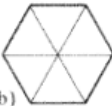
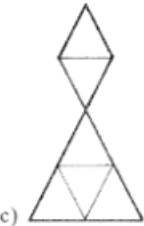
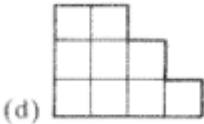
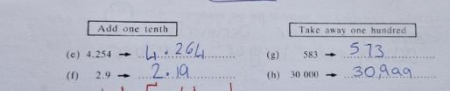
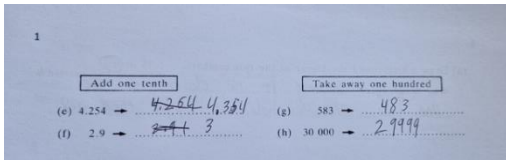
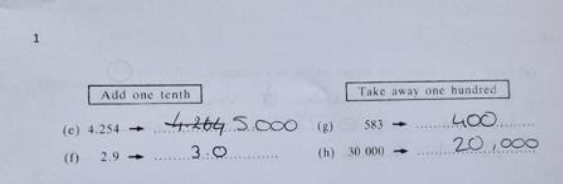
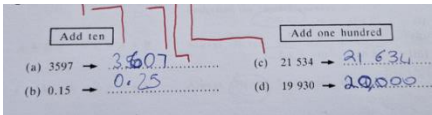
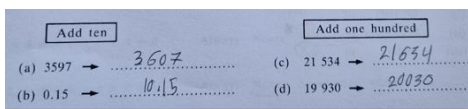
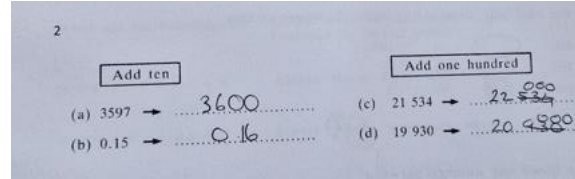
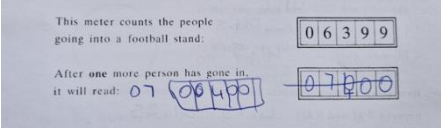
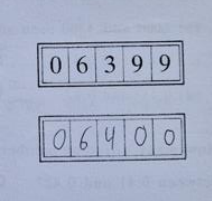
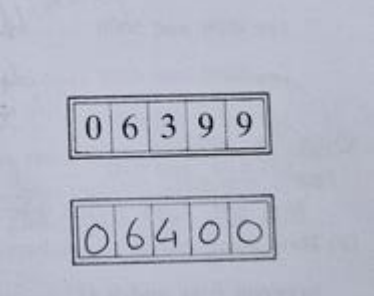
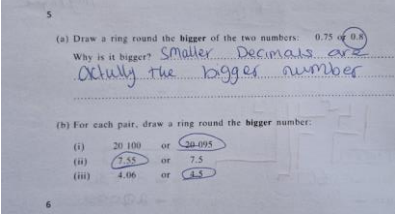
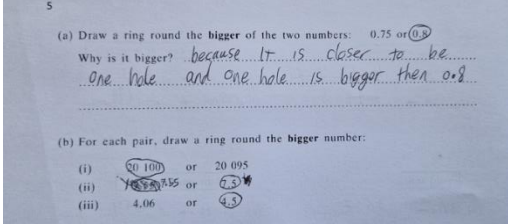
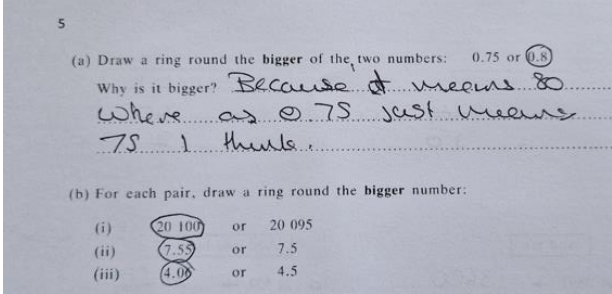
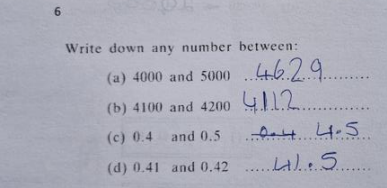
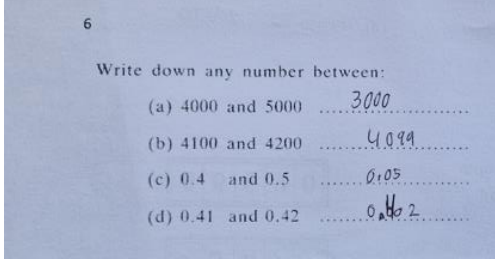
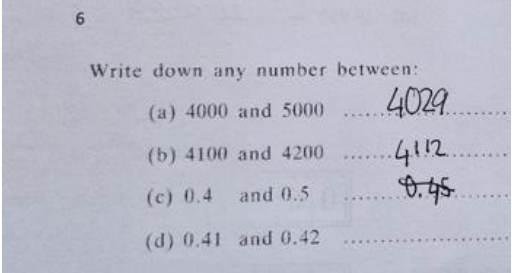
Question		Pre-test% correct	Post-test % correct
Q5b (i)	Draw a ring around the bigger number 20 100 or 20 095	92	96
Q5b (ii)	Draw a ring around the bigger number 7.55 or 7.5	75	67
Q5b (iii)	Draw a ring around the bigger number 4.06 or 4.5	63	89
Q6 a	Write down any number between 4 000 and 5 000	84	86
Q6b	4 100 and 4 200	83	88
Q6c	0.4 and 0.5	71	75
Q6d	0.41 and 0.42	52	67
Q6e	How many different numbers are there between 0.41 and 0.42	16	9
Q10	Is this circle split into 2 halves 	81	89
Q11a	Shade in two thirds of each of the shapes 	95	93
Q11b	Shade in two thirds of each of the shapes 	30	46
Q11c	Shade in two thirds of each of the shapes 	22	37
Q11d	Shade in two thirds of each of the shapes 	24	48

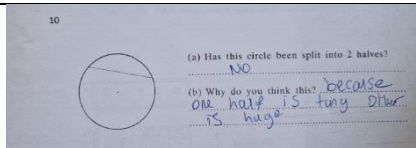
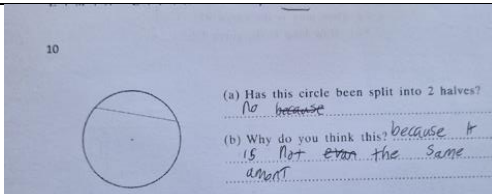
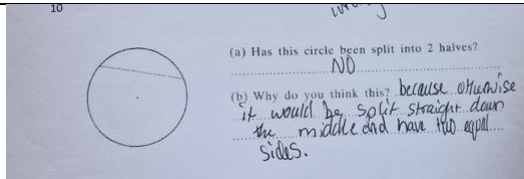
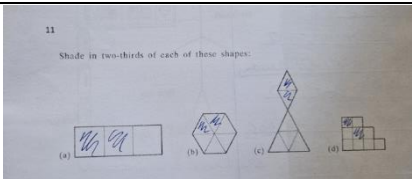
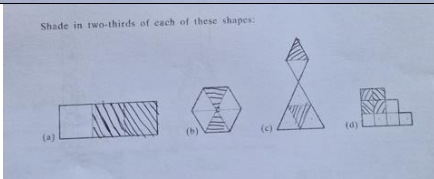
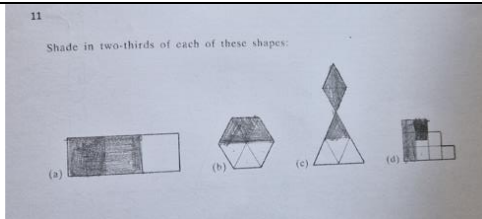
Table 6-16 Pre-test vs Post-test for the intervention year

6.5.2 Illustrative Cases

Illustrative Case 6-5 details the results of the post-test for Tasha, Indai and Rachel and compares the post-test answers with their pre-test answers.

	Tasha	India	Rachel
	Post-test		
Post test	11 (out of 24) (8 pretest)	13 (out of 24) (12 pretest)	11 (out of 24) (13 pretest)
Post test Q1	 This is similar to the pretest, with only g different. In the pre-test Tasha wrote 483 (correct)	 For e, the answer that India has crossed out is the answer she gave in the pretest. Her answer in the post-test is correct. For f India had written 2.91 in the pre-test. For h India had the correct answer in the pre-test	 Rachel did not get e correct in either the pre-test or the post-test. The answer she has crossed out matches the answer from the pretest. f is correct and was incorrect in the pre-test. G was correct in the pre-test. Rachel has answered h the same way for both tests.
	 This is correct in the post-test but the pre-test showed that Tasha was not competent with place value. In a she had written 35107, indicating that the 'tens' column could contain 10 rather than the maximum of 9. She did the same on d with an answer of 19 1030. B and c were answered in a similar manner both pre- and post-test.	 India answered all four questions the same pre- and post- test.	 In the pre-test Rachel used a column addition algorithm to answer these questions. In the post-test it appears that Rachel is not answering the question and is perhaps answering using rounding methods. There is no clear answer. Rachel in lesson often relied on what appeared to be complete algorithms, but does not appear to have used one here.

	Tasha	India	Rachel
	 Tasha answered this correctly in the pre-test.		
	 In the pre-test Tasha displayed a longer is larger – where here Tasha appears to be displaying a smaller is larger misconception in part a, though she has answered part b correctly for decimals (though not part b i)	 India could not answer part a in the pre-test. She has answered part b in the same way as she did for the pre-test	 Here Rachel describes the annexe zero algorithm that she tended to rely on in lessons. She has not applied it correctly to b iii. In the pre-test she explained part a by converting to percentages.
	 Tasha has correctly answered a,b and c. d appears to have been considered as a whole number.	 None of these is correctly answered. This question was not attempted in the pre-test. d could possibly read 0.41.2 meaning that India appears to have tried to find a number between 0.41 and 0.42	 Rachel has answered a, b and c successfully, though then crossing out c. She has not attempted d. On the pre-test she selected a correct answer for all four questions.

	Tasha	India	Rachel
			
	 This is a similar response to the pre-test	 India's answer in the pre-test was very unclear. Here she has coloured in two for each shape (other than d) whatever the denominator is.	 For the pretest Rachel coloured in two for each shape. In the post-test she has coloured in $\frac{2}{3}$ for a and $\frac{1}{2}$ for the other shapes. This perhaps shows that she has an understanding of equivalent fractions but has found one-half of each shape rather than two-thirds.

Illustrative Case 6-5

6.6 Intervention 1 interviews

I selected students for the interview. I only considered students who had been present for both interventions and who had a misconception in their DCT analysis.

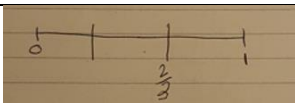
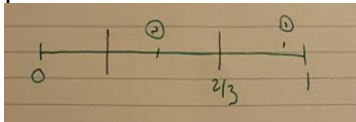
I selected Rachel, as she appeared to have an ordering algorithm in lessons, but still showed a misconception on testing. The other five were randomly selected from the students who met the initial criteria. All six selected agreed to take part and I obtained written permission from all six. By the time I asked the students to come to the interview, Chantelle changed her mind and withdrew her permission.

In the interviews, I wanted to know what the students thought a fraction was. This had been discussed during the lessons and I wanted to know what they now expressed as their understanding. I wanted them to show me representations of a fraction and whether they could place them on a number line. I wanted to see if they could compare fractions and decimals and whether they could use a decimal to help them compare decimals.

The questions I asked in the interviews were.

- What is a fraction?
- Show me $\frac{2}{3}$
- Show me $\frac{2}{3}$ on a number line.
- Show me 0.236 on a decimal.
- Which is bigger 0.236 or 0.24 (also depending on the answer which is bigger 0.225 or 0.21)?
- Show me on the number line.
- Which is bigger $\frac{1}{3}$ or $\frac{1}{4}$?

A summary of the answers to the questions are below in Table 6-17

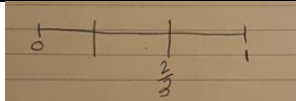
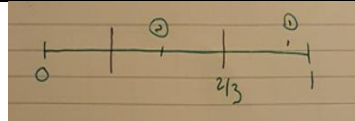
	Student				
Question	James	Rachel	India	Tasha	Ruth
What is a fraction?	it's got a denominator and a numerator and there's a line that separates them it can be like $\frac{1}{2}$ or $\frac{1}{4}$	something where you simplify a number to make it easier. To understand... so if like you had like a box of 10 and it was split into 5 and only one box was coloured in it would be $\frac{1}{5}$	a fraction is like one and a half, or 1 over 4 as a quarter or one over 2 that's a half. One over three that's a third.	a one and a line underneath and a 2	it's a half of something.
Place $\frac{2}{3}$ on a number line	James managed this by considering the number line as a shape and splitting it into 3				First indicated $\frac{2}{3}$ was at the point numbered 1 
Using decimal	Remembered the decimal but needed reminding on how to use it. Used it properly once reminded	Remembered the decimal but needed reminding on how to use it. Used it properly once reminded	Remembered the decimal but needed reminding on how to use it. Used it properly once reminded	Remembered the decimal but needed reminding on how to use it. Used it properly once reminded	Remembered the decimal but needed reminding on how to use it. Used it properly once reminded

Question	James	Rachel	India	Tasha	Ruth
Ordering decimals	James: because there the further it gets away from zero the smaller it gets. Teacher: so how do you know that this one is bigger than that one so how do you know 0.24 is bigger than 0.236? What tells you? James: Because there's three numbers after the decimal point in that one and there's only two in that one.	Rachel: because I think that if you take like (pause). Because that's 0.236 and that's 0.24 and if you added a zero onto the end of 0.24 it would be 0.240 Used an algorithm accurately. (This was the same algorithm Rachel used in lessons)	India: because if... 4 is bigger than 3 so that's 0.04 and 0.03 – and because it's 3 and 6 isn't a whole number yet it will be less. So, it means 0.24 will be bigger – it's got more pieces.	Tasha: because it is 24 instead of 23.	Indicated 0.236 is larger. Ruth: Because it has more numbers than the other one.
Ordering fractions	Successful	Rachel: because you would colour in more for one third than for one quarter.	India: because if you have it in a circle a quarter will be 4 pieces but a third will be three and the three pieces will be bigger than a quarter	Tasha: because 4 is bigger than three.	Correctly identifies $\frac{1}{3}$ Ruth: It has... it has three.... Three like bigger shapes
DCT	Shorter-is-larger	Longer-is-larger	Longer-is-larger	Longer-is-larger	Longer-is-larger

Table 6-17 - Summary of interview answers

6.6.1 Illustrative Cases

Illustrative Case 6-6 details the results of the interviews with Tasha, India and Rachel. All were able to recall the decimat but did not use it as a tool without being prompted. India and Rachel were able to order $\frac{1}{3}$ and $\frac{1}{4}$, though Tasha still displayed a natural number bias.

	Tasha	India	Rachel
	Interview		
What is a fraction	a one and a line underneath and a two	A fraction is like one and a half, or one over four or one over two, that's half. One over three that's a third	Something where you simplify a number to make it easier. To understand.. so id you had like a box of 10 and it was split into 5 and only one box was coloured in it would be $\frac{1}{5}$
Place $\frac{2}{3}$ on a number line			 First indicated $\frac{2}{3}$ was at the point numbered 1
Using the decimat	Remembered the decimat but needed reminding on how to use it. Used it properly once reminded	Remembered the decimat but needed reminding on how to use it. Used it properly once reminded	Remembered the decimat but needed reminding on how to use it. Used it properly once reminded
Ordering Decimals	Tasha: because it is 24 instead of 23. .	India: because if... 4 is bigger than 3 so that's 0.04 and 0.03 – and because it's 3 and 6 isn't a whole number yet it will be less. So, it means 0.24 will be bigger – it's got more pieces.	Rachel: because I think that if you take like (pause). Because that's 0.236 and that's 0.24 and if you added a zero onto the end of 0.24 it would be 0.240 Used an algorithm accurately. (This was the same algorithm Rachel used in lessons)
Ordering fractions	Tasha: because 4 is bigger than three.	India: because if you have it in a circle a quarter will be 4 pieces but a third will be three and the three pieces will be bigger than a quarter	Rachel: because you would colour in more for one third than for one quarter.

Illustrative Case 6-6

James appeared, in the interview, to express a shorter-is-larger misconception, although he successfully ordered the fractions. Rachel used the annexe zero algorithm and successfully ordered the fractions by considering an area model. India used a left-to-right comparison method (see more below) to order the decimals and was successful in ordering fractions by considering sharing. Tasha appeared to use a left to right comparison method though this may have been incomplete. She then incorrectly ordered the fraction using whole number thinking to decide on the size of the fraction. Ruth incorrectly used a longer-is-larger strategy to order the decimals, but was able to order the fractions by considering sharing. Some of the answers are quite confusing, they do not really indicate that the students are fully clear in their own mind as to why they have made the decisions they have made. For example, when India is explaining why 0.24 is larger than 0.236 her reply is.

India: Because if... 4 is bigger than 3 so that's 0.04 and 0.03 – and because it's 3 and 6 isn't a whole number yet it will be less. So, it means 0.24 will be bigger – it's got more pieces.

India's reasoning appeared to be a comparison left to right of the two decimals. 4 is bigger than 3, then considering the next number (the 6) this is less than a whole and so does not increase the 3 past being bigger than a 4. Although the reasoning does not appear clear, India appears to have a complete algorithm for left to right digit comparison. It does not fall apart when there is a digit in one decimal and not in the other, which is one of Steinle's (2004a) warnings about incomplete algorithms.

Having reviewed the results of the testing and the interviews, there were questions that I wanted to ask the students 1) to see what they had taken from the post-test questions and 2) whether they could answer questions and remember the intervention. I decided to interview the students in the academic year 2019- 2020

approximately one year after the post-test to see whether they held onto their previous misconceptions.

In the second set of interviews my intention was to investigate persistence and why students had reverted back to previous misconceptions. Every time I analysed the pre- and post-test I wondered why some questions (e.g. 1e add one tenth to 4.254) were answered so badly. In interview I decided to take a selection of the questions and ask the students to complete them in front of me, talking about what they were doing. In addition, I wanted to see how their expression of the concept of a fraction had changed as well as comparing decimals and fractions.

The questions I asked were.

- What a fraction was
- To compare two decimals (0.24 and 0.236 also if needed 0.225 and 0.21)
- To compare two fractions ($\frac{1}{4}$ and $\frac{1}{3}$)
- To ask the students to complete a small portion of the post-test with me to talk about their difficulties.

6.7 Intervention 1 further interviews

The first interviews showed that there were significant misconceptions in the students' thinking that they had not shown at the end of the intervention. Questioning did, however, reveal that students could recall the intervention and could apply what had been taught. I decided to interview them again to see if I could find out more about their reasoning and the misconceptions held. The students were due to be interviewed in March and April 2020. Due to COVID and other factors, only Tasha was reinterviewed before schools were closed to the majority of students.

Tasha expressed her understanding of a fraction in terms of notation. When asked what a fraction was Tasha replied, '1 over 2'. She could remember and referred to the term denominator and recognised (though had forgotten) numerator.

When asked to order decimals (0.24 and 0.236) although she ordered them correctly, Tasha appeared to revert to an intuitive, incorrect, reasoning.

Tasha: because if there's more. If there's a bigger number, there's less. If there's a smaller number, there's like more.

This possibly indicates a shorter-is-larger misconception. This was supported by her choice of 0.21 as larger than 0.225.

Tasha could remember using the decimat but needed prompting on how to use it.

Tasha appeared to have an insecure grasp of place value though she also found the language hard e.g., add one tenth.

When asked which was bigger 0.8 or 0.75, Tasha correctly identified 0.8 but then her reason was

Tasha: if it was like a cake, you would get more if you cut it into 8 that you would if you cut it into 75

This adds to the idea that Tasha is holding a shorter-is-larger misconception. But potentially shines a light onto why she is holding onto this misconception as she appears to be attaching her explanation to the magnitudes of fractions. This would be classed as reciprocal thinking by Steinle and Stacey (2004), although I would expect this thinking to apply to decimals closer in magnitude. Interestingly, in the initial DCT student Tasha showed a longer-is-larger misconception.

When asked which was bigger, 0.24 or 0.236, Tasha gave the right answer, but her explanation indicated that there was still misunderstanding. A single question, especially when not followed up with an explanation, can mask a misconception. 0.24 is bigger than 0.236 if you use correct reasoning but also if you reason that the longer a decimal is the smaller it is, or if you believe that decimals are like negative numbers: that they are to the left of 0 on the number line. By asking the follow up questions it is easier to see which misconception is held.

6.7.1 Illustrative Cases

Illustrative Case 6-7 details the second interview. I was only able to interview Tasha because schools closed due to COVID before I could complete them. Tasha was not able to order fractions in the first interview. In the second interview she displayed denominator thinking by applying fractional thinking to decimals.

	Tasha	India	Rachel
	Follow – up Interview		
Second Interview	Tasha displayed a shorter is larger misconception in the interview. She also stated that 0.8 was greater than 0.75 because if you shared a pizza in 8 you would have more than if you shared It between 75.		

Illustrative Case 6-7

6.8 What is a fraction?

At the start of the first two lessons in September, the first lesson in January and in the interviews, I asked students what a fraction was and why it had two numbers.

The question is from Niemi (1996). The answers for the five students who were interviewed are as follows: -

Student	Lesson 1	Lesson 2	Lesson 1 January	1st interview	2nd interview
Tasha		A fraction is something like $\frac{1}{12}$ instead of it being 1 12 that looks confusing and / makes it better	It is something that is a half like a $\frac{2}{1}$ or $\frac{1}{2}$ or $\frac{2}{4}$	a one and a line underneath and a 2	1 over 2
India	A fraction is taking away	A fraction is like one over ten 1/10	It is a half	A fraction is like one and a half, or 1 over 4 as a quarter or one over 2; that's a half one over three that's a third	
Rachel	A fraction is a	A fraction is a number that has a whole on the bottom that you can split into different numbers	A fraction is a number that goes over a number than can (this has been crossed out and replaced with) A fraction is a percentage of a number	something where you simplify a number to make it easier. To understand. so, if like you had a like a box of 10 and it was split into 5 and only one box was coloured in it would be $\frac{1}{5}$	
James	A fraction is a number over a number		A fraction is a % of a number	it's got a denominator and a numerator and there's a line that separates them it can be like $\frac{1}{2}$ or $\frac{1}{4}$	
Ruth	A fraction is a half of something	A fraction is something split into pieces	A fraction is a half	it's a half of something	

Table 6-18 - fraction concepts

From Table 6-18 the students hold three basic concepts of a fraction.

- An operator – e.g., half of something (yellow)
- Notation – e.g., one over two (green)
- Shape – e.g., a part of a shape (blue)

The most common answer was the one connected to notation.

Reviewing all 19 answers from the first and second lesson there is a similar spread of answers.

	Lesson1	Lesson 2	Lesson 1 January
An operator	1	1	3
Notation / type of number	2	9	8
Shape	1	3	0
No answer	2	1	2

Table 6-19 answers to 'What is a fraction?'

As part of this task, I asked the class what the difference between a decimal and a fraction was. All but one said they were the same, however, one child said that a decimal was a number and a fraction needed to be a fraction of something. This is an interesting answer considering the nature of fractions.

6.9 Intervention 1 discussion

The results of the pre-test and post-test, at a first glance, appear to indicate that the intervention was not successful. However, at the end of the intervention students were able to answer questions about decimals and fractions. The post-test was deliberately left until later to see the longevity of any positive gains. The post-test seems to say that there is no significant gain and yet in the interview students recalled the lessons and were able to recall when their memory was activated. Most were able to order the decimals, when asked. It is also worth

considering that the students in the intervention were those who have perhaps had difficulties with mathematics in the past.

The intervention had taken place in September and in January. At the end of this teaching section all students were ordering sets of decimals using the number line, the decimals or their own algorithm. In the post-test the same students were not correctly ordering decimals, this was repeated in the interview. It appeared that the positive benefits of the intervention had not lasted. This led me to investigate the persistence of errors and the influence of intuition on the retention of misconceptions.

This second round of interviews was approximately a year after the intervention was complete. I interviewed one of the five students but could not continue as the school was then closed due to the first COVID lockdown. At first the school did not have the technology for virtual interviews. By the time it did, too much time had gone past. I had analysed the first interview though, and alongside the other interviews I needed to investigate whether a short-term intervention can help and why the students were jumping back to their initial understandings.

Although Stylianides and Stylianides (2014) found that a positive impact could be made by an short duration instructional intervention, the circumstances in that study are different. The students involved were adults, the improvement was in their confidence and the review of the outcomes was shortly after the intervention concluded. The students in my intervention have long held misconceptions and the review was up to a year after the intervention.

6.10 Summary from Intervention 1

- Students have not developed a secure concept of what a fraction is.
- Students were ordering fractional numbers by the end of the intervention but not repeating this in the post-test or interview.
- When reminded in the interview they were recalling the intervention.
- Students have persistent misconceptions.

It seemed to me that there was a persistence of errors and that there was some sort of intuitive response to questions, with an intuition that pre-dated the intervention but was persisting past the intervention. This conjecture resonated with my experience as a teacher. I decided, therefore, to review literature on understanding and intuition, including the persistence of errors. My intention was that once that was done I would develop a second intervention to try to address these persistent errors

6.11 illustrative Cases Summary

Illustrative Case 6-8 summarises the journey of Tasha, India and Rachel through the intervention.

	Tasha	India	Rachel
	Summary		
	In the pre-test, Tasha appears to display an insecure grasp of place value and a longer is larger misconception. The DCT showed that Tasha is likely to have held a longer-is-larger misconception with a whole number bias. In lessons, Tasha was able to order decimals, fractions and represent both. In the post-test Tasha has displayed a mixture of misconceptions. In the interview, Tasha could recall and apply what had been learned in lessons. She was able to order decimals, by noting that 4 is bigger than 3 when considering 0.24 and 0.236. This is indicative of a full left-to-right comparison which, used properly, is a full algorithm. I interviewed Tasha again almost a year after the first interview. At this stage Tasha displayed a shorter is larger misconception. Her understanding of decimals appeared to be affected by fractions as she described comparing 0.8 and 0.75 by splitting a pizza into 8 pieced and 75 pieces.	In the pre-test India displayed a shorter-is- larger misconception. This was on a small number of questions. The DCT, however, showed that she was likely to hold a longer is larger misconception. India was able to order three decimals towards the start of the intervention, stating that the “numbers were bigger”. In lessons, India was able to order fractions and decimals and match the different instantiations. In the post-test, India has reverted to the same answers she gave for the pre-test. In the interview, India appeared to be using a left to right comparison to compare decimals. She correctly explained why 0. 24 was bigger than 0.236 by first noting that 4 was bigger than 3. She was also able t compare unit fractions and explain why $\frac{1}{3}$ is larger than $\frac{1}{4}$.	In the pretest, Rachel displayed some difficulties with place value and magnitude of fractional amounts. She uses methods in the pre-test rather than relying on the structure of the decimal system. These methods were not complete and led to errors. Where Rachel did not appear to rely on a method she appeared to have a longer is larger misconception. Rachel was not able to represent fractions equivalent to $\frac{2}{3}$. The DCT showed that Rachel was likely to have a longer is larger misconception with a whole number bias. All but one of her responses would be selected by considering the decimal part as a whole number (eg $0.63 > 0.8$ because $63 > 8$). At the start of the intervention, when she didn’t use the annexe-zero algorithm, Rachel selected the shortest decimal as the largest. At the start of the intervention, Rachel was able to order fractions with the same denominator but not with different denominators. In lessons, Rachel was quick to complete tasks. Given a choice, she used the annexe-zero algorithm to compare decimals rather than relying on other representations or instantiations. Rachel matched different instantiations and representations of decimals and fractions and ordered them. In the post-test, Rachel, displayed a longer is larger misconception. She was less reliant on algorithms, though her answers did not seem to answer the questions asked. In the interview, Rachel again relied on an algorithm and applied it accurately. Before the intervention Rachel had a longer is larger misconception. During the intervention she was able to order decimals and fractions both with and without an annexe zero algorithm. In the post-test and the interview she reverted to a longer is larger misconception, although she answered some questions correctly using the algorithm.

Illustrative Case 6-8

6.12 The Effects of COVID-19 on Intervention 1

In December 2019, I had completed the review of first intervention. I decided to interview the students again to investigate the persistence of errors and what the students still remembered from what I had taught them. This set of interviews was planned for February and March 2020, approximately one year after the teaching episode and the post test. One student was interviewed before schools were closed to most students. The school did not have the technology for online interviews, and I felt that much that I had discovered from the first interviews had been in watching the students and in interacting over the same resource. I decided to wait until the school reopened to complete the interviews. In the end, the school did not open again for the students that academic year and so I used what I had found to inform the second intervention.

6.13 Research Questions

My research question is: Can the mathematical understanding of students be modified by encouraging students to engage with underlying mathematical structures?

The interpretation of the results of the intervention did not show long lasting results. The students appeared to change their responses in lessons and were able to answer questions. However, when tested in the delayed post-test, there was little difference from the pre-test. In the interviews the student recalled sayings like 'Because it has more numbers than the other one' (Ruth in section 6.5.2) or 'because if there's more. If there's a bigger number, there's less. If there's a smaller number, there's like more' (Tasha in section 6.7). These were not phrases from the intervention lessons and appeared to predate those lessons. They had a feel of automatic thinking. In terms of continuing the study, I needed to ask the following questions:

- Where had the statements originated?
- Was there any new learning retained from the intervention?
- How sticky are these automated statements?
- Can they be overcome?
- Can errors be reclassified to help identify these intuitive thoughts?

Chapter 7

7 Cycle 2

7.1 Literature Review 2

The first intervention showed me that where I thought there was understanding within the lesson the results of the post-test showed me that the understandings were not necessarily long-lasting, and the interviews showed me that students may have reverted to previously held intuitive notions. Whilst, in lesson, they held more accurate notions, and these could still be recalled in the interview, they did not replace previously held beliefs. There seemed to be an intuitive reaction to my questions. I searched literature for answers to the following questions:

- What is understanding?
- Can understanding be measured?
- What is the relationship between understanding and concepts?
- What is intuition?
- Can errors be reclassified to assist in identifying these intuitive errors?
- How can tasks and interventions be designed so that students' intuitions are modified?

7.1.1 What is understanding?

7.1.1.1 Introduction

Everyone has a notion of what understanding is; anyone can say 'I get it' or 'I understand' and internally have a strong idea of what they mean. Understanding is, however, subjective. A student in a mathematics lesson, working on a task might say 'I understand that' but the utterance could have many layers of meaning. It could be that they understand how to apply the algorithm, or that they

understand why the algorithm is applicable, or maybe that they understand the derivation of the algorithm. Each of these could be considered a level of understanding but each shows a different appreciation of the concepts involved. In the section on mathematical structure (chapter 3), I discussed that mathematics is many layered and so, too, is understanding. It has always been problematic to intend to teach that students should understand X; this is vague and not measurable. There is an everyday notion of understanding, but a technical definition is harder to find. Below I review the literature that intended to define understanding.

7.1.1.2 Levels of understanding

Various authors considered different levels of understanding. Skemp (1976) distinguished between relational and instrumental understanding of mathematics. He identified these as almost distinct intentions that existed in parallel, unaware of each other. He went as far as to say that there were, perhaps, two types of mathematics in play. If a student can correctly apply the quadratic formula to solve a quadratic equation, then they have, at least, an instrumental understanding. They can apply and follow a procedure. The student may not, however, have an underlying understanding of why they perform that procedure. For Skemp, knowing how to do something was instrumental; but knowing why was relational.

Whereas Skemp had two layers of understanding Byers V. Herscovics (1977) had four levels “Instrumental”, “Relational”, “Intuitive” and “Formal” (V. Byers & Herscovics, 1977, p. 26). They extended Skemp’s work stating that the initial relational and instrumental did not encompass instinctive, immediate understanding (intuitive) nor a formal understanding where students not only understand “mathematical results but also of symbolic and logical form” (V. Byers & Herscovics, 1977, p. 25). These four levels are similar to Buxton’s (1978) four

levels: Level 1 is rote learning, much like Skemp's instrumental learning. Level 2 is observational, a richer version of Skemp's instrumental, at which point patterns are seen; Level 3 is insightful. This is the beginning of relational understanding for Buxton and is, for example, where the student starts explaining why there are patterns in the nine times table; and Level 4 is a formal understanding. This is where, if a person is faced with a fact, they will search out a proof for it. Both Buxton and V. Byers and Herscovics talk about a formal level of understanding but the problem here is a formal proof can be rote learned and it is not clear whether this would be "level 4" (Buxton, 1978) "formal" (V. Byers & Herscovics, 1977) or Level 1 (Buxton, 1978) "instrumental" (V. Byers & Herscovics, 1977).

Sierpinska (1994), too, utilises four levels of understanding. She identifies four mental operations in understanding: "Identification", "Discrimination", "Generalisation" and "Syntheses" (Sierpinska, 1994, p. 56). Sierpinska (1994, p. 32) also discusses that being able to understand something is accompanied by a sense of completeness: a sense of closure which she describes as "the feeling that 'it fits'". Below (see section 7.1.4.2) I have linked this to 'cognitive ease' and a proposed explanation of why intuitions are so difficult to address in a classroom.

Although the levels of understanding resonate with my experience in the classroom, these models do not appear nuanced enough. Two other approaches counter this lack of nuance in different ways.

Pirie and Kieren (1994) consider a multitude of layers of understanding. Their model is represented in Figure 7-1 below.

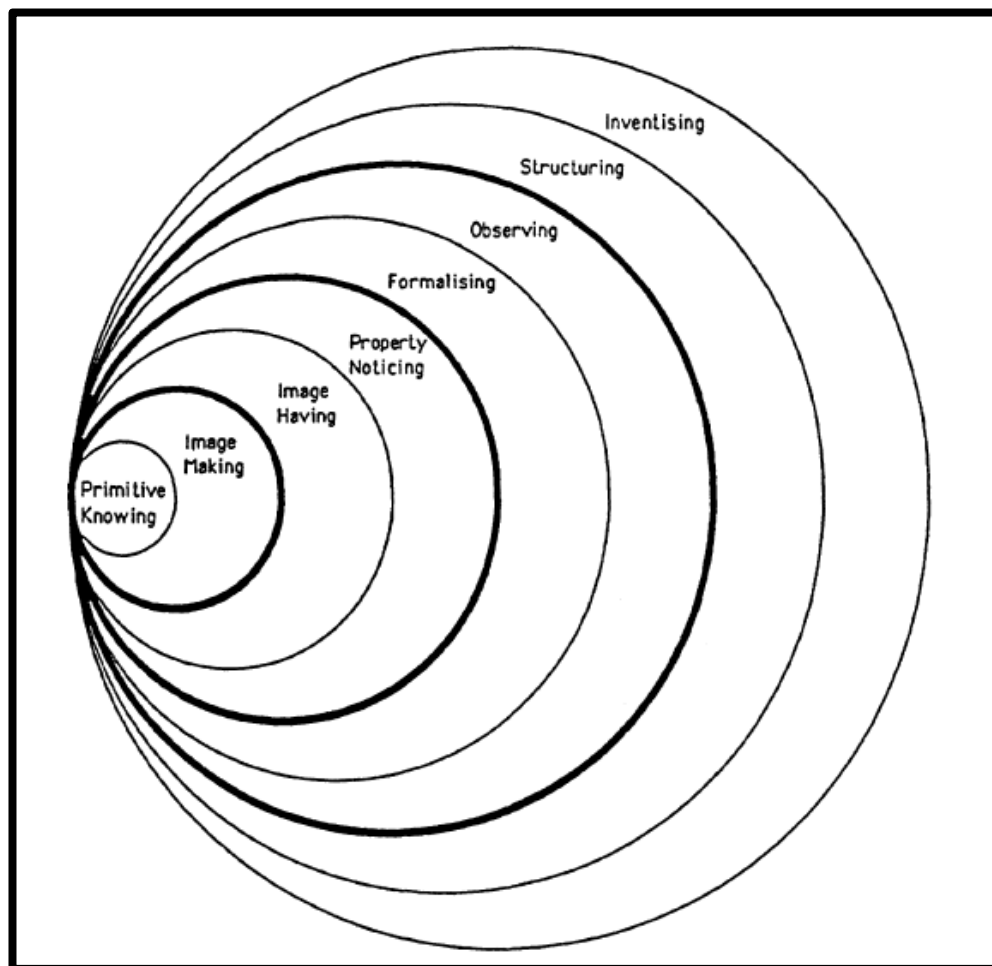


Figure 7-1 Pirie and Kieren (1994, p.167)

The original description of the model included the first layer as doing (Pirie & Kieren, 1989) rather than primitive knowing. The example in the 1989 version of the model was being able to plot specific quadratic curves and in the 1994 refined version of the model primitive knowing refers to what is already understood by the learner before the new concept is taught. The primitive level of knowing may be incomplete or erroneous. It is not considered that someone will traverse through the levels but will fold back with each new concept so that primitive knowing at the start of learning about a new concept will be different from primitive knowing later on. This model then counters the earlier criticism of V. Byers and Herscovics and Buxton; knowing a formal proof by rote, potentially formal understanding in the Buxton or V. Byers and Herscovics models, could be primitive knowing in the Pirie and Kieren model.

Haylock (1982) and Hiebert and Carpenter (1992), all define depth of understanding by considering connections.

to understand something means to make (cognitive) connections. The more connections the learner can make between the new experience and previous experiences, the greater and consequentially the more useful the understanding. (Haylock, 1982, p. 54)

Haylock (1982) describes rote learning as unconnected learning: facts learned in isolation. A deeper understanding is formed by making more and more connections. This idea is similarly explained by Hiebert and Carpenter.

A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of the connections. A mathematical idea, procedure or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections (Hiebert & Carpenter, 1992, p. 67).

Hiebert and Carpenter's statement here is that the strength of understanding of a concept is somehow measured by the strength of the links between it and previous learning. Considering again whether understanding a formal proof would constitute a greater depth of understanding, it seems unlikely in these models that a rote-learned proof would indicate higher levels of understanding. In this case the rote-learned proof would not be likely to have the connections to indicate a greater depth of understanding.

Both of these views of understanding resonate with my conjecture that by selecting tasks that require students to work with different instantiations and to move between the instantiations they will develop an appreciation of structure and will improve their mathematical thinking and understanding.

7.1.1.3 Resistance

From my intervention it was clear that even when students, in the lesson, accepted concepts which contradicted their intuitive beliefs they were resistant to replacing those incorrect intuitions. I reviewed the literature to find out about resistance.

Skemp (1976), V. Byers and Herscovics (1977) and Buxton (1978) all discuss a resistance from students to deepen their understanding once they have reached a level of cognitive certainty: once they believe that they understand.

‘Well is the enemy of better’ and if pupils can get the right answers by the kind of thinking they are used to, they will not take kindly to suggestions that they should try for something beyond this (Skemp, 1976, p. 22).

An example of this might be asking students, with a longer-is-larger misconception, which is larger 0.74 or 0.5. The student will decide that 0.74 is larger, this is indeed the correct answer, however their reasoning may be wrong. This might confirm for the student that their reasoning is correct, because it gave them the correct answer. Where a method has worked in the past students will rely on that method and see no reason to change it. This leads to the conclusion that any task needs to be designed carefully to ensure that the correct answer cannot be accidentally found with the incorrect reasoning. Using badly behaved examples gives students a chance to reflect on the structure of the concept at hand.

In addition, both Skemp (1976) and V. Byers and Herscovics (1977) discuss the fact that teachers might wish to teach relationally but there are constraints, other than student motivation, that restrict them. Just as Skemp (1976) discussed the unwillingness for some students to gain relational understanding, Buxton (1978) conjectures that many students will be satisfied with how to answer a question and will not seek his level 3 understanding. A student will feel that they have understanding at the “Oh, I see...” (Buxton, 1978, p. 36) moment. Buxton places

this moment in the level 3 understanding: his first level of relational understanding. Skemp (1976) points out that a student may well have this break-through at purely instrumental levels and a response of 'no, you don't' would be unwelcome – both because the student at this point believes they understand and also because the student might feel that the teacher doubting their understanding would devalue their work. This interaction resonates with my experience as a teacher when I realise that a student is able to complete a task but might not know why. I have experienced students being irritated by my questions when they feel they know what they are doing and just want to be left alone to get on. Sometimes, in this case, I want to ask further questions; either because I want to check the understanding, or because one case is not enough to check, or because the student's answer or work leads me to believe that they have not understood. There is, therefore, a potential gap, between when a student feels as if they understand and when a teacher, wanting to ensure a good depth of understanding, believes that the student has understood. Whether the understanding of a student can be measured is discussed below. As a teacher, though, you can be left wondering what to do next when a student appears to have all the pieces to attempt a task but for them the pieces have not clicked into place and they still say, 'I don't get it'.

V. Byers and Herscovics (1977) report the findings of discussions about the Skemp (1976) model with teachers: where teachers identify as teaching for relational understanding but sometimes have to resort to instrumental understanding; depending on the class and the concepts being taught. This is recognised in Skemp (1976), along with curriculum and assessment needs. It is also discussed by Siegler *et al.* (2011) who noticed that procedural understanding

of fraction arithmetic could lead to relational understanding of the fractional magnitude.

7.1.2 Can understanding be measured?

7.1.2.1 Measuring Understanding

It is very difficult to assess exactly what someone else is perceiving. It can only be done by saying student A is demonstrating understanding by performing task X. Even this, however, could either be the result of a student applying what they know to a novel situation or following a set of learnt procedures without a grasp of why each step is performed. Looking at the example above where a student had an awareness of the links between the quadratic formula and completing the square, indicating a possible relational understanding, the student might be able to derive the quadratic formula from completing the square. However, both completing the square and the derivation are procedures that could be learnt.

...we are unable to observe our pupils' feelings and schemas directly and look for confirmatory behaviour. As evidence that A understands X, we accept the fact that A applies X in situations different (in greater or lesser degree) from that in which it was learned. We might also accept a description by A of how X was related to other mathematical knowledge but this is not so frequently done (Backhouse, 1978, p. 39)

Haylock (1982) explains that recognising understanding is a case of looking for behaviours. Such behaviours are concerned with making connections; for example, transforming a concrete situation into symbols. Haylock (1982) further considered that learning without connecting new knowledge to old was rote learning. Assessing that there is some degree of understanding is, therefore, done by looking for behaviours which show that a learner can make connections, for example, by turning a word problem into a symbolic problem.

7.1.3 What is the relationship between understanding and concepts?

7.1.3.1 Learning and Schema

The most widely accepted model for learning is that of schema (Piaget, 1952), and is used by cognitive scientists. Current practice in schools includes developing teachers' concepts of cognitive load, spacing and interleaving (e.g. Lovell, 2020). These are based on the theory of schema.

Piaget (1952) developed a theory of schema and learning. A schema is a building block of information. It holds all the experiences and memories about a certain concept. For example, a child will have a schema around what a table looks like. When they see another table they can assimilate, accommodate or reject the item as a table. When they see a table which fits their previous experiences, they recognise it as a table. This is assimilation. If, however, the table has three legs a child can either assimilate the new item as a table, accommodate, which will alter the schema via equilibration or reject the new item as a table. Equilibration is the process by which schemata are altered. When faced with a new item that cannot be assimilated the child is in disequilibrium and either the schemata must be altered to allow for the three-legged item to be classified as a table or the item will be rejected as not being a table. There is nothing to say that the child's classification will match generally accepted classifications. My daughter classified anyone with grey (or blond) hair as a 'grandpa' regardless of age or gender. On seeing a picture of Diana, Princess of Wales, my daughter declared 'Grandpa'. She had a schema for 'Grandpa', and on seeing the picture of a mid-30s blonde woman she assimilated the new item into that schema. The older the child gets, and the more schemata they have, the more precise the classifications. Schemata are constantly built and altered.

Sweller *et al.* (1998) devised cognitive load theory, which centres around the limitations of working memory, compared to long term memory. Working memory is the immediately accessible memory which can hold up to, maybe, seven items at a time. However, Sweller *et al.* consider that a schema is a single element, generated from “chunking” constituent elements, that can be held in working memory and so “understanding occurs when high element interactivity material can be held simultaneously in working memory” (Sweller *et al.*, 1998, p. 261).

7.1.3.2 Concept image

I return to the notion of “concept image” first described in section 3.1. Tall and Vinner (1981) differentiated between a “concept” in mathematics, formally defined by the mathematics community and a “concept image”; what an individual learner perceives as the concept. The concept image is the entirety of the learner’s thoughts and experiences of the concept.

My experience tells me that when concept images are strong students can be reluctant to take on board the agreed definition. Students, in my experience, are reluctant to accept that a square is a rectangle, because they have in their concept image that a rectangle has two long and two short sides.

Whilst recognising that the Tall and Vinner’s concept image is restricted to mathematical concepts and schemata are a mechanism for describing all concepts, it appears that schemata and concept images are very similar. They are both models for expressing the layers of experience that lead a learner to their own interpretation of a concept. I asked David Tall about the differences his answer included an intentional vagueness:

There are many issues involved. For instance, I see 'concept image' as related to the operation of the human brain. Shlomo Vinner interprets it as a philosophical concept and rejects my interpretation. (D. Tall, personal communication, 2020).

The schema may describe the cognitive and biological processes that lead to learning and recalling information, the mechanism by which people accept or reject pieces of information, whereas the concept image is the information that the student can draw upon when considering the concept and describes how internally held truths are connected to the concept. It seems a fine point and outside the scope of this project. The schema and the concept image are both the result of the entirety of a learner's experiences with that concept. Either model allows us to compare the formally defined concept as opposed to what the learner experiences as the concept. From the perspective of my work, it seems that there is little difference between the two. Both schema and concept image describe the entirety of what a learner believes about the concept so I will use either model interchangeably. From my perspective either model illustrates how students might build their own understanding by new experiences building on previous experiences and prior learning.

A student's concept image might not be correct, in that it is likely to differ from the concept definition. More interestingly, it need not be internally consistent. A learner may hold mutually exclusive ideas within their concept image but only the part of the image being evoked will be attended to and so it is only when both of the conflicting parts of the image are invoked simultaneously that a learner might experience cognitive conflict (Zaslavsky, 2005). An example given in Tall and Vinner (1981) is where student will happily add $\frac{1}{2} + \frac{1}{2}$ correctly but then faced with $\frac{1}{2} + \frac{1}{3}$ will make the error of concluding it is $\frac{2}{5}$.

7.1.4 What is intuition?

7.1.4.1 *Intuition*

In everyday thinking, intuition is a sixth sense. The Cambridge Online Dictionary defines it as “an ability to understand or know something without needing to think about it or use reason to discover it, or a feeling that shows this ability” (Cambridge, n.d.).

In psychology and mathematics education literature, intuition is not a sixth sense but where a realisation is obvious, immediate and involves no rational thought. This is how any conclusion is perceived but does in no way indicate whether that conclusion is correct. Bruner is quoted as saying.

Intuition implies the act of grasping the meaning or significance or structure of a problem without explicit reliance on the analytical apparatus of one's craft (Bruner, cited in Noddings and Shore, (1984, p. 37).

V. Byers and Herscovics (1977) considered that Skemp's (1976) model of understanding did not sufficiently account for the instinctive “just knowing” that they labelled intuitive understanding. Features of intuitive understanding are:

Implicit perception of the problem as a whole, reliance on imagery, the absence of well-defined steps, little awareness of the solving process and difficulty in reporting it (V. Byers & Herscovics, 1977, p. 25).

In section 3.2 I alluded to my experience that students can solve for x ; $3x = 9$ but cannot solve $3x = 10$. Considering that example again, it could be that their intuition kicks in to solve the first instance and not the second. They are unaware that they are solving $3x = 9$ by dividing by 3 and therefore have the skills to solve $ax = b$ for any values of a and b . These two examples are different instantiations (see section 3.2), one being able to be solved by an intuitive response and the other leading to an intuitive (incorrect) response that there is no answer.

Intuition can refer to “non-deductive parts of mathematical thinking” (V. Byers & Herscovics, 1977, p. 24); this could be mathematics taught without proof using instead heuristic and real-life explanations. Intuitive understanding can be concepts brought with the child to the school, it can be unorthodox solutions, often incomplete. Intuition is often equated with guessing, but this is not a fair characterisation of students. If asked what 3×4 is, I *know* the answer. I do not need to justify it or be able to explain it. There are many things in life that I just *know*, and some turn out later to be wrong. *Just guessing* would be where I did not know the answer and threw out a conjecture. Students who answer intuitively are answering about what they think they know. Pirie and Kieren (1994) include in their tiers of understanding “Primitive Knowing”. These are personal axioms: things which are just known, not derived or inspected. They are part of a student’s concept image on arrival to a lesson. Haylock (1982), Hiebert and Carpenter (1992), and Sierpiska (1994) all include prior knowledge in part of the description for understanding. Students arrive to a lesson with prior knowledge and that prior knowledge will include intuitions (both accurate and erroneous). It is worth noting that their understanding may not be internally consistent, in other words the prior knowledge might include erroneous intuitions and accurate concepts which would contradict the intuition.

Ironically, Fischbein (1999), in his seminal writing about intuition states that “the meaning of intuition is implicitly considered as being intuitively evident” (Fischbein, 1999, p. 12).

Intuitions themselves appear and develop as schemata develop. For example there is a widely considered intuition “more A implies more B” (Stavy & Tirosh, 2000, p. 1). Here students will feel that when one variable A increases, so must another variable which might appear to be dependent but is not. One example that

Stavy and Tirosh (2000) discuss is comparing two lines of marbles. When shown two lines of marbles young children are likely to associate the shorter line length with the smaller number of marbles. They will consider the overall length of the line rather than counting the number of marbles. As students grow older, however, they are more likely to count the number of marbles to check. At this stage, through formal teaching and more life experience they have developed another intuition “same A same B”: a form of conservation. If the areas of two shapes are the same, then this intuition might lead to the misconception that the perimeters are also the same. Intuitions will initially be considered in advance of more logical deliberation. They appear to be the basis for many ‘sticky errors’, see below. In terms of my classroom intervention $\frac{1}{3}$ is smaller than $\frac{1}{4}$ could be an instance of more A more B. 4 is bigger than 3 so $\frac{1}{4}$ is bigger than $\frac{1}{3}$. Students growing out of more A more B is consistent with Steinle’s (2004a) longer-is-larger misconception being outgrown.

Given the changes in intuition as students receive more formal mathematics training it needs to be realised that a correct answer may often be no more than happenstance. To explain this let us consider a third intuition “everything can be divided endlessly” (Babai, 2010, p. 204). Younger students have an intuition that there is a finite limit to how many times something can be divided. In terms of their practical experience this is an entirely reasonable assumption. Older students, maybe as a result of formal teaching, will recognise that, in the abstract, anything can be continuously divided. When asked if a physical substance can be continuously divided and whether a mathematical object (for example, a line segment) can be continuously divided, younger students will say neither can and older students, both can (Tirosh & Stavy, 1996). The younger students answered correctly for the physical object, but incorrectly for the mathematical object. The

older students conversely answered correctly for the abstract but not the concrete. The correct answers from both, are as a result of an intuition that needs to be modified by reasoned thought.

Stavy and Tirosh (2000) also discuss the incorrect intuition if A then B implies if not A then not B. Students knowing that A implies B, erroneously extend this to not A implies not B. An example might be that I tell a child that any number divisible by 6 is also divisible by 3, and they select 9 and say that as not divisible by 6 it is not divisible by 3. The more complex solution is actually if A then B implies if not B then not A. In the case before being divisible by 6 implies that a number is divisible by 3, but not being divisible by 3 implies that the number is not divisible by 6. This certainly resonates with my experience in the classroom along with students overgeneralising that if A then B, then if B then A should be a result. My experience shows me that without specific guidance students are unlikely to differentiate between the statements “all squares are rectangles” and “all rectangles are squares”. Stavy and Tirosh (2000) state that the origin of intuitive rules is the tendency to extrapolate beyond the appropriate domain. “Generalizing” (Burton, 1984; Johnston-Wilder & Mason, 2006) is one of characteristics of mathematical thinking and, therefore, it is a useful tendency in mathematics but it does need to be applied with critical scepticism. It needs the user to step back and ask ‘does this work in this instance?’ but the issue is that intuition is, by definition, without thought. It adapts as later schema take hold. In terms of the discussion on structure in section 3.3, this is where students consider all instantiations to be isomorphic; believing that all necessary properties are essential.

7.1.4.2 Fast and Slow thinking

Kahnemann (2012, pp. 59–70) describes a sense of “*cognitive ease*”, where the thinker has a sense of well-being around the thoughts, with little need to mobilise

effort. The consequences of cognitive ease are that the thought “feels familiar”, “feels true”, “feels good” and “feels effortless” (Kahneman, 2012, p. 60).

Kahnemann (2012) discusses two types of thought: type 1 and type 2. Type 1 is quick and effortless, it is connected with cognitive ease and is connected to a feeling of being right. Type 2 is thinking that is deliberate, takes time and is associated with cognitive difficulty. Any level of increased cognitive load appears to call on type 2 thinking. Type 1 thinking can lead us to a false sense of security. Type 1 changes with life experiences and lessons. Although rational thought can overcome type 1 thinking, this lies with a level of unease.

Type 1 thinking appears to have all the hall marks of intuition and type 2 of cognitive effort. Kahneman (2012) states that cognitive ease will tend to make an answer more likely to be chosen. This certainly resonates with intuition and the two theories (types of thinking and intuition) seem to agree that intuitive, fast thinking, based on a sense of cognitive ease, can lead an individual to make an incorrect decision. Obersteiner *et al.* (2016, p. 541) points out that a type 1 answer “comes to mind easily” and can give an incorrect response “unless type 2 processing inhibits the intuitive response tendency”. In this, it appears that requiring a student to think, rather than select an intuitive answer will be more likely to lead to correct answers, aligning with the ideas from Kahnemann.

7.1.4.3 Tacit models and Intuition

Tacit models are internal models that allow us to work with abstract concepts (Fischbein, 1989). Fischbein describes tacit models. If A is an abstract notion,

B is defined as a model of A if it is possible to translate properties of A in terms of B so as to produce consistent descriptions of A in terms of B, or to solve problems – originally formulated in terms of A – by resorting to a translation in terms of B (Fischbein, 2001, p. 312)

Moving between A and B appears to be a similar process to moving between instantiations. Fischbein continues.

The concept of mental model refers to mental representations which replace, in the reasoning process, the original entities, usually in order to stimulate and to ease the solving endeavor (sic). (Fischbein, 2001, p. 312)

Fischbein discusses the fact that even where students are aware of the abstract nature of geometrical concepts they continue to use the models.

Although *we know perfectly well* that mathematical points have no dimensions, we continue to think *tacitly*, unconsciously, in terms of small spots (Fischbein, 2001, p. 315).

Sometimes a model allows learners to support correct mathematical inferences. In fact, without a model it is often impossible to reason. At other times, though, conclusions maybe correct for the model but not for the underlying abstract concept.

Fischbein points out that tacit models are robust. In other words, students revert back to tacit, primitive models long after they should have no use. He discusses how high school students make the same mistakes as much younger children and how this may be because of their reliance on tacit models. He discusses why this might be the case: 1) the models are used as part of the reasoning process; 2) because the model is tacit the user does not even know that their thinking is being controlled by it and therefore does not realise that there is a need to change anything; and 3) part of their strength will be the fact that the models are simple and easy (Fischbein, 1989). Similar to an intuition, the models are automated and give a sense of 'cognitive ease' and students do not realise that they are being overtaken by an intuition because they do not even know it is an intuition that they are using. Fischbein suggested solution is the use of metacognition to turn attention onto the fact that the model exists (Fischbein, 1989).

Kidron (2011) discusses the tacit models in Fischbein's work. He notes that these are models and images that even seasoned mathematicians use to appreciate mathematical contexts and abstract notions. Kidron describes students who are aware of the conflict between the abstract and the model; even though they know about the difference they still feel that the model is right, even when they know that it does not properly describe the abstract. Descriptions, rather than definitions, can call on existing intuitions. An example with school students might be a rectangle. Given the definition of a rectangle students should be able to say that squares are rectangles, however, the students have an image of a rectangle, and they will often fall back to the perceived fact that rectangles need to have unequal adjacent lengths. Tsamir *et al.* (2008) find that students often think that once something is named it is unique and therefore find it difficult to accept that a square is also a rectangle and a quadrilateral. The use of non-examples has been shown to make students talk out loud, making it easier for teachers to examine their thinking.

Although students are able to appreciate an infinite process, they are less likely to believe that the result is finite. They can believe approaches to, but not is equal to. Kidron (2011) describes this as the fact that a potentially infinite object has a behaviour meaning (for example slicing a cake in half) whereas an actual infinity does not have a behavioural meaning and, therefore, is not congruent with an intuitive interpretation. The example in the literature (Fischbein, 2001) is whether $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$. Students are happy to accept that it approaches 1, but are less inclined to believe that the two are equal. My own experience tells me that in discussions with students about limits or recurring decimals, they intimate that they can have a different last digit on a recurring decimal. They might say something like '4.9 recurring 8'.

Tacit models can be so strong that people have confidence in them even in the face of their known limitations (Kidron, 2011). This is another example of people being able to hold internally inconsistent concept images (Zaslavsky, 2005, p. 318).

In terms of my intervention and students' understanding of the number line, Fischbein (2001, p. 315) discusses comparing the number of points on two line segments of different length. The abstract, formal answer is that the two-line segments have an equal, infinite number of points; that there is a one-to-one relationship between the two lines. It is difficult to reconcile this, though, with the intuitive understanding of the line as a set of points, with a very small size; this leaves us with a sense of "difficulty, of contradiction or paradox" which cannot be easily overcome.

Much of the work on the intuitive nature of thought shows that even trained mathematicians are subject to intuitions which cloud their judgement, meaning that incorrect conclusions are reached. Many of these have the learners saying 'I know it should be A but B feels right'. My work is with students who are far less confident in the mathematics, in their manipulations of abstract definitions or in reasoning about their own thinking and their intuitions. There is, though, an infinite nature to the number line. Any interval in the line, however small, can be expanded to contain an infinite number of numbers. I would argue that this is at the basis of understanding place value; that without this understanding it is impossible to understand the infinite nature of the number line. Knowing that 0.27 is in between 0.2 and 0.3 and that 0.272 is in between 0.27 and 0.28 ...etc.

Gillard *et al.* (2009) compares the different theories: Kahnemann and Fischbein. In this they discuss heuristic and analytical answers. The heuristic response is

immediate and is processed in parallel. Analytical responses are sequential and take longer. With an increase in cognitive load Fischbein's heuristic responses become more likely. This conflicts with Kahnemann (2012) where any increase in cognitive strain was more likely to activate analytical thought processes. Obersteiner *et al.* (2016, p. 541) refer to "effortless type 1 processing, and analytic, effortful type 2 processing".

7.1.4.4 Intuition vs schemata

Fischbein (1999) believes that intuitions are schemata. Sweller *et al.* (1998) talk about highly automated schemata. It seems likely that intuitions would be highly automated schemata. Intuitions are developmental and Fischbein claims that their accuracy improves with age, if supported by adequate schemata because structural schemata progress with age. If, however, the schemata are inadequate then the intuition will worsen with age. This supports my conjecture that teaching should be based around structure to support the development of adequate schemata.

Mathematicians and mathematics students are often able to take short cuts and compress ideas. Fischbein concludes that although a compression does not necessarily lead to an intuition, an intuition is generally caused by a compression. (Fischbein, 1999). This is similar to an overgeneralisation causing a misconception.

7.1.5 Errors

Although I reviewed errors in section 2.7, the results of my intervention and the proliferation of automated, intuitive errors made me realise I needed to look again at the literature around errors. There is frequently an element of judgement around the word 'error': someone has gone wrong. Within the scope of this work an error

is where the student's answer or response does not conform to the generally accepted answer for the question, or in some sense what the teacher had intended the answer to be. In my experience the teacher sometimes has an answer in mind to a question and the answer that they consider incorrect is more often a miscommunication than misconception. Sfard (2007, pp. 573–576) used the term “commognitive conflict”. Commognitive refers to the interaction between communication and cognitive, and commognitive conflict arises where two people are using different rules of discourse. The issue with such perceived errors is that there may be no error at all: both teacher and student are making complete sense within their own rules of discourse. Neither realises that the other is using a different set of rules to communicate. Considering this is beyond what I am discussing here; here I restrict this to assuming a well-formed question with a clear set of correct answers. Even when the difference between the expected answer and the given response is not due to communication difference, most erroneous responses have their own logic and there is no judgement to the fact that the student has given the wrong answer. In fact, here I aim to analyse and address reasons for errors and find a way to move our pedagogy forward to help students overcome these errors. For this reason, errors are often called misconceptions: recognising that there was a conception, but that it was misplaced. There is much spoken and written about saying that in teaching one of the teacher's main jobs is to recognise misconceptions so that they can be addressed or to teach in a way so that misconceptions can be avoided in the first place. I would argue that the process is more complicated than that and, potentially partly due to commognitive conflict (Sfard, 2007) it is not possible to analyse a misconception from a single verbal response or answer in a test. Section 2.7 was a review of literature on errors in mathematics. Having reviewed the results of the first intervention I have found that knowing a student's answer to a question is not enough information to

really know what they are saying or why an error is occurring. As Ryan and Williams (2007) state it is of little use using an assessment to diagnose the reason for errors. This would require a discussion with the child to find out more about their thinking.

7.1.5.1 Diagnosing Errors

Much is written about diagnosing the errors that students make: identifying the misconception so that a teacher is able to adapt teaching to address the misconception (Barton, 2018). Such questions have limits and, as discussed in Wylie and Wiliam (2006), using the results of diagnostic questions assumes that conclusions about the cognitive processes of students can be inferred. Wylie and Wiliam point out that the more focused the question, the more valid the inference.

Whilst I concur that the misconceptions at play should be investigated, I propose that to be able to address the misconception properly it is important to classify the reason for the error *from the point of view of the student*.

For example, a diagnostic question might be:

$$\frac{1}{2} + \frac{1}{3} = ?$$

With answers a) $\frac{2}{5}$, b) $\frac{5}{6}$, c) $\frac{2}{6}$, d) $\frac{1}{6}$

All four answers provide a plausible method for adding fractions with answer a) having a level of cognitive ease that mean many students will select it. But if the student selects answer a) what has been learned about the student's thought processes? Assuming that a student does not just guess and there is a positive choice of answer made the question allows us to know what error has been made but not why. If the student selects a) then it might be thought that the student

believes that to add fractions, the numerators and the denominators are added. At this stage, this is no more than a working assumption. To know for certain more similar questions would need to be asked or the student would need to be talked to. The issue is that to address the misconception it is important to be able to identify why the student has applied a misconception as well as what that misconception is. I propose a classification of errors, not based on what the misconception is, but instead on how it has formed. This classification of errors will attend to the intention of the student.

Considering our diagnostic question above, students who do not know anything about adding fractions are likely to pick $\frac{2}{5}$ because this will fit in with their model for addition, thereby applying the necessary properties of integer addition, erroneously to the addition of proper fractions.

7.1.5.2 Classification of errors

Here, I refine the classification of other writers to include the intent or conscious awareness of the student attending to the intentions. In all this there is assumed to be an intention to answer correctly. I propose the following classification which allows me to analyse the errors which I have encountered in the project so far.

- Careless slip
- Type 1 error
- Intuitive error
- Structural error
- Assessment error
- Cognitive difficulty

7.1.5.3 Careless slips

Careless slips happen often, they may be due to a number of factors e.g., stress, being interrupted, reading the question incorrectly. The distinguishable feature of slip is that it is not predictable. When the student sees the error, they can immediately correct it. For example:

Susie looks at her work and sees $16-12 = 5$. She immediately wonders how on earth she wrote that.

There is little more that needs to be said about careless slips in this work. The student knows the response is wrong and no more teaching is needed to alleviate it. There are careless slips which appear predictable. For example, when multiplying the binomials $(x + 2)(x + 3)$ students often multiply 2 and 3 to make 5; $(x^2 + 5x + 5)$. The important thing to note is that classifying this as a careless slip is in response to the *student's reaction*. If the student makes this mistake repeatedly or does not see the error and immediately know it is wrong, then this would not be a careless slip.

7.1.5.4 Type 1 error

There are two types of error that follow from a feeling of cognitive ease, an intuitive response to a question. I have named the first of these *Type 1 errors* after Type 1 thinking in Kahneman (2012) and the second an intuitive error. They are similar, but not quite the same.

Type 1 errors are the result of an immediate answer giving the student a sense of cognitive ease. They are hard to shift because the feeling of cognitive ease is just there, even though conscious thought means that they can be overcome (Kahneman, 2012).

There are many errors which are seen regularly in many different classrooms and contexts that appear repeatable but look more like careless errors. The incorrect answers feel like they are correct and give the student a sense of rightness leaving them feeling content that there is no need to check their work. An example is 'simplify $2u^3 \times 3u^4$ '. A question similar to this is set in many Edexcel GCSE papers from 2008 to date. As a teacher, I have noticed that the answer often given by students is often $5u^7$. This is not what happens in a lesson solely about simplifying algebra, but does happen when reviewing exam papers or doing exams. Here the type 1 answer, the simply obvious choice that 'has to be right' and requires no thought, is that you add numbers. In a lesson on manipulating algebra the student is likely to be more attuned to noticing the difference between numbers that are indices and numbers that are coefficients. Because they are answering a number of similar questions, they are more likely to be attending to the detail. As a teacher, my experience tells me that this is also more likely to happen with the numbers 2 and 3, where the product and the sum are close. The distinction between the type 1 error and a careless slip is that the type 1 error is reproducible and has some misapplied logic or structure. The distinction between a type 1 error and an intuition error is that when discovered the response by the student is likely to be 'why did I write that?'. In other words, when a cognitive light is shone on a type 1 error the student requires no more guidance to see the error or correct it.

An example of a type 1 error from the study would be where students self-corrected swapping 0.4 with 0.04, just by the teacher drawing attention to the question (see section 6.4.10).

7.1.5.5 *Intuitive error*

The second error that results from cognitive ease is the *intuitive error*. This has many of the hallmarks of a type 1 error, they are predictable, it gives the student a feeling of cognitive ease and certainty and the answers are immediate.

The difference between a type 1 and an intuitive error is the reaction of the student when they see it. When a student sees a type 1 error, they have a similar response to seeing a slip, but an intuitive error still feels right until they have worked through some cognitive process to produce a cognitive conflict. A student might have a perfectly good method for solving the question but having jumped to an intuitive answer they did not bother to carry out the method. When reviewing their error initially they might not spot the error. Only when they have had to work through some process would they find the error. An example of this might be adding fractions $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ is an intuitive error. It feels right to the student. If the student represented the fractions in a drawing or converted each of the fractions on the left to fractions with a common denominator, they might spot the error. Intuitive errors are hard to shift, they are sticky. Stavy *et al.* (2006) talk about the persistence of intuitive errors and how these errors resist formal learning contradicting it.

7.1.5.6 *Structural error*

Students are introduced to mathematics in layers. They will learn about one instantiation and then may learn about the underlying structure. Having learnt about arithmetic on the natural numbers they will then be introduced to the integers, then the rationals then the reals. They will be taught about addition before learning about subtraction. If they apply what they have learnt in one instantiation to another it might work. If they are dealing with an essential property then they will be able to transfer that knowledge. On the other hand, if they apply a

strategy that relies on a property that is not an essential property then they will create a structural error.

Addition is commutative, subtraction is not. Dan recognises that $3+4=4+3$ and so erroneously extends that thinking to $3-4$, believing that $3-4$ is that same as $4-3$ and so writes that $3-4=1$. This would be an example of a structural error.

7.1.5.7 Errors made during Assessment tasks

It is not clear if errors in high stakes testing would be reproduced in “real life” (Ryan & Williams, 2007, p. 14). Assessment errors are those occurring during high stakes tests under high pressure. They are potentially relevant for the study because there are tests in the methodology. However, the errors and misconceptions which are being analysed are generally those where the students are working in a lesson or talking to me in an interview, and so I will not consider assessment errors further.

7.1.5.8 Cognitive difficulty

There are students who have cognitive difficulties, such as dyslexia, dyscalculia or many other SEND needs. At the study school around 29% of students will have some form of SEND need and, as such, this will impact on the learning and the teaching in the classroom. In the case of making an error, the errors may well be predictable if a student has cognitive difficulties. For example, students with dyslexia may not spell accurately, those with dyscalculia may have difficulties counting in twos. Overcoming these difficulties is beyond this project but students with these difficulties are in the classroom and can still be taught with the same ideas, so it is good to be aware of their needs. Tayla in intervention 1, needed one-to-one intervention and I needed to be aware of the questions I was asking Jenny in intervention 2 (see section 7.3).

7.1.5.9 Examples

In this section I give examples of classifying errors from the perspective of a student:

Case A. Jimmy writes in an exam that $4 \times 3 = 9$. Jimmy knows his tables well. The moment he sees his paper he groans; this was a slip.

Case B. Timmy frequently writes in an exam that $4 \times 3 = 9$ although he knows his tables well. The moment he sees his paper he groans; this was a type 1 error. The distinction between this and Case A was the fact that Timmy frequently makes this mistake. He does not truly believe that $4 \times 3 = 9$ but he often makes that particular mistake. Subjectively there is some cognitive ease surrounding the statement, but the moment he looks at the situation he knows that it is wrong. It maybe that there is an underlying reason that Timmy often makes this mistake. It may or may not be based in logic. An example outside of mathematics is where someone frequently calls someone the same wrong name. They know the person's name but use the wrong one much of the time.

Case C. Jocie has been learning about how to order decimals. She has been taught to use the number line to place the decimals and use that to order. She is asked which is bigger 0.37 or 0.4 and immediately answers 0.37. On further investigation the teacher asks Jocie if there is a way she can check. Jocie uses the number line and finds her error. This is not a type 1 error because it did not immediately become clear to Jocie that she was wrong or why she was wrong. To find the error she had to apply her knowledge and, in some sense, create a cognitive conflict. The reason this is an intuition and not type 1 is the learner's response.

Case D. May is squaring a binomial $(x + 1)^2$ her answer is $x^2 + 1$. How this error is classified depends on May's response. This could be a one off, a slip, however, it is a common error that is reproducible in other learners. Assuming that May makes this mistake more than once, then the classification is dependent on her reaction. If she looks and immediately spots the mistake then it would be a type 1 error. If, when asked to consider the problem as two brackets, she expanded the brackets to the correct answer then that would be an intuition error and if she were unable to expand the brackets it is a structural error. May has perhaps overgeneralised her knowledge of multiplying. There is a logic in her application. She has squared both the x and the 1. This though, is not the correct structure for this question.

7.1.6 What to do in the classroom

Slips are possible at any stage, by anybody in the classroom, student or teacher. Slips are not particularly relevant to the purposes of this study. Slips by their very nature, have no underlying cause. If the slips repeated, or were predictable then they would not be slips, they would be type 1 errors.

Type 1 errors are interesting. It could be that this is something stuck in the student's head; like when someone always calls someone the wrong name or remembers one number fact incorrectly. Again, this is not necessarily within the scope of the project. On the other hand, a type 1 error could follow from an intuition, that maybe the student has almost overcome. For example, they write down $\frac{1}{3} < \frac{1}{4}$ but immediately look at it and realise their error. The teacher would need to bring the student's attention to this and no more.

An intuitive error would not be overcome merely by attention being brought to the question. It feels of little value just to explain that the answer is wrong. In this case

the student would need to work through, either building up the answer from what is known, or, if appropriate, by activating cognitive conflict.

Structural errors can be situations where the structure has not been taught yet, in which case the question is not appropriate. Or it could be where the structure has been misapplied and, in this case, similar to above, the teacher will either have to help the student to build up the answer from what is known, carefully using the necessary properties of the instantiation in questions, or again use cognitive conflict to show that the properties which the student has used are not, in this case, necessary.

In all of this, the intuitive solution could be correct, and often is. The problem is that it will feel correct to the student, *whether or not it is*. The student, actively thinking, would be less likely to pick the incorrect intuitive answer but appears almost impossible for the student to know that they have selected an incorrect intuitive answer. There is certainly a case for ensuring that students are aware of intuitive answers, though, as discussed above, students, knowing that they have a tacit model, still had a gut feel for an incorrect solution. 'I know that the answer is A but I want to say B'.

It seems that an important part of this project is how to overcome this, how to change embedded intuitive thinking. Stavy and Tirosh (2000) suggest various ways to teach to overcome errors caused by intuitive rules

- Teaching by analogy. First students are given an “anchoring task” which is unlikely to trigger the incorrect intuitive rules and provides students with a way to avoid the incorrect intuitive response, then further “bridging” tasks, which are increasingly likely to trigger the incorrect intuitive response then finally a target task (Stavy & Tirosh, 2000, p. 90). This approach builds up a

correct response by first avoiding the incorrect intuitive response. This is similar to Fischbein's (1989) suggested response to the overuse of tacit models – where he suggested the use of metacognition to turn a light on the fact that a tacit model exists.

- Conflict Teaching approach – first students are given a task known to trigger an erroneous intuitive response. Then a task is presented which will give the student a contradictory response. This can be split into
 - Concrete evidence
 - Different representations
 - Extreme cases
 - Attention to relevant variables. (Stavy & Tirosh, 2000).

This is similar to the suggestion in Swan (1983), who compared conflict with positive only approaches

Teaching by analogy, in some sense, appears to be the inverse of teaching by conflict. Teaching by conflict starts with engineering situations likely to provoke an intuitive response which will provide students with a potential conflict that can be activated, whereas teaching by analogy starts with situations which are engineered to avoid intuitions. It builds up a series of tasks which are increasingly more likely to trigger an intuition. The build up of tasks is intended to ensure that students become competent and are less likely to rely on the intuition when faced with a task that would have been likely to trigger an intuition without the build up of tasks. An example might be to ask a student with a longer-is-larger whether 0.34 or 0.4 was larger to trigger their intuition. If instead, the lesson were to be taught by analogy the teacher could ask which was larger 1.34 or 0.3 and then carefully selecting pairs of decimals that the larger decimal can be found using the same strategies as the previous pair until the student is confronted by selecting the

larger between 0.34 and 0.4. The idea is that the intuition would be less likely to be triggered because of the bridging tasks.

Allaire-Duquette *et al.* (2019) and Babai *et al.* (2015) found that warning learners about the likelihood of an intuitive result helps to reduce the intuitive interference. Similarly, Babai *et al.* (2015) found that an intervention which specifically warned students about intuitive interference produced more accurate results than in the control group, though it did not eliminate the intuitive interference altogether. Allaire-Duquette *et al.* (2019) discuss using this warning intervention alongside a cognitive conflict approach.

There are then three approaches to combatting intuitive errors that can be considered in the classroom

- Cognitive conflict
- Analogy
- Warning about the intuitive errors

It is very important to note that no judgement should be attached to an error. A teacher recognising an error can use it as a diagnostic tool rather than a judgement on the student.

7.1.6.1 *Three ways to use structure to teach.*

Cognitive Conflict

As well as being a possible way to combat intuitive errors cognitive conflict is also one of the ways to use structure to teach. When using the approach of cognitive conflict, students are faced with contradictory situations which shine a light on the fact that one of their beliefs must be false. Swan (1983) compared a cognitive conflict

approach with a 'positive only' approach to teaching about decimal place value and found that the cognitive conflict approach produced longer lasting results.

Imagine, for example, that a student holds a longer-is-larger misconception and so, asked whether 0.345 or 0.42 is bigger, they will select 0.345. They have applied the logic that 345 is larger than 42 and so 0.345 is larger than 0.42. At this stage a teacher can use structure to create cognitive conflict. That might be by using an alternative representation for example the decimal or the number line. The student will see that both of the answers cannot be true, and the teacher can intervene to help the student sort out the conflict. If the alternative representation has a stronger influence on the student, then further intervention from the teacher might not be needed. It is important to note that the existence of the contradictory images is not, in itself, enough. Zaslavsky (2005, p. 318) discusses "people living at peace with mathematical inconsistencies". Zazkis and Chernoff (2008) warn that conflicts are potential and may not be effectively invoked. It is in this case that the teacher will need to craft the episode carefully so that the conflict is apparent and activated. It is also not the case, in my experience, that this will necessarily have a long-lasting effect, particularly if it is overcoming embedded intuitions. This is exemplified in the fact that after the first cycle where cognitive conflict tasks were used the students reverted to intuitive understanding by the time of the post-test and the interview.

Comparison

By comparing instantiations, the strength of connections can be increased. In this sense the students will be introduced to the essential properties which hold within each instantiation. Students are usually able to collect like terms and simplify $3x$ plus $2x$ as $5x$. But an indication that the strength of understanding is not secure

comes from the fact that that the same students will consider that $2x^2 + 3x^2 = 5x^4$ or that $\frac{2}{7} + \frac{3}{7} = \frac{5}{14}$ but by moving from one to the other structure can be more fully embedded. In this case considering different instantiations of addition (addition of algebra and of fractions) there is an essential property that what is being added does not change. In terms of how it might be understood by a young student: 2 of anything added to 3 of the same thing, will give us five of whatever we were adding.

Contrast

It is also important to contrast, to show that the necessary properties of one instantiation does not necessarily translate to a necessary property of another instantiation, this property is not an essential property across these instantiations. For example, when considering addition and subtraction, addition is commutative and many young students knowing this will overgeneralise and will apply this property to subtraction. It is very important, therefore, to show the differences in instantiations for example, by contrast $2 + 1 = 1 + 2$ but $2 - 1 \neq 1 - 2$.

7.2 Applying the literature to the mathematical concepts in the study

Students arrive at the classroom with an existing set of conceptions. It would seem that the word preconceptions would encompass this. However, there are negative connotations that make me uneasy to use this term. Instead, for the following I will use the term ante-conceptions. Ante-conceptions are the entirety of all the concept images that students have before arriving in the classroom. For each given area the ante-conceptions might align with the concept definition or may be a misconception.

Sierpinska's suggestion was that understanding gives learners a sense of "it fits" (Sierpinska, 1994, p. 32) (see section 7.1.1.2). This is subjective and learners can hold this sense of completeness whether or not their understanding is what would be generally accepted. This could be described as Kahnemann's sense of "cognitive ease" (Kahneman, 2012, p. 60). This could then be why intuitions are so sticky. Students have a sense of cognitive ease and therefore believe they have full understanding. This also offers an explanation as to why students resist change (see section 7.1.1.3)

If understanding is a sense of completeness as suggested by Sierpinska a 'cognitive ease', then the job of learning is to create that cognitive ease. If I am teaching Danish to students and I explain that the word 'small' in English is 'lille' in Danish, this should pretty quickly fall into the category of things that can be understood. If, though, I explain that the word for 'poison' is 'gift' this might cause more problems as the students will already have a meaning assigned to the English word gift. When teaching mathematics, if there are no ante-conceptions which interfere with the concepts being taught then the job of learning is merely to produce that sense of ease. However, where the ante-conceptions include misconceptions then the job of learning is not only to create a sense of cognitive ease for this concept but also for this to be strong enough to replace the ante-conception. This is particularly difficult where the ante-conception is intuitive. Often the child is not able to fully articulate such an ante-conception.

From reviewing the literature around intuition and tacit models, one thought I have is around the infinite nature of the decimal representation and the number line (Fischbein, 2001). One issue with teaching students about ordering decimals is infinity. You can, as the National Curriculum (Department of Education, 2013b) expects, look at decimals of the same length and students will be able to explain

that 4.5 is bigger than 4.4. This, though, can be seen by looking at ordering whole numbers. To become truly fluent with ordering decimals and to have an understanding, that is to consider this a cognitively easy task, where ordering decimals is automatic rather than the application of an algorithm, students need to appreciate the infinite nature of the natural number line. For example, they need to appreciate that the sequence 2.1, 2.11, 2.111, 2.1111 , is infinite and that each 'extra' 1 is ten times smaller than the previous one, that this sequence can continue forever, but that at the end of the day every single term in the sequence is less than 2.12.

I am looking at this project through the lens of understanding the logic of the learner trying to sense make of the mathematics they are engaging with. It is important to note that in this there are at least three different notions. There is the *concept under investigation*. This is the one as defined and generally accepted by the wider mathematical community: the concept definition (Tall & Vinner, 1981). There is also the *concept image of the student*. This is the result of all the student's experiences of this concept to the point of the lesson (Tall & Vinner, 1981). However, it needs to be remembered that teachers themselves may not hold the full concept definition and so the *concept image of the teacher* also needs to be considered. This is particularly important when trying to unpick prior learning. Primary school teachers in the UK are not subject specialists in all the subjects that they teach and although they will have a good understanding of what they actually teach they are less likely to have the knowledge of the layers of structure beyond what is being taught. An example of this is that primary school teachers are likely to consider that it is not possible to find the square root of a negative number. Indeed, secondary school teachers are also likely to believe that all multiplication is commutative.

7.3 Intervention 2

7.3.1 Overview

With intervention 1, I had intended to show that by teachers and students engaging with different instantiations, students would be more confident and competent with working with the magnitudes of decimals and fractions. Although the students were able to demonstrate that they could order decimals and discuss magnitudes in the lessons, the post-test and the interviews showed that many reverted to previously held intuitive concepts.

The purpose of intervention 2 was to investigate these intuitive understandings and to see the effects of adapting teaching to specifically address them.

Intervention 2 took place two years after intervention 1. I selected a similar class, a Year 7, set 3 (out of 3) class. Because this class had been in Year 6 at the time of the first lockdown due to the pandemic, they had not taken KS2 SATS and so there was no data available to us about their attainment. Because of this, the year group was taught in mixed attainment groups until the October half term (approximately seven weeks). Students were placed into sets at the half term break in late October. This was done using the results of an assessment undertaken by all students in the year group as well as teacher opinion.

The intervention itself was shorter, four lessons, and the focus was refined to the magnitude of decimals, although I still used the instantiation of fractions in the intervention, there was not an intention specifically to ask students to order fractions. Again, I used different instantiations of decimals; the notation, the number line and the decimats. I ensured that both the continuous and the discrete number lines were used. I also incorporated the strategies to prevent students from relying on intuitions by conflict and by analogy (Stavy & Tirosh, 2000) and by

warning students about possible misconceptions formed by intuition (Allaire-Duquette *et al.*, 2019; Babai *et al.*, 2015).

7.3.2 The tasks

Because intervention 2 was shorter and more focussed, I selected tasks from intervention 1 which, on reflection, best suited the new focus. Each task was adapted, in light of what I had learned from intervention 1 and from the review of literature as well as in line with COVID restrictions. Each alteration is discussed within the task description.

Task	Description	Purpose	Instantiation
Decimal Comparison Test	Taking Steinle and Stacey's DCT (Steinle & Stacey, 2004).	To gauge whether the students hold misconceptions at the start and the end of the intervention	Notation
Number line	Placing decimals	To be able to place decimals on a number line (Siegler <i>et al.</i> , 2011)	Notation Number line
Decimat (Roche, 2005, 2010)	The students were introduced to the decimat; a pictorial way of representing decimals.	To be able to represent a decimal by tenths, hundredths, and thousandths – to be able to compare decimals up to three decimal places	Notation Decimat
Practice	Moving between decimat, notation and the number line using an increasingly challenging worksheet.	To be able to represent decimals on the decimat and the number line	Notation Number line Decimat
Maths 4 life	Comparing decimals, via a number line	To be able to order decimals via a number line. The task specifically requires students to move between the instantiations.	Number line Notation Decimats

Table 7-1 tasks in Intervention 2

7.3.3 Decimal Comparison Test

In intervention 1, the students undertook the Decimal Comparison Test (DCT) (Steinle & Stacey, 2004). The purpose of the test is that it gives an indication of likely misconceptions following a short set of 30 questions. It showed me that most of the students had held misconceptions around ordering decimals at the start of the intervention. Because there was already a post-test in intervention 1, I had not asked the students to complete the DCT at the end of the intervention. During intervention 1, students were able to correctly order decimals without intuitive misconceptions, but by the time they were tested and interviewed they were displaying misconceptions and utilising intuitive responses. Given the fact I had no

pre- or post-test planned for intervention 2, I decided to repeat the DCT at the end of the intervention to give an idea of any immediate gains or changes in thinking. The results of the test also informed my selection of who to interview (see section 7.4.1). Longer term gains or losses would be considered in the interview, the DCT would help inform me about what gains had, or had not, been made over the term of the intervention. Although Steinle and Stacey (2004) found that there are gains made by students just by sitting the DCT again even without any review or feedback, the gains were not significant, so I still felt it would be useful to see if there were any changes in the misconceptions held.

The students took the test both at the beginning and the end of the intervention. I did not give any feedback on the task and did not give any support to the students whilst they were completing the task. Even with the improvement phenomena, the DCT indicates that most of the students who took the DCT at the end of the intervention were apparent experts compared to only three at the start. Students were selected for interview to provide a cross section of those who had changed their understanding and those who had not.

Not all DCT tests result in a Steinle and Stacey classification; results can be inconclusive. The classification and the number of correct answers out of the number of questions answered are recorded below in Table 7-2. Not all students completed the DCT at the end of the intervention, because they were absent on the day. It can also be seen that some students started but did not finish the test.

Student	Before Intervention DCT classification	Number of correct answers	After intervention- DCT classification	Number of correct answers	Interview
Ryan	Inconclusive	24/30	Inconclusive	27/28	
Max	apparent – expert	28/30	apparent – expert	30/30	
Georgina	apparent – expert	17/30	Inconclusive	5 /8	
Annie	inconclusive	20/30	apparent – expert	30/30	
Jack	shorter-larger	14/30	Did not sit		
Abby	shorter-larger	20/30	apparent – expert	29/30	
Theo	longer-larger	20/30	Did not sit		
Oscar	apparent - expert	30/30	apparent – expert	28/29	
Grace	shorter-larger	11/30	longer-larger	18/30	absent
Jenny	longer-larger	21/30	shorter-larger	6/10	Yes
Rory	longer-larger	19/30	Inconclusive	8/10	
Brooke	shorter-larger	12/30	Did not sit		
Olivia	shorter-larger	12/30	longer-larger	16/30	Yes
Charles	shorter-larger	13/30	apparent – expert	19/20	Yes
Anya	shorter-larger	18/30	apparent – expert	29/30	Yes
Ayden	Inconclusive	19/30	apparent – expert	28/30	
Phoebe	longer-larger	16/30	apparent - expert	15/30	
Jacob	longer-larger	18/30	apparent - expert	28/29	No permission received
Adam	did not sit.		apparent - expert	20/26	

Table 7-2 DCT results

7.3.4 Number Line

The aim of the number line task was for students to be able to place any decimal up to at least three decimal places on the number line. By being able to do this, students are able to compare magnitudes and appreciate the structure of the number line instantiation of the real numbers.

In intervention 1, I had started with the human number line, then progressed to number lines on white boards and on paper. The human number line was not a choice that was available to me for this intervention, due to COVID restrictions that

students needed to be sitting in rows facing the front with minimal contact between staff and students.

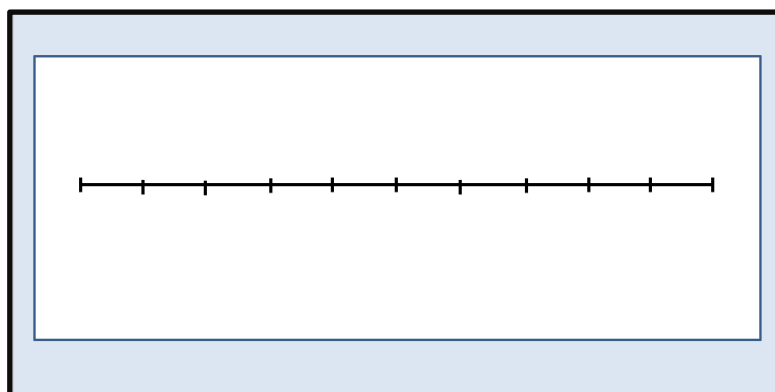


Figure 7-2 IWB Number line

Instead of the human number line, I drew a number line with 10 pips with no numbers marked on. All the students had whiteboards. I asked what was at the 5th pip (and pointed to it). The students responded to the question by writing their answer on their whiteboard. One student said 25, one said -5, 2 said 0.5 the rest said 5.

Me: why have you said 5? What do you think each mark is?

Student 1: each mark is 1

I talked to the student who had written 25

Me: why have you put 25?

Student 2: each mark is 5

I repeated the question but this time I marked in the ends as 0 and 1.

Most students replied that the 5th mark was 0.5.

Looking at the picture of the number line between 0 and 1, students marked in

$\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$ etc. Then 0.1, 0.2, 0.3. I asked the students what they could say about 0.3

and $\frac{3}{10}$. One student stated that they were equal and others concurred.

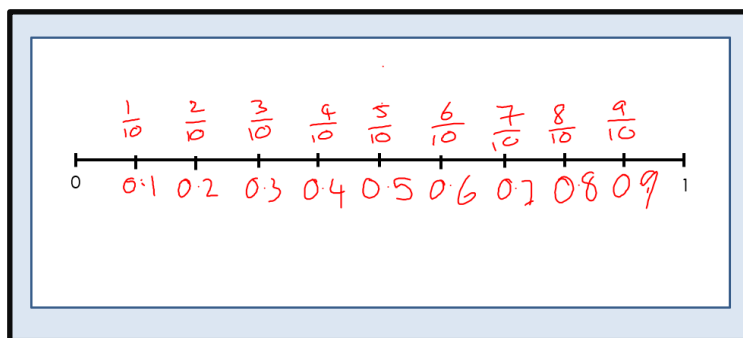


Figure 7-3 completed number line

By building up slowly from a position of security, I was teaching by analogy (Stavy & Tirosh, 2000). I was doing this slowly and carefully with the intent of avoiding triggering intuitive responses.

I asked the students which was bigger 0.241 or 0.34, and collected answers from students but then, following the advice to warn about misconceptions (Allaire-Duquette *et al.*, 2019; Babai *et al.*, 2015) the following interaction happened:

Me: Which is bigger, 0.34 or 0.241?

Anya: 0.34 is bigger than 0.241 because 3 is bigger than 2.

Me: Some people think that 0.241 is bigger than 0.34. Why might that be?

Jenny: I was going to say that

Student: they think 0.241 is bigger because 241 is bigger than 34.

Using a number, I asked the students where 0.01 should go. One student said that it should go to the left of the 0. I asked why and his answer was that 0.01 is smaller than 0 because it has two zeros. Although this reflects a finding in Steinle and Stacey (2001) it is only the second time as a teacher that I have been aware of this misconception; the first time is reported in section 9.5.3 below. Both times I have resorted to explaining by using money. This is not wholly satisfactory as I believe that it encourages whole number thinking. Certainly in this second case I would have been better off explaining it with reference to the decimal.

7.3.5 Decimat

I felt that in the first intervention I had not spent enough time using the decimat with the students. The students had appeared to pick up the idea very quickly. They had used them well to help them order decimals, however, the interviews showed me that it was not a tool they were reaching for, and they appeared to have forgotten the detail around using the decimat. I believed that they needed more practice and to use it in more situations.

Rather than asking the students to draw the decimat on their whiteboards, which is what I had done for the first intervention, I laminated some diagrams to create a mini whiteboard for each student to use. One side had the simplified decimat: a rectangle split into 10, and the other side had the full decimat. This allowed students to practise using the decimat and allowed me to see from a distance whether they needed more support.

I built up the concept of how to use a decimat slowly, working from a point where students would know the answer without reverting to any intuitive misconceptions. (Stavy & Tirosh, 2000).

I introduced the idea with a rectangle which represented 1. Then I split it into 2 equal parts; $\frac{1}{2}$ as each part.

I then moved onto the simplified decimat (Figure 7-4). I showed the students that each block was $\frac{1}{10}$ or was 0.1 and linked the concept with division. Students were able to show me 0.2, $\frac{4}{10}$ etc on their whiteboards. I found that one student (Jenny) seemed to have the idea but kept making counting errors. She was the only student not to realise that the decimat in Figure 7-4 was 0.6. She initially thought it was 0.7. I slowed her down and she counted carefully she realised it was not 0.7.

From this interaction, I assumed that this was a careless slip (7.1.5.3), however, I quickly became aware that this was a common theme with Jenny. A pattern arose that her answers appeared careless when there were numbers involved but were very careful when they required logical explanation. Although at the time, Jenny had no SEND needs identified, she has been tested since and it is likely that she has dyscalculia.

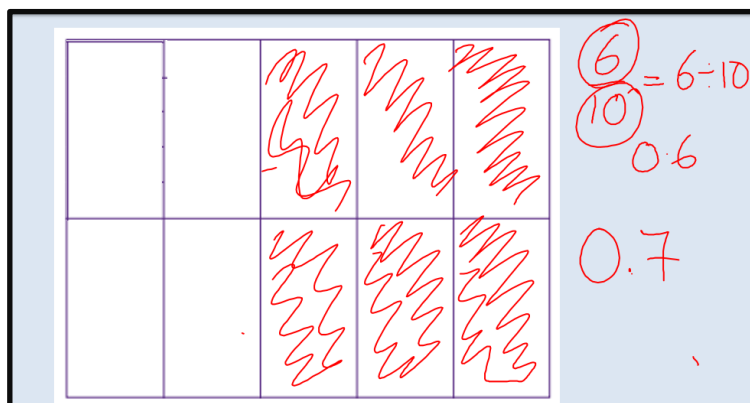


Figure 7-4 simplified decimat.

I moved onto the decimat with the top left region being split into 10. This was linked to each small region being $\frac{1}{100}$.

During the interviews following intervention 1, students had been able to recall the decimat, but they were as likely to mix up the decimal places, so when asked to represent 0.236, they were likely to colour in 2 small rectangles rather than 2 large rectangles. I decided that it was important to emphasize more the connection between the size of the area of the decimat with the size of the decimal, making sure that it was clear that the representation of $\frac{1}{10}$ was, for example, 10 times bigger than the representation for $\frac{1}{100}$. Although this had been part of the discussion in intervention 1, it had not stuck as a defining feature of a decimat. Because of this, I spent longer working with the students on this aspect of the decimat.

I wanted to compare the ways decimals can be ordered using the different instantiations. I asked the students how they would know which decimal was bigger if they had the decimat. One student replied that more would be coloured in. At the time I took this to be a good answer, on reflection, however, I could have questioned the use of the word 'more' to ensure that they were referring to the area and not the number of regions.

I asked the student how they knew a number was bigger when it was represented on a number line. One student answered that the larger number was further to 1. This is an example of a correct answer for one instantiation that may not be correct in a different number line. For example, if the number line had been from 1 to 2. My reflection on this interaction is that it was easy for me to assume that the student meant that being further towards 1 meant further along the positive number line. However, given the myriad misconceptions around understanding decimal magnitude it is important to follow up such answers, although in this case I did not.

I asked the students how they would know which decimal was bigger if they were looking at the number written on the page. One student answered that you would compare the whole number first then the first decimal place and so on.

I asked the students to show me different decimals on their mini decimats until I was satisfied all were answering accurately.

7.3.6 Practice

Having reflected on the results of intervention 1, I wanted students to engage in practice, using decimats and moving from representing decimals on decimats and number lines, so I created four progressively complex worksheets. These can be seen in Figure 7-5. This allowed students to draw the decimats and place the

decimals on the number line. I felt that this level of practice had been missing from the first intervention.

The worksheets built up from one decimal place on the first, two on the second, three on the third, to ragged decimals on the fourth.

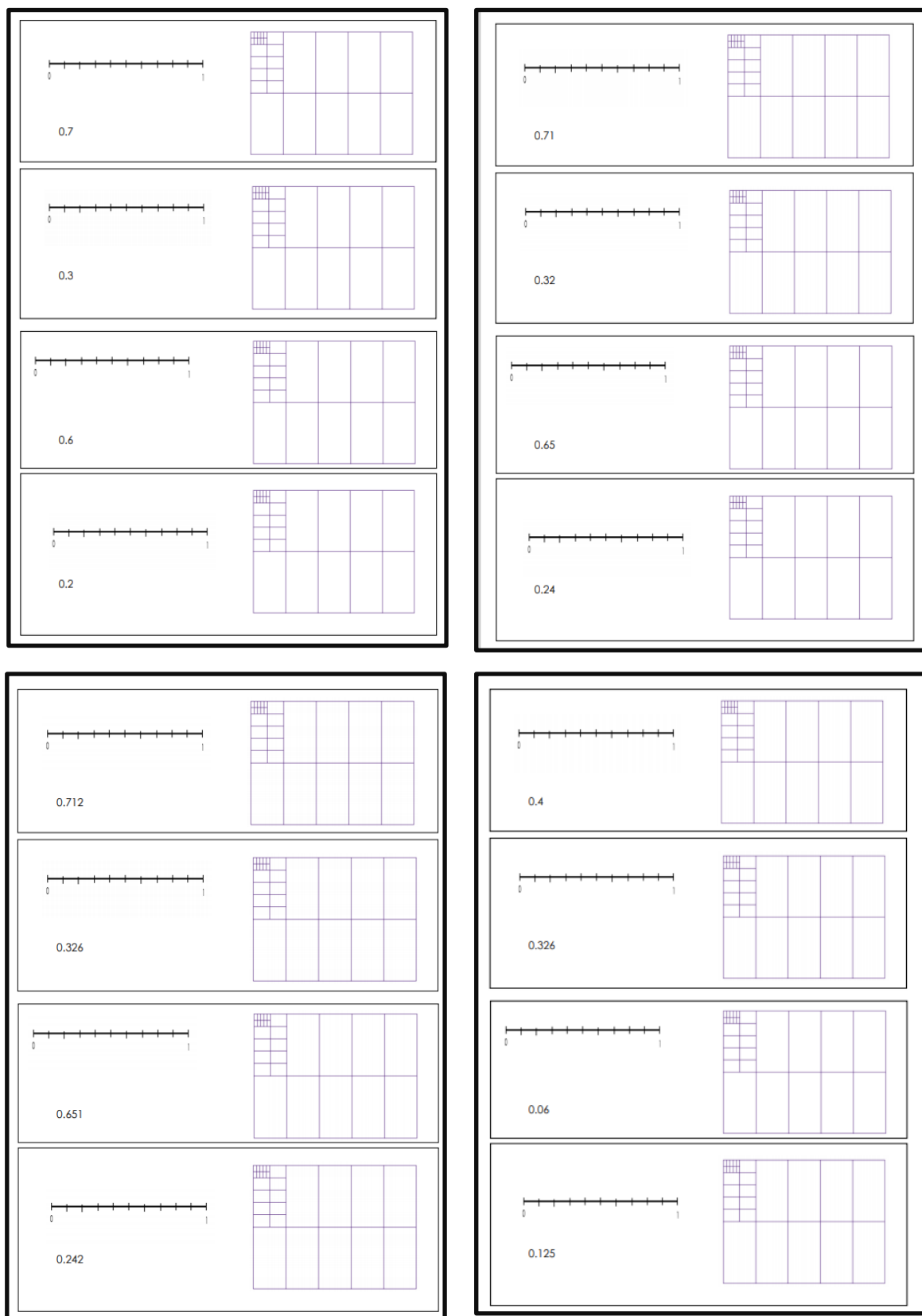


Figure 7-5 decimal worksheets

The students were all able to complete the worksheets; some very fast, though some needed a little support. Support was in the form of an adult questioning; asking students to recall how to fill in the decimat. No student needed more than a gentle reminder, and all completed the decimat part of the sheets correctly.

7.3.7 Maths4Life

I had used the Maths4Life task in the September part of intervention 1. It had initiated good discussions between students and had introduced discussions on some common errors, for example, whether $0.04 = 0.4$ and whether 0.4 or 0.25 was $\frac{1}{4}$. The task, as intended, had included a 100-square to give an area representation of decimals. In Intervention 1, I gave the students a sheet with blank decimats. Although they used the decimat, it felt like it was workings at the side rather than an integral part of the task and so, in this intervention, I decided to create an extra set of cards with a completed decimat to match to the other two instantiations. The decimats can be seen matched to the other cards in Figure 7-6.

The task was designed by Swan and is specifically designed to have potential cognitive conflicts (Swan, 1983). Whereas the previous tasks were more associated with teaching by analogy the purpose of this task was to include some teaching by cognitive conflict following the advice from (Allaire-Duquette *et al.*, 2019; Babai *et al.*, 2006; Stavy & Tirosh, 2000).

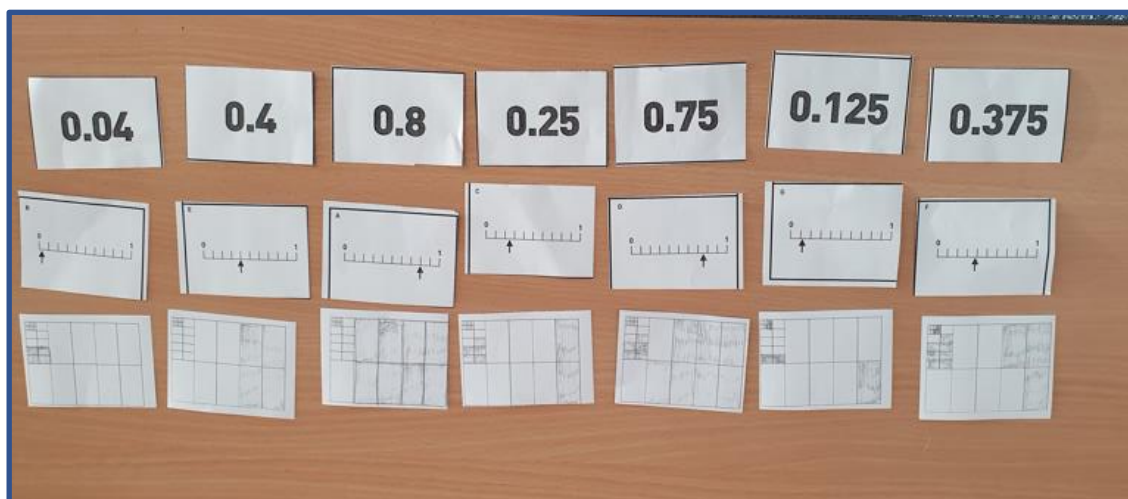


Figure 7-6 Maths 4 life - matched

Although students could not work in groups because of COVID restrictions, they were sitting in pairs and so were able to discuss their work. Each student managed to match and order the three instantiations. Some were able to do this independently and some required a little adult help. All matched the different instantiations first before they ordered them.

One student was struggling with 0.04 and 0.4. This phenomenon had also been seen in both the ordering decimals task and the ordering decimals and fractions task in intervention 1

Student: (pointing and the 0.4 and 0.04) are these the same?

Me: Are they?

Student: yes

Me: (pointing) what is this?

Student: nought point four

Me: what is this?

Student: nought point nought four

Me: Are they the same?

Student: No

Me: which one is bigger?

Student points at 0.4

Me: How do you know?

Student: because of the zero.

This is an example where this task can work with cognitive conflict. It should be clear that a decimal cannot be both 0.4 and 0.04. This, however, is only a potential conflict (Zazkis & Chernoff, 2008). It only becomes a cognitive conflict if it is invoked. As this was quickly resolved this was an example of a Type 1 error (see section 7.1.5.4). As students were working independently, I cannot say how often students' understanding was triggered, or helped, by cognitive conflict or if they resolved this conflict without requiring help. I do know, though, that many conversations were like the one above. The students either had an inkling that something was wrong and could not quite express why they knew that or had put down solutions which were inconsistent, however, they did not notice until I asked a question that highlighted to them the inconsistency and activated the conflict. I rarely had to do more than shine a spotlight on the inconsistency.

Whilst circulating around this task. I noticed that the students were very good at matching the number and the decimal: and using that could order the decimals. They were less happy to match the number with the number line, though I feel in this intervention I spent less time on the number line than I had with intervention 1. Sometimes, though, having matched with the decimal they were more able to match all three instantiations. Once they had matched, they were able to order the decimals. Where students found this difficult, they had adult support. This support was always in the form of a question such as 'how could you check this?' or 'what did we do when we compared these two?'. In general, they did not have a problem comparing any two individual numbers, but I did realise that there is quite a jump

from comparing two decimals to ordering a set of them. This jump had not seemed such an issue in the first intervention.

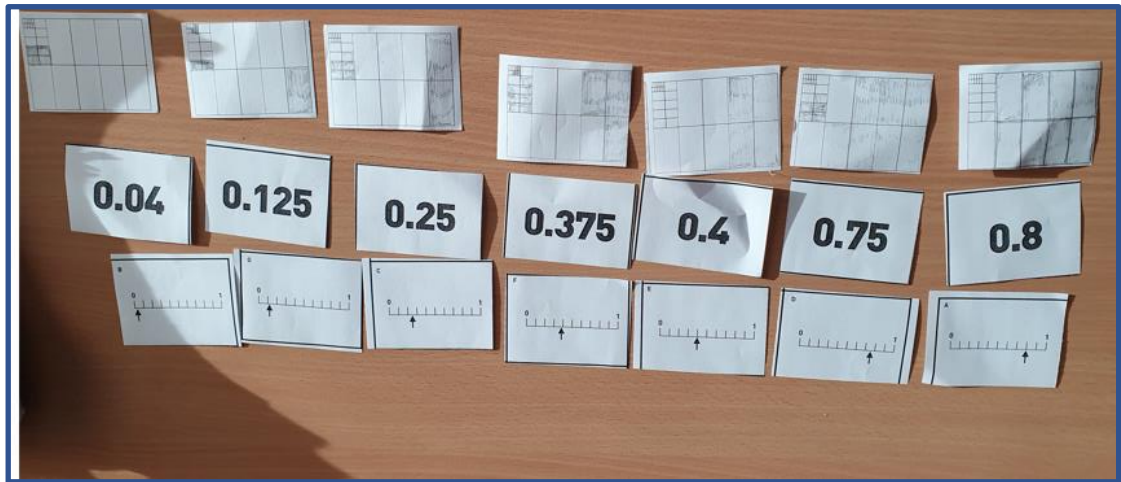
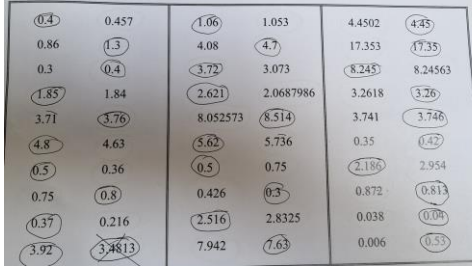
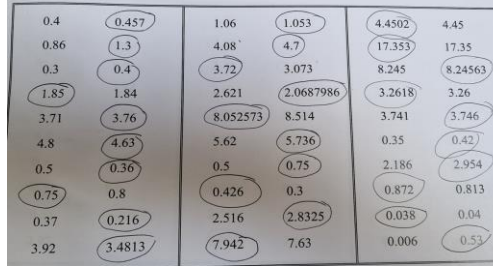
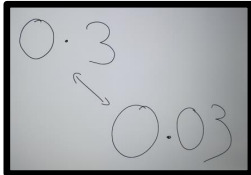


Figure 7-7 Maths 4 life. Matched and ordered

Illustrative Cases

For intervention 2 and 3, the illustrative cases are Anya and Jenny. Illustrative Case 7-1 details their DCT results, at the start and the end of the intervention along with observations about them in lessons.

	Anya	Jenny
	Decimal Comparison Test (DCT)	
DCT initial (before the intervention)	 18/20 classified as shorter-is-larger	 21/30 classified as longer-is-larger
	Second Intervention	
Class discussions	Anya was very keen to ensure she answered questions and appeared disappointed when she was not correct.	Jenny stated that a decimal was 0.7. With support she realised this was 0.6.
		Jenny was aware of why someone might believe 0.234 was greater than 0.3. She stated that this is a mistake she would make
Final DCT	29/30 classified as apparent-expert	Did not complete – however, for the 10 questions she did complete the likely classification is shorter is larger
		 When asked what 0.3 was Jenny asked if it was 0.03.

Illustrative Case 7-1

7.4 Intervention 2 interviews

7.4.1 Selection

I selected Grace, Olivia, Charles, Anya, Jenny and Jacob for the interviews. I received permission for all but Jacob. Although Jacob consented to be interviewed his parents did not return the permission slip. Grace gave her permission – but was then absent on the day of the interviews. This was the last day of term in December 2020 and on the first day of the next term schools were again closed to the majority of students. Grace was, therefore, not back in school until March 2021, too late to take part in the interviews.

My criteria for selection were that I wanted to select three students who had moved their understanding to that of an apparent expert and three who either retained the same misconception or moved to a different misconception. I wanted to include Jenny and Anya in the sample because of the way they had answered questions in class. Anya, being so keen to answer and so unhappy with an incorrect answer, and Jenny because of the contrast between how well she answered logical questions and how often she made mistakes when numbers or counting were involved. I had selected Grace, Jenny and Olivia because they displayed a different misconception at the end of the intervention from the start. I had selected Charles, Anya and Jacob because they moved from holding a misconception to being apparent-experts.

7.4.2 Interview Analysis

Student	Initial DCT	Final DCT	Interview findings	Learning
Grace	shorter-larger	longer-larger	Absent	
Jenny	longer-larger	shorter-larger	Jenny, in the lessons, showed good logic and was eloquent in her thoughts but she made frequent mistakes with numbers. It seemed more than slips. In the interview it is clear that decimal place value is not fully understood. She remembered the work from the lessons and was able to apply it with prompting. But longer-is-larger intuitions were strong. (longer-is-larger)	Intuition overtakes learning and Jenny's understanding in the interview indicates a longer-is-larger misconception.
Olivia	shorter-larger	longer-larger	Olivia had some significant gaps in terms of place value even with whole numbers – she was not sure how to add or subtract with carries or borrows – even though this work had been recently revisited with her class teacher. She had a shorter-is-larger intuition – which she could overcome with discussion and analogy. (shorter in larger)	Intuition overtakes learning and Olivia's understanding in the interview indicates a shorter-is-larger misconception.
Charles	shorter-larger	apparent - expert	Charles completed the pre-test work in the interview and got it completely correct. He knows the terminology of place value and can add and subtract. What he appears to be missing is the understanding of the magnitude of decimal places – he has his own logic the longer the decimal the smaller it is. He could remember the decimal and with some prompting was able to use them to compare sizes. (shorter-is-larger)	Intuition overtakes learning and Charles' understanding in the interview indicates a shorter-is-larger misconception

Student	Initial DCT	Final DCT	Interview findings	Learning
Anya	shorter-larger	apparent - expert	Anya is a child who appears to be very unwilling to make mistakes. In the interview, she had remembered how to compare numbers and had an efficient and correct algorithm for this. The language confused her (tenths, hundredths, and wholes) and she was not secure in place value. She knew that each number to the right was less important but did not know the naming of the columns. She remembered the decimat – and knew each smaller number was the next sized box – but not that the tenths were the biggest box. With a very quick reminder she was able to read the decimat. (apparent expert)	The interview indicates that the learning has been retained and Anya appears to still be an apparent expert but there was also clear evidence of some insecurities with decimal places.
Jacob	longer-larger	apparent - expert	No form returned	

Table 7-3 intervention 2 interviews

Only Anya retained the learning from the intervention. The other students reverted back to the intuitions that they displayed when they did the DCT at the start of the intervention. None of the students were secure in being able to discuss place value; they all had difficulties describing that each decimal place was 10 times the size of the one to its right. It seems possible that much of what they have learned in terms of decimals has been procedures that they have learnt by rote. The errors seen in the interview were mostly intuitive errors (section 7.1.5.5). An example is an interaction with Charles. He was asked to select the bigger decimal between 0.24 and 0.235: -

Me: I'd like you to do to start with is tell me which of these is bigger. Can you circle the bigger one and how do you know that's bigger?

Charles: (circling the 0.24) Because that's like going back in the wrong way.

Me: Say it again?

Charles: It's like going backwards.

Although 0.24 is the correct answer, Charles reasoning is not clear and so I asked him whether 0.21 or 0.225 was larger and he said 0.21. To decide whether to classify this as a slip, a type 1 error, an intuitive error or a structural slip, depends on how Charles overcomes the error. He did not immediately recognise the error; when a cognitive light was shone on it, he talked about numbers going backward. He was able however, to change his answer when I asked him to draw two decimals and answer the question that way. This is, therefore, most likely an intuitive error. This is perhaps what would have been expected given that Charles had been classified as an apparent-expert in the DCT and his answers reflect the intuitive misconceptions that he held at the start of the intervention.

7.4.3 Illustrative Cases

Illustrative Case 7-2 summarises the interviews for Anya and Jenny.

	Anya	Jenny
	Interview	
Interview	The interview indicates that the learning has been retained and Anya appears to still be an apparent expert but there was also clear evidence of some insecurities with decimal places.	In the interview Jenny was able to recall and apply what had been taught in lessons but she still displayed a longer is larger misconception.

Illustrative Case 7-2

7.5 The Effects of COVID-19 on Intervention 2

The second intervention had to take place under the most stringent classroom controls against COVID. Students had to sit in rows facing the front, teachers had to maintain a 2-metre distance from students, so for all but the shortest of interactions were not easily able to interact with them. I used mini whiteboards to ensure that I could interrogate what the students were thinking. It allowed me to select students to talk to who held misconceptions. Although I circulated round the room, I had to minimise the time I spent with students. Group discussions were limited and so, to counteract this, I brought in as much whole class discussion as I could.

The second interviews were in December 2020. For these I had maintained a 2-metre distance from the students, and I made sure students did not share resources. I had intended to follow this up with the third intervention in January. Unfortunately, the closure of schools on 4th January meant I had to delay this until after the reopening of schools in March 2021. Although there were still stringent restrictions on place by the third intervention, only having 4 students in the room meant that social distancing was more manageable, and students were able to work in pairs.

One less direct effect of COVID on the project at this stage was my own personal working pattern. My role in school includes being in charge of curriculum, assessment and timetabling. My workload at school increased during the COVID pandemic because, in addition to my usual roles, I was overseeing the remote learning, the roll-out of devices to students, the move to a cloud-based environment, as well as being in charge of the in-school procedures that the government introduced in 2020 and 2021 to decide grades instead of formal exams along with the quality assurance of these grades. All of these factors limited the available capacity for research.

7.6 Summary from Intervention 2

Ten out of the 16 students who sat the DCT were classified by the DCT at the end of the intervention as apparent–experts. Only three were given a classification that was not apparent-expert. These three, alongside three others, were selected for the interview.

The lessons and the DCT show that the students were able to talk about the magnitude of decimals and compare them at the end of the intervention. The interview however, only four weeks later, showed that three out of the four interviewed had reverted back to their pre-intervention misconception and none of the four interviewed were really able explain decimal place value.

A third intervention was planned to work only with the four students from the interview and to try to engage them with the magnitude of decimal and an understanding of decimal place value.

Chapter 8

8 Cycle 3

8.1 Intervention 3

For Intervention 3, I worked for two lessons with the four students from the interviews: Jenny, Anya, Charles, and Olivia. From the interviews, I had realised that the students did not have a good grasp of place value. Although they all had some algorithms allowing them to solve problems procedurally, the interviews showed me that these were not all complete and that the procedures were not always implemented accurately. The aim of the intervention was that the students should hold a better image of place value and magnitude of decimals. I had noticed in the interviews and previous intervention that the students were more accurate representing decimals on the decimat than judging where they might lie on the number line. I wanted to ensure that this intervention included Number to Position (NP) and Position to Number (PN) practice (Iuculano & Butterworth, 2011). This would require moving from the instantiation of the number line to and from the instantiation of the symbolic representation of the number, potentially with the support of the decimat.

8.1.1 Tasks for intervention 3

The tasks for intervention 3 were somewhat different from the ones in intervention 1 and 2. The intervention was intended to address the potential underlying misconceptions about place value as well as to work with the magnitude of decimals. I wanted to focus on the language used when discussing decimals and also use NP tasks (Iuculano & Butterworth, 2011). I still wanted to use the decimat as a visual instantiation of decimal numbers, but wanted a way for students to practise with these in a way where they could practise many times and have

instant feedback so that they knew that they had the right answer. This relates to the notion of building up confidence and competence by analogy (Stavy & Tirosh, 2000).

Task	Description	Purpose	Instantiation
Vocabulary	Discussing the words that are used to describe the value of digits in numbers	To give students a way to understand that for example 0.03 is 3 hundredths and 0.3 is 3 tenths.	Notation Words
Decimat	Reminding students about the decimat and linking to decimals, fractions, and the number line	To give the students a visual representation of decimals and to connect the sizes of the decimat sections with the relevant decimal place.	Notation Number line Decimat Fractions
NP task (Number to Position)	Placing decimals on the number line (NP – number to position)	Estimating the position of a decimal on the number line.	Notation Number line
Excel Decimat	Using an excel version of the decimat	To allow students to work at their own pace to compare instantiations and estimate and check the magnitude of decimals.	Notation Number line Decimat

Table 8-1 Intervention 3 tasks

8.1.2 Vocabulary

The aim of this task was to discuss with the students how decimals are named and how the language does not help with magnitudes of decimals in the same way that it does for larger numbers. For example, naming 0.0003 (zero point zero zero zero three) requires one to count the number of zeros to recognise the magnitude of the three, whereas naming 3000 as three thousand, the magnitude is in the name. In understanding decimals, the language can hinder the identification of magnitude. For 0.3 (zero point three) and 0.27 (zero point two seven) there is nothing in the words that gives a sense of size.

To appreciate the magnitude of a fraction e.g., $\frac{3}{4}$ it is necessary to attend to the numerator, the denominator and the relationship between them. If $\frac{3}{4}$ is called three-

quarters there is a sense of the magnitude in the name (so long as you understand the word quarter) whereas when called 'three over four' there is less of a sense of the relative size of the number. It also has to be said that the word 'half' does not convey a two-ness, nor does a quarter convey four-ness. As often happens in English the most commonly used words have an irregular feel to them. The word 'sixth' conveys a feeling of a shape being split into six. The American English 'fourth' is perhaps more helpful than the British English 'quarter' in this respect.

I started by asking the students what 0.3 was.

Me: (writing 0.3 on the board) What is this?

Charles: err nought point three.

Me: yes, but what is it?

Jenny: What is nought point three? err nought point three (spoken)

Me: OK yes but what is it (lots of silence and uncertainty).

Jenny: is it nought point nought three?

Charles: 3 tenths

In response to Jenny's reply, I put this on the board.

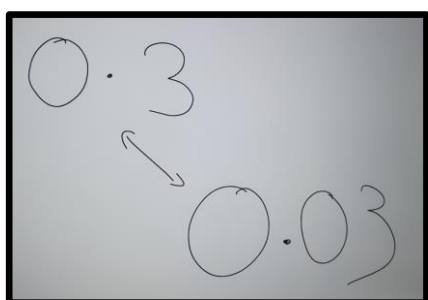


Figure 8-1 Jenny's response to what is 0.03?

Jenny realised that 0.3 did not equal 0.03 but she did not appear to have an answer that was more acceptable to her. After this, I discussed with the group the language around place value. So that the 3 in thirty-two is not the same as the 3 in three hundred and twenty-one, but that the name for the number takes that into account. I asked students what 3 stood for in various numbers in Figure 8-2. I

started by writing 3 and asking all four students what the number was. Then I wrote 31 and asked again what the 3 was. The students responded 'thirty'. With only four students it was relatively easy to ensure that all were answering the question.

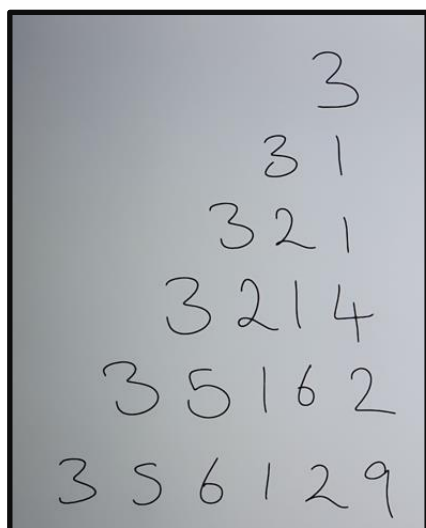


Figure 8-2 Different values of 3

As the students named the 3, 'three', 'thirty', 'three hundred', Charles noted that the 3 in each represented a number ten times bigger than the previous number. One interesting point was that all four students read the 3 in 35162 as 3 million. This is a phenomenon I have seen before in lessons. I have always considered that this is because the students are considering a sequence of words, rather than thinking that 30000 actually represents three million (see section 2.3.5).

Having considered 356129, I then asked about 0.3, what it was called, 0.03 etc. The students, including Jenny, were able to tell me the words to describe, in terms of decimals ('zero point three', 'zero point zero three' etc). The advice of Roche (2005) is to encourage students to use the language around the fractional representation to describe decimals so that 0.75 should be read as '75 hundredths' rather than 'zero point seven five'. In response to this I wanted the

students to be able to link the decimal with the fractional amount, so I asked them to compare 0.67 and 0.7.

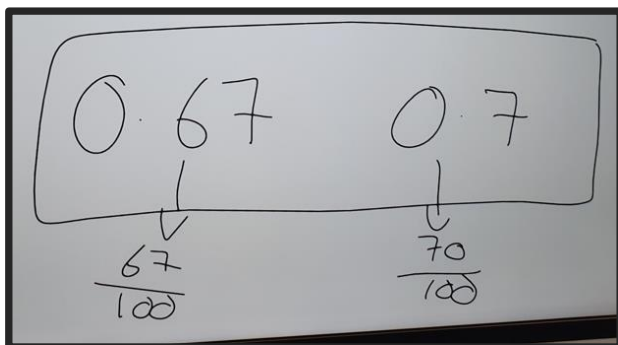


Figure 8-3 linking decimals with fractions

I wrote 0.67 and 0.7 on the board (see Figure 8-3) and asked the students to tell me what these would be in fractions $\frac{7}{10}$ and $\frac{67}{100}$. I asked the students to consider what 0.7 would be in hundredths and was given the answer $\frac{70}{100}$. Taking on the advice from research to warn students actively against intuitive responses (Allaire-Duquette *et al.*, 2019; Babai *et al.*, 2015) I asked them why someone *might* think 0.67 was bigger than 0.7.

Me: Why might someone say 0.67 was bigger than 0.7

Jenny: Because it has two numbers.

Here Jenny is displaying that she can recognise a longer-is-larger misconception, although there is nothing to say that she will avoid such a misconception. Jenny was the student who was able to recognise the possibility of the misconception in intervention 2 as well.

8.1.3 Decimat

All the students had been in intervention 2 and had all been able to use the decimat during the interview, the aim of this task, therefore, was to remind students about how to use the decimat and also to connect the decimat with the language that had been used in the previous task. I moved onto the simplified

decimat and then the full decimat. At each stage, I emphasized the equivalence of the fractions, the number line and the decimat representation and the precise language.

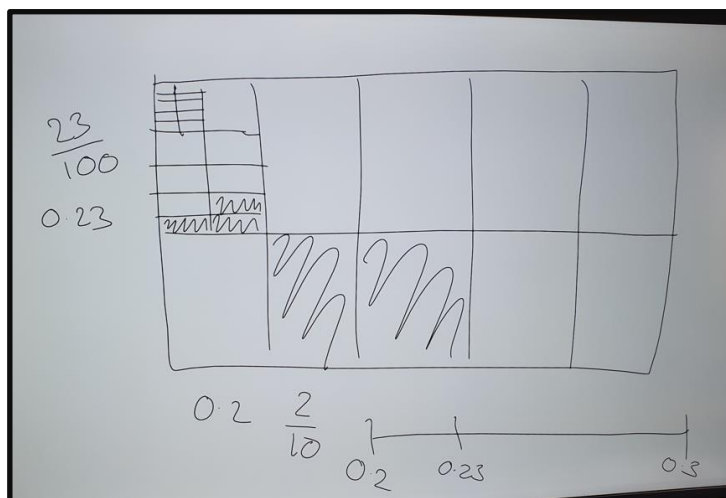


Figure 8-4 0.23

8.1.4 NP Task

I introduced the Number to Position (NP) task by using a discrete and a continuous number line and asking the students to mark where 0.4 would be (Iuculano & Butterworth, 2011).

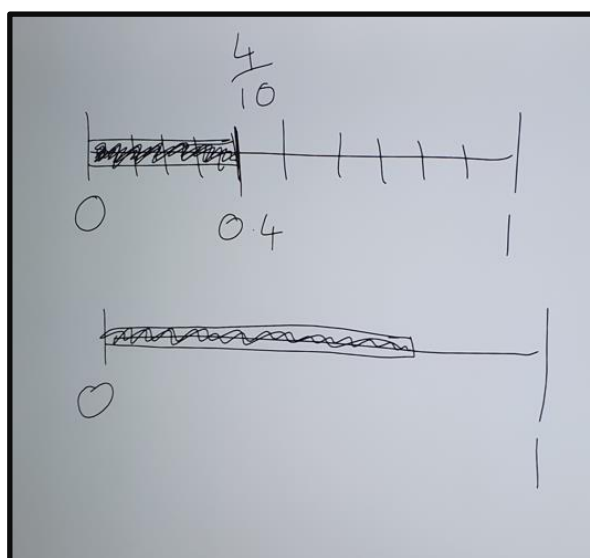


Figure 8-5 NP task

Even with the reference of the discrete line above it the students were less happy with the continuous task. All through all the interventions, unless marking $\frac{1}{2}$ the

students all seemed to use the same strategy when approaching the continuous number line. They split a continuous line into tenths before indicating where a number would be. This is one of the student strategies discussed in Siegler *et al.* (2011) for estimating the placement of a fraction on a number line.

I then asked the students to place 0.23 on the number line. The number line I gave them was from 0 to 1 with pips at each 0.1 but with nothing else labelled. The students each came to the board and marked their name by where they thought 0.23 was. Seeing a wide variety of answers, I decided not to try to unpick the answers. Instead, I labelled the pips with the number and their initial answers with the name of the student and then I asked them if they wanted to change their answers. The diagrams in Figure 8-6 represents the original answer (above) and the revised answer (below). The blue arrow represents Charles' answers, red Anya's, green Jenny's, and purple Olivia's.

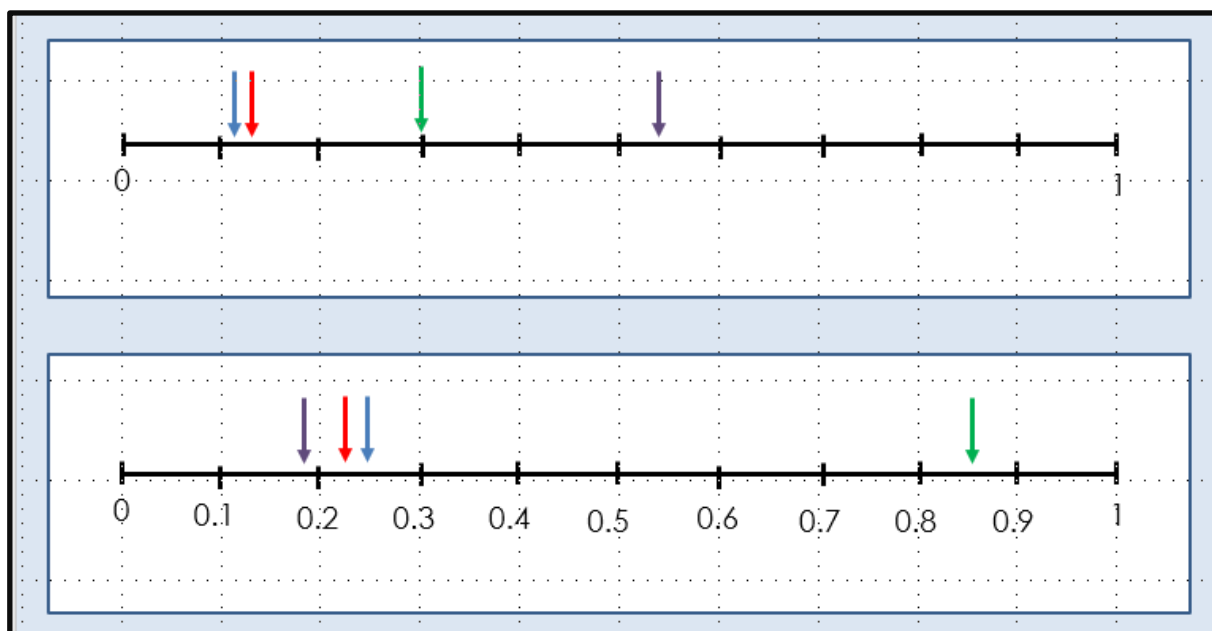


Figure 8-6 0.23 NP task

Charles and Olivia changed the answer to one that was close to being correct.

Olivia's answer improved from about 0.55 to 0.18. Jenny's answer went from being about 0.3 to being about 0.83.

8.1.5 Excel Decimat

For this intervention, I created a version of the decimat on Excel. A picture of this is at Figure 8-7. When a number is put into a designated cell (yellow) excel displays a decimat, a continuous and a discrete number line. I showed the students how it worked on the interactive screen. I gave the students a couple of examples then asked them to give me a number to put into the yellow cell. Before I hit enter to display the decimat, I asked them to tell me what they expected to see. Once I had done this a few times, I asked the students to work in pairs. One pair used the interactive whiteboard and the other pair used a laptop.

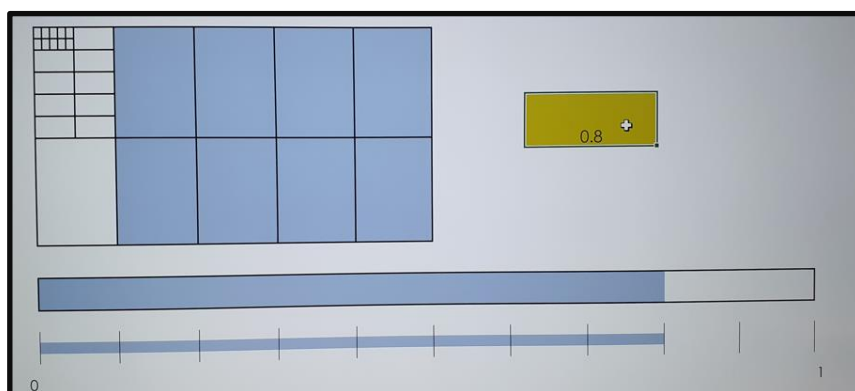


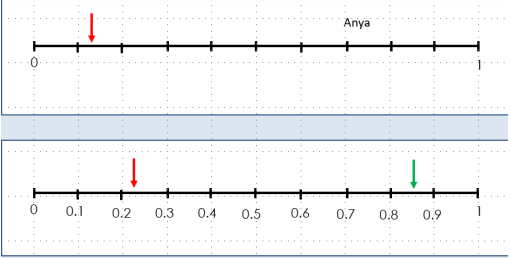
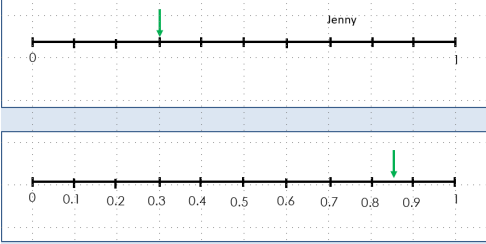
Figure 8-7 Excel decimat

One student in each pair thought of a decimal (up to three decimal places) and challenged the other to say what the decimat and the number line would look like. The first student either agreed or explained what they would expect to see and then they would use the Excel decimat to check. They then swapped roles. After a few minutes, I moved between the pairs to monitor what was happening. All the students were able to say how the decimat would be coloured. It took longer for the students to be able to say what the number line representation would look like. But, after another 5 minutes, all were able to tell each other. The advantage of this task over other tasks I had used with the students was that the students were able to select their own questions and that they were able to repeat the exercise many times in a short space of time whilst receiving instant feedback. The design

principle here was moving between the instantiations and multiple representations. The students received instant feedback, allowing them to form, test and reconsider conjectures quickly.

8.1.6 Illustrative Cases

Illustrative case 8-1 shows that Anya was able to place a decimal on a number line but Jenny was not able to.

	Anya	Jenny
	Third Intervention	
NP Task – mark 0.23 on a number line	 Asked to place 0.23 on the number line – first with unmarked then with marked pips, Anya has managed to mark it correctly. There were no comments about the original placements, the students were just asked if they wanted to change after the numbers were added to the diagram.	 This activity shows little evidence that Jenny is able to use the number line or that she has a clear understanding of the size of decimals

Illustrative case 8-1

8.2 Follow up questions

This intervention was intentionally very short. I had already seen that changes in one lesson do not translate to a long-term change in memory or application. I wanted to see if there was any longer-term impact on the students after this final intervention. For the four students who had taken part in the interviews I gave them a question twice between the end of the interview (December 2020) (Table 8-2) and the end of the school year (July 2021) (Table 8-3). I gave them a paper version of what they had seen on the Excel Decimat and asked them to represent the number. I did not explain the task to them or talk to them about it once it was finished.

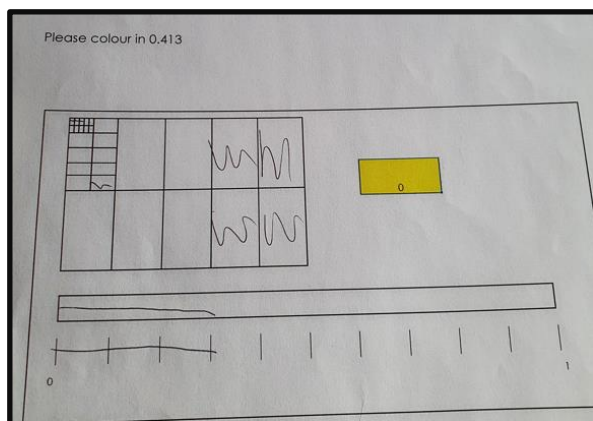


Figure 8-8 post intervention task

For each of the questions the results were as follows

0.312	Decimat	Number Line
Anya	Correct	Did not attempt
Charles	Absent	Absent
Olivia	Correct	Correct
Jenny	Correct	Did not attempt

Table 8-2 Follow up question results 1

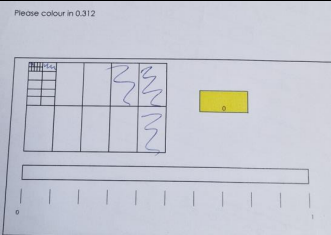
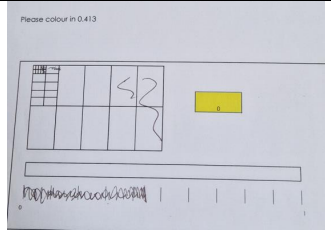
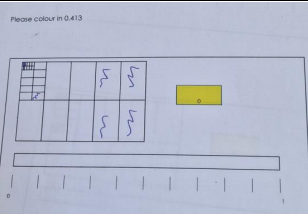
0.413	Decimat	Number Line
Anya	Incorrect	Correct
Charles	Incorrect	Close
Olivia	Correct	Correct
Jenny	Correct	Did not attempt

Table 8-3 Follow up question results 2

The format of the decimat has appeared to persist with the students. The number line is not so secure. At the end of the second intervention, they were all able to place a decimal on the number line but were not as confident doing this towards the end of the year. Even with procedures that allow the students to use the number line, this is still difficult. The decimat appears to be a more accessible representation for these students. It is closely linked to the shape and area models often used for teaching fractions but built to focus on the ten-ness of the number system.

8.2.1 Illustrative Cases

Illustrative Case 8-2 shows the post-interview tasks and a summary of Anya and Jenny's journey through intervention 2 and 3

	Anya	Jenny
	Post-Interview task	
First task	 Anya has not attempted the number line. She has correctly coloured in the decimat to represent 0.312	 Jenny has not attempted the number line. She has correctly coloured in the decimat to represent 0.312
Second task	 Anya has incorrectly coloured in the decimat. It is not clear whether this is 0.314 or 0.313 that has been coloured in. This time Anya has attempted the number line and it is relatively close to a representation of 0.413	 Jenny has not attempted the number line. She has correctly coloured in the decimat to represent 0.413

	Anya	Jenny
	Summary	
	Anya moved from a shorter is larger misconception before the intervention to be classified as an apparent expert at the end. In lessons, Anya was able to order decimals and match different decimal representations. Although the interview showed that Anya still exhibited some difficulties with fluency in decimal place she appeared to have retained what she had learned in the lessons and was able to apply it. In the third intervention Anya was able to place decimals on a number line and succeeded in representing numbers in a decimat and on a number line.	Jenny displayed a longer-is-larger misconception before the intervention. During the lessons she was able to talk about intuitive responses and recognise that she might be susceptible to them. In lessons, Jenny often made mistakes which appeared to be type 1 or intuitive errors. They appeared to be intuitive errors rather than type 1 or slips because they recurred and needed support from the teacher to see that it was an error. In the interview, Jenny was able to recall the lessons and what had been learned but she still displayed a longer is larger misconception. In the third intervention, Jenny displayed an insecure concept of place value, thinking that 0.3 and 0.03 were the same and placing 0.23 near 0.8 on the number line. Using the Excel decimat, Jenny was able to represent a decimal in the form of a decimat and on a number line.

Illustrative Case 8-2

Chapter 9

9 Discussion

9.1 Frameworks

One of the contributions from this study is the introduction of two frameworks, one a way of considering structure via instantiations and properties (see section 3) and one to consider errors (section 7.1.5). Figure 9-1 and the description below brings together the two frameworks along with a consideration about intuitive errors to show how they can work together to inform lesson planning and task design. A key aspect of both of the frameworks is that they are considered through the lens of the learner's experience. They are integrated in Figure 9-1 as a "Framework for planning teaching through the lens of the learner's experience".

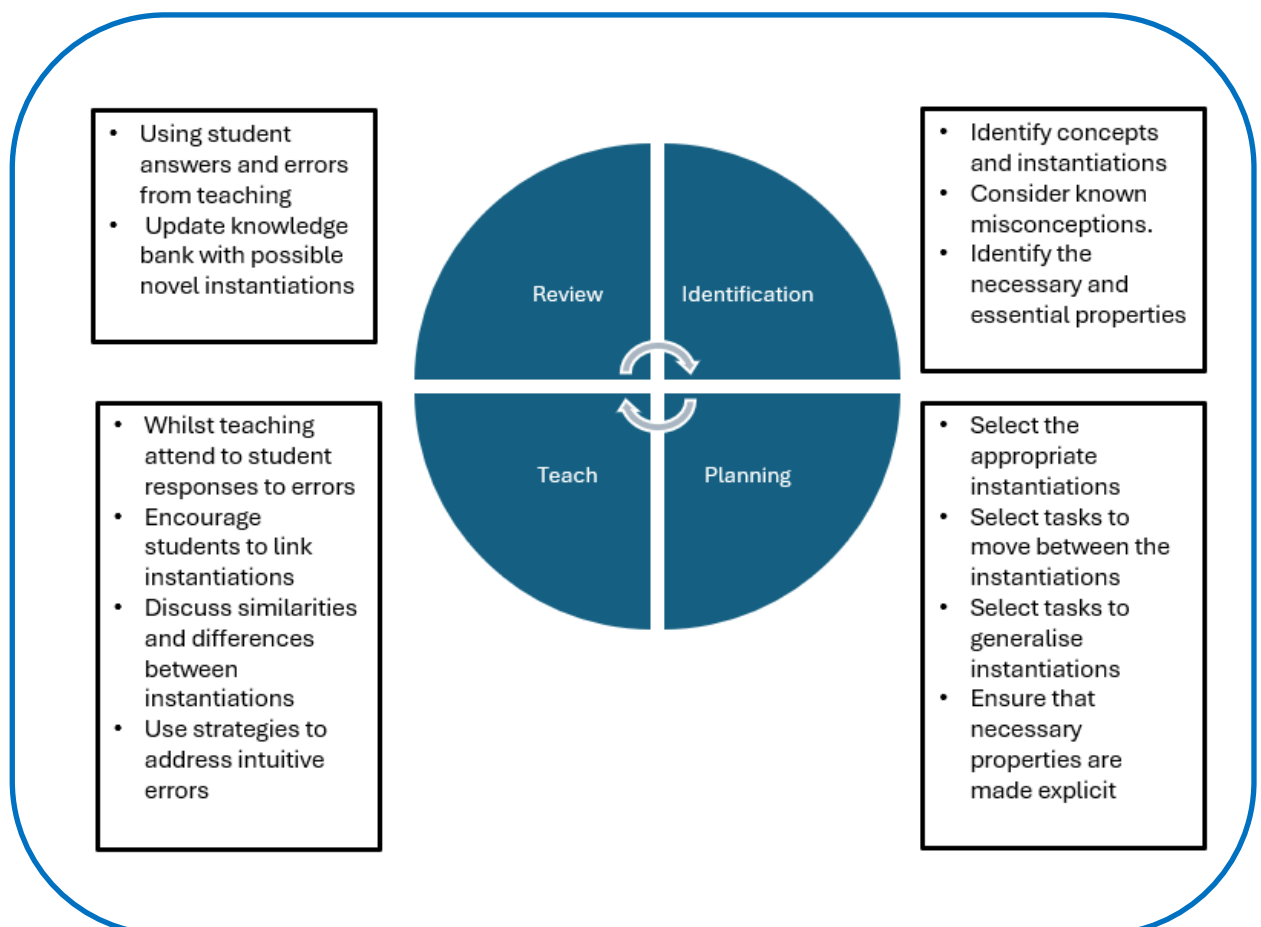


Figure 9-1 Framework for planning teaching through the lens of the learner's experience

Using the integrated framework introduced in this study has four phases:-

1. Identification
2. Planning
3. Teaching
4. Review

For each section I have included what needs to be considered and an example of teaching about fraction magnitude. So that the example can illustrate many of the points I am making it is necessarily forced and therefore does not necessarily represent a best away to teach fraction magnitude.

Identification

At this stage the teacher identifies the concept that is going to be taught. They will consider which instantiations they want to present to the students. Consideration of which instantiations are presented should include careful consideration of future learning and how current learning will support future learning. This exploration of later study is for the teacher's awareness only. In this the teacher is engaging with the mathematical structure in which the chosen concept sits.

Example teaching fraction magnitude.

The instantiations to be included are:-

- 1) The number line
- 2) The notation
- 3) A discrete circle
- 4) A discrete line
- 5) A continuous circle
- 6) A continuous line
- 7) Fraction diagrams with unequal pieces
- 8) Decimal representation of the fraction

(The instantiations 3,4 5 and 6 are introduced in section 6.4.16)

Once the instantiations are selected the teacher needs to explore what the necessary properties are and the essential properties between the instantiations being considered.

Example considerations:-

Instantiations 1,5 and 6 represent real numbers whereas representations 2,3, 4, 7 and 8 represent only rational numbers.

Instantiations 3,4 (and 1, if there are pips) have limiters to mark the sections. In this way the students are able to use counting to find the fractions.

Instantiations 5,6 (and 1, if there are no pips) will require students to use judgements around scaling and approximation.

(I have stated that the decimal represents a rational number rather than a real number because any decimal representation presented to a child (eg 0.235) is necessarily rational)

Finally, the teacher needs to consider any known misconceptions or difficulties that students have and consider if there are more instantiations than they had considered.

Students often have a whole number bias. Because of this when they look at $\frac{1}{4}$ and at $\frac{1}{3}$ they think that $\frac{1}{4}$ is bigger because 4 is bigger than 3. It is important that the instantiations (and tasks) chosen address this.

Where possible learning should support rather than inhibit future learning.

To support future learning, it would be better to avoid using only counting strategies for the numerator and the denominator. Having the continuous version of any diagrams will support this.

Often shapes used for teaching fractions are cut into equal sized pieces and the students count the number of pieces out of the total possible without asking the question “are these equal pieces?” (An example of this is at XXX). It would be better to have examples where the piece sizes are not the same.

Lesson Planning and Task design

Tasks need to be selected to:-

- introduce new instantiations
- compare different instantiations
- explicitly work with the properties of different instantiations and where they are the same across instantiations (essential) and where they are not.
- address known intuitive responses by using a combinations of warnings, analogy, and cognitive conflict
- ensure that the learning supports future learning

Example tasks are:-

- Placing fractions on the number line
- Matching fractions with their decimal equivalents
- Matching different visual representations
- Ordering fractions using visual representations
- Ordering fractions using the standard notation

Teaching

Whilst teaching the teacher should:-

- be explicit about the different instantiations
- ask students to move from one instantiation to another
- attend carefully to student responses to see if the student is using a different instantiation from the teacher
- attend carefully to student responses to errors to see if the student holds a misconception or an intuitive response
- specifically warn students about intuitive errors
- build up counter examples to intuitive errors by analogy.
- offer a counter example to an intuitive error that create

Many actions during the lesson will be a reaction to the actions of the students and so the following is a list of examples which might be expected :-

- careful use of vocabulary to support the instantiations.
- students might be asked to explain the link between 0.2 and $\frac{1}{5}$.
- students would be warned that some people think $\frac{1}{4}$ is greater than $\frac{1}{3}$
- if a student makes an error matching a fraction with a notation the teacher can build it from scratch (analogy) or offer a counter example
- the teacher should be watching for student responses to discover if there are misconceptions that have not been addressed

Review

Once the teaching episode is over teachers consider the errors, reactions to errors and any misconceptions found. They might also identify new instantiations. They can use this to inform their knowledge bank for the next time this topic is taught.

9.2 Contributions

My work offers three contributions to the body of knowledge. These are: -

- Frameworks for considering structure in mathematics and for analysing students' errors from the point of view of the student.
- The fact that these frameworks can form the basis of lesson planning in a classroom.
- The 'stickiness' of intuitions and intuitive errors

Frameworks

When first considering structure I was influenced by the work of Siegler *et al.* (2011), Steinle and Stacey (2004), Swan (1983), Askew (1997), and others. In determining how to define structure, I considered different ways of viewing the same concept. For example, trigonometry as a ratio on right-angled triangles or as a function on the real numbers or viewing division as partitive or quotative. I introduced a framework where each of these different ways of viewing a concept is an 'instantiation'.

- An instantiation is one way, possibly among several, of viewing a concept
- Each instantiation has properties that are necessarily true. I call these necessary properties.
- When comparing two instantiations, necessary properties which hold between instantiations are called essential properties.

- When students view a piece of mathematics there may be extra instantiations that the teacher had not considered.

It is by viewing the mathematical structures through this lens of the students' experience that the teacher can start to help the student to improve their mathematical knowledge. For example, students who have been taught that to divide by 10 'you take away a zero' have an instantiation where a number can only be divided by 10 when they are a multiple of 10. When faced with $67 \div 10$ their response might be that this is not possible. In this case the teacher is teaching an instantiation where a necessary property is that the solution will be a rational number whereas the student is viewing the problem through an instantiation with a necessary property that the solution is an integer. By understanding this difference between the viewpoint of the teacher and the student, the teacher is able to adapt their teaching to address this difference.

The second framework is how to classify student errors. Whereas previous classifications of errors appear to impose a cause onto an erroneous result (e.g. Movshovitz-Hadar *et al.* (1987, pp. 8–9) categorises high school mathematics errors as 'missed data', 'misinterpreted language', 'logically invalid reference', 'distorted theorem or definition', 'unverified solution' or 'technical error'), my classification looks to see how the student responds to the error. Knowing that a student makes a particular error does not tell us how the error came about or indeed what we need to do to help the student overcome and avoid the error. My classification looks at how the student responds to see why the error occurred and what the next step should be. For example, seeing $9 \times 3 = 24$ might imply that the student does not know their multiplication facts fully or that, in the moment, they have written something 'silly'. As teachers we can assume the answer, but only the student will actually know. In section 7.3.5, Jenny said that there were 7 boxes

coloured in on a decimat rather than 6 and, in the moment, in the lesson, I dismissed this as a slip. Later, as I taught Jenny more, I realised that this sort of error happened often and maybe there was more to it.

My categorisation of errors is as follows:-

- Slips: - Everyday mistakes with no pattern or cause. As soon as the student looks at the error, they realise they have made a mistake. A typical response might be “why did I write that?”
- Type 1 errors: - Named after Type 1 thinking in Kahneman (2012) which are automated thoughts without cognitive effort. This error looks like a slip but is an error that the student often makes or that many students make. An example could be $1 \times 1 = 2$. The student looking at this will wonder why they made that mistake. A typical response might be “Why do I always write that?”
- An intuitive error: - A mistake that the student does not spot immediately but when it is talked through logically the student can see their mistake. An example might be $(-3)^2 = -9$. The student might ask why this is wrong until a conversation about the multiplication of directed numbers reminds them.
- Structural error : - In this case addressing the error is beyond what the student knows. Their structural knowledge of mathematics does not encompass the mathematics being considered. For example, a student might apply right-angled trigonometric ratios to a non-right-angled triangle.

Lesson Planning and Task Design

Both of the above frameworks view situations from the point of view of the learner. On first glance it would appear that using these frameworks as part of a teacher's

regular practice would be impossible as it seems like the frameworks are part of a one-to-one strategy. This is not the case.

When planning lessons and selecting tasks to support learning a teacher can anticipate 1) the instantiations that are desirable to be taught and 2) the instantiations that students are likely to consider. If teachers were considering teaching students how to solve linear equations desirable instantiations might include the using the standard notation, a model based on function machines or a bar model. For each of these essential properties will include the solution for x . Using the example I have looked at many times in the thesis $ax = b$ the teacher may have in mind the necessary property that x is a real number. My experience tells me, though, that the students often hold the property that x is an integer. Although the teacher might not have prepared for the fact that this instantiation is under consideration, common errors and reactions in the classroom can make the teacher aware of the instantiation. Students might say “I can’t solve $3x = 10$ “. This is an indication that there is an instantiation other than the one the teacher has planned for. By considering the errors and comments from students then the teacher becomes aware that work needs to be done to bring these two instantiations together.

By considering common misconceptions before the lesson, the teacher might become aware of new instantiations. Other new instantiations might only become obvious during the lesson itself.

Both planning for misconceptions and adapting teaching in response to students’ responses are expected parts of a teacher’s role, therefore, considering structure from this viewpoint is a nuance on the work already being done.

Although it will never be possible in a classroom situation to interrogate every error on a one-to-one basis, it is possible to use the second framework in a classroom setting. Firstly, when talking one-to-one with a student, asking them to react to corrections and attending to their reply. Secondly, when talking to a group of students it is possible to ask them to assess their own errors. Students are able to identify and correct slips and type 1 errors easily. These do not need further corrective work and this gives the teacher time to address errors which do require corrective work.

I have created an integrated framework see Figure 9-1 Framework for planning teaching through the lens of the learner's experience. This details how a teacher can plan and teach using both the ideas of instantiations and analysing students' errors.

Stickiness of Intuitions

I provide insight into the stickiness of misconceptions. In my study, although students adapted their thinking and their methods in lessons, delayed post-tests and interviews often showed that the students had reverted to their initial, intuitive, answers. Intuitions, by their very nature, give students a sense of cognitive ease: a certainty that they are right. This is also the feeling that is reported when a student thoroughly understands something. In the mind of a student, therefore, an intuitive response is the same as one where they understand the generally accepted concept. If a student learns about a new instantiation that contradicts an intuition can learn about it and using cognitive effort can apply the ideas correctly. However, there is no difference to the student to knowing that something is correct or is the result of an intuitive error. The new ideas need to be revisited and

practised enough that they replace the intuitive error. This is linked to the classification of errors.

If a student learning about equations of straight lines believes that $y = 5$ is parallel to the y -axis they would have a 'gut' feel that this is right. If, having been taught that $y = 5$ is parallel to the x -axis, they drew it parallel to the y -axis their initial response is likely to be that they don't know why it's wrong. This would be an intuitive error, if, when asked to plot some points where the y value is 5 they realise that the line should have been horizontal. As they practise the skill more then there should be a point where they see the error and realise the mistake without further prompting, this is a type 1 error, and finally the student will draw the line horizontally once. At this point either the sense of ease of the correct answer has replaced the sense of ease of the intuition or neither answer gives the student the sense of ease and so the answer is actively thought about. In my own mathematical experience and listening to students in my class there are ideas where the learner says "I always get this wrong". Often these are situations like the above where an initial intuitive result is wrong and the learner is aware that of that their 'gut feeling' might be wrong and actively works to overcome it.

I investigated different ways to address erroneous intuitions, teaching by analogy (Stavy & Tirosh, 2000) was one way. This allows the students to see the sense in the generally accepted concept by building up slowly. What it does not do is flag up to a child that this truth cannot coexist with their intuitive response. The literature (Zaslavsky, 2005) and my experience both show me that students can happily hold contradictory truths. This does not in itself necessary produce a conflict. Zazkis and Chernoff (2008), describes the need to identify a pivotal

example, one that creates a dissonance for the student. Much like the concept of instantiations or my classification of errors, the pivotal example has to be from the point of view of the student. Students then rely on what they know to be true to realise that their intuitive understanding is false. Swan (1983) promotes this use of cognitive conflict for teaching about decimal values. One thing that helps the teacher who is trying to overcome intuitive responses is that they are mostly predictable, and they can arise from overgeneralising, that is applying the properties of one instantiation to other instantiations where those properties do not hold, or as a result of intuitions which appear to arise as part of the usual cognitive development of children. Because the intuitive responses do tend to be predictable, teachers can plan for some of them, and, most importantly, can warn against them. However, not all of these intuitive responses are predicted and as a teacher it is important to be aware of the possibility of an unexpected instantiation when talking to students.

9.3 My Research Question

My research question was: -

Can the mathematical understanding of students be modified by encouraging students to engage with underlying mathematical structures?

I started this research because my experience shows me that students engage with mathematics with what might be described as a siloed view, with concepts and topics being taught in isolation, rather than as a connected whole. It appears to me that schemes of work and textbooks do the same. Schools usually teach mathematics with units of work (this might be called different things in different schools). Each unit is a selection of the concepts brought together as a connected entity. For example, a unit might be fractional arithmetic and cover concepts such as fraction of an amount, addition and subtraction of fractions and mixed numbers,

multiplication and division of fractions and mixed numbers. The concepts tend to be sequenced so that links can be made between them within a unit. In my experience though, there is little linking of topics from concepts outside the units and when these links are discussed it appears that what is found is considered to be amazing rather than the norm. I am not stating that every school has a curriculum that looks like this, but am considering my own experience, the experience of other teachers I have talked to and the majority of textbooks or published curricula that I have looked at. An example of unconnected mathematics teaching might be that when simplifying fractions, students might realise that to get to the simplest form, they would have divided the numerator and denominator by their highest common factor, however the link with highest common factor would not be central to the lesson or to what students were supposed to take away from the lesson. When students learn about finding the highest common factor, it becomes an end point rather than a tool that can be used throughout different parts of mathematics e.g. simplifying fractions, factorising algebraic expressions. To my mind there is no purpose for the child to be learning all these distinct concepts and tools if it is not intended that they apply them later on. Haylock (1982, p. 54) describes learning where “new material exists in his cognitive structure as an isolated, unconnected experience” as “rote learning”.

The aim of my research was to see if I could use structure and a sense of connectedness in mathematics to improve the mathematical understanding of students.

My study showed that students’ understanding was modified and that by my viewing the mathematical structure by using the instantiations I was able to adapt my teaching to react to how the students were engaging with mathematics. The

study also showed that intuitive errors are very difficult to shift and although understanding can be modified it is not easy or immediate.

In summary, my research question was:-

Can the mathematical understanding of students be modified by encouraging students to engage with underlying mathematical structures?

In my study I used mathematical structure, in the form of instantiations and properties to plan to teach the students. By using different instantiations, I was able to engage the students in mathematical structure. My study showed that students' understanding was modified and that viewing the mathematical structure by using the instantiations enabled me to adapt my teaching to react to how the students were engaging with mathematics. However, the study also showed that intuitive errors are very difficult to shift and a short-term change in understanding does not necessarily lead to long term change.

9.4 Background

9.4.1 Overview

9.4.1.1 Development of number concept and structure in mathematics

Development of understanding of number is staged, with children relatively quickly able to understand natural numbers but not being as competent with fractional quantities and rational numbers.

The literature points to some of the difficulties arising from a natural number bias a tendency to treat all numbers as if they were whole numbers or a preference for working with the logic provided by whole numbers. Interestingly, all the studies that investigate, show that this bias is not just present in novices and students, but also in experts. With experts, for example postgraduate mathematics students, the bias

shows in speed of processing rather than in errors and some of the bias appears to be able to overcome by utilising expert knowledge to reframe questions so that the differences between natural number and rational number do not need to be considered. This is discussed in more detail in section 2.5.2.

My consideration of literature around structure in mathematics (section 2.8) led to my definition of instantiations, a way of looking at structure in mathematics from the view point of the learner, and that by teaching students and asking them to engage with the similarities and differences between instantiations they would have a better understanding of the mathematics they were learning. How the instantiations are defined is detailed in section 3.2.

To be able to think about how learners' understanding could improve, I investigated the relevant literature (section 7.1) and found many links between understanding and intuition (section 7.1.4). The literature suggests that the feeling of certainty that is present when someone thoroughly understands something, is the same as the feeling of certainty that intuition provides.

9.4.1.2 Review of the interventions

Initially I was going to teach several different topics during the intervention. The results of the pilot indicated that I should limit the topics taught in the intervention to focus on understanding of decimal and fractional magnitudes and place value, since these presented bigger problems than I had anticipated. Fortuitously, this fitted with changes in the constraints of my fieldwork.

There were three interventions, each designed to take account of the results and the further literature research arising from the previous interventions. I found that asking students to engage with the structure and instantiations did enable them to talk about magnitudes and place value which they had previously not been able to

do. They were also able to overcome pre-existing misconceptions. When I first start teaching these students, they appeared to have mantras, which they referred to in their learning. There were specific mantras like 'as decimals get bigger, they get smaller' and more general ones like 'decimals go the other way'. Some of these were consistent from child-to-child e.g. 'to multiply by 10 just add a zero' and others are vague or less well-formed. As a teacher it appeared that the consistently phrased mantras had been taught to the student and the vague mantras might have developed through the students' own experiences. Students used these until they had better reasoning to fall back on. I observed that students used these mantras without any recourse to check their answer. Many did not have a 'feel' for what the answer should be. Answers could seem extreme to me and when questioned the students immediately spotted that they were wrong, but this response did not happen without my prompting them. The mantras appeared to be the truth to the students and even when they could not express them well, they used them in a predictable way. During the intervention, the students were able to use cognitive conflict to move from one understanding to another, however this needed to be activated. It was possible for students to have conflicting beliefs and not appear to recognise that this was the case. During the first intervention, I wanted to show that the teaching meant that better structures replaced the mantras. Through the interviews and into the second intervention, I saw that this was not the case. During the second intervention I taught ideas using the structures that I had used in the first intervention. I also taught the students how to avoid the mantras and intuitions. There were three methods for this: 1) by warning them specifically about the intuitions 2) by building up tasks so that each step was entirely logical and leading the students to pick answers which would have been against their intuition 3) by using cognitive conflict to show the students that their intuition could not coexist with other beliefs.

9.4.1.3 Review of the interviews

The interviews showed me the strength and persistence of the intuitions in the students I was working with. Students were able to order decimals correctly in the lesson and explain their thinking and even explain why intuitions were wrong. However, they reverted back to those same intuitions a few weeks later in the follow-up interview.

I also discovered how poorly some of the students understood place value and the relationship between adjacent columns in our numerical representation system.

9.4.2 Understanding

The use of the word ‘understanding’ in the research question ‘Can the mathematical understanding of students be modified...’ is problematic. It is not clear what understanding means or whether it can be measured. In order to address this, I started by reading Skemp’s (1976) work on understanding. He distinguished relational from instrumental understanding and I expanded my interpretation of understanding using references which discuss different models of understanding (for example: Hiebert & Carpenter, 1992; Pirie & Kieren, 1989, 1994). These models are discussed in detail in section 7.1.1. The definition which resonated most with me was from Hiebert and Carpenter (1992) and linked the strength of understanding to the strength of the connections to existing knowledge.

Looking at this definition alongside my definition of structure it seems that the strength and number of the connections, by which I define structure, is how Hiebert and Carpenter (1992) measure the strength of understanding.

Haylock (1982) too measured the strength of understanding by considering the number and strength of connections. As I worked through the literature and with the students, it became clear that my use of the word ‘understanding’ was in itself

defined by structure. For me, a student ‘understands’ a concept when they are able to apply it to novel situations, or to connect it with other concepts or ideas. In terms of my work, a student would understand decimal magnitude when they could represent it using different instantiations and could move easily between one instantiation to another.

9.4.3 Mantras

I wanted to explore whether using structure would make it easier for students to remember and apply concepts correctly, rather than to learn and be expected to apply algorithm after algorithm. I suspect that most people have shortcuts they use in mathematics rather than going back to first principles and I do not find that problematic as long as the person knows why they are doing what they are doing. There are many mantras, for example, ‘two negatives make a positive’, ‘you can’t take a bigger number from a smaller number’, ‘to multiply by 10 just add a zero’, ‘you can’t take a square root of a negative number’, ‘multiplication is always commutative’ and many more implicit ones ‘multiplying makes bigger’, ‘rectangles have two longer sides’. Some of these appear to arise from generalising about early specific cases and some appear to have been taught to students to make early mathematics more accessible.

Where mantras are taught to students, it might be that, in some cases, the teachers do not know that such phrases are not “forward-looking” (Gardiner, 2016, p. 12) (see section 2.6.2). For example, it is reasonable for a primary school teacher to state that multiplication is always commutative and consider this to be forward-looking, the multiplication of matrices is currently beyond school mathematics and so it is unreasonable to expect someone to know that this property does not always hold. If teachers realise that their mantras are not forward-looking, they might not realise how these early ideas stick. It might be that

teachers consider that students will grow out of mantras such as 'add a zero' or 'you can't take a bigger number from a smaller number' a bit like students move on from spelling using letter sounds rather than letter names. It might be that in most cases the students do grow out of them, but some mantras stick or, more accurately, some mantras stick for some students. The students in my study exhibit examples of this.

Some of the mantras I heard are well-formed and they appear consistent from child to child (e.g., 'two negatives make a positive'), making it likely that they have heard it from an expert or have actively been taught it. Some are less well formed, e.g. 'decimals go the opposite way' and these seem likely to be the sense that individual students have made of their learning. They appear to be internal, because the students act in a predictable manner but cannot express the mantras clearly enough for someone else to follow. These mantras often appear to be a result of overgeneralising, having found a new method and applying it everywhere.

There are a few topics in mathematics which, in my experience and from talking to other teachers, students regularly appear to find harder than others. One of those is fractional arithmetic and when I reflect on lessons where I have taught this, I find that students often ask questions like 'Is this the one we add or multiply?' This happens so frequently it made me reflect on fractions and what is taught.

To be able to add fractions algorithmically you follow these steps:-

- find a common denominator.
- find equivalent fractions using the common denominator.
- add the fractions.
- simplify the result.

Adding fractions this way is long-winded but it is logical and accurate.

I realised I needed to consider it from the student's point of view. For a student who is learning the methods and the algorithms only:-

- find a common denominator (either by multiplying the denominators or by finding the lowest common multiple of the denominators)
- find equivalent fractions using the common denominator – multiply the numerators by what you have had to multiply the denominator by
- add the numerators, whilst keeping the denominator the same
- simplify the result by dividing numerator and denominator by their highest common factor
- when you look at the answer if the numerator is bigger than the denominator, then divide the numerator by the denominator, this gives you the whole number, what is left over you put over the common denominator.

This is incredibly complicated, and, without anything to hang those steps onto, they are often not remembered. The steps either need to be practised until they are automated or the students need to understand why the steps happen. The difficulties around remembering and applying these algorithms could explain why students say 'is this the one where we add or where we multiply' (see section 2.5.8)

9.5 Main Findings

9.5.1 Cycles

As I was designing the study, I wanted to work with Year 7. In my experience Year 7 students usually are more open to new ideas and are more open to different ways of working than older students. In addition, my intention was to select areas of mathematics that could have a large impact on the mathematical journey of the students, these are often studied in Year 7 and would have the biggest impact if

they were addressed early in the secondary career. I initially selected five topics to consider; place value, decimals, fractions, algebraic representation and multiplicative reasoning. These were all topics which were tested in the CSMS study (Hart, 1981). I selected these because it appeared to me that these topics are the building blocks of secondary mathematics. As stated before, some of these areas are also predictors for later mathematical achievement (Siegler *et al.*, 2011).

In the pilot of the assessment instrument, I discovered that some students were not able to find the magnitude of fractional amounts and were not able to use place value confidently. As a teacher my first priority was to the mathematical development of the children and not the abstract research interest and so I narrowed the focus of my study. I decided only to consider magnitudes of fractional amounts and place value, and I would restrict the study to the students in school who were the previously lowest attaining.

I ran the intervention for two weeks in September (see 6.4.3) and two weeks in January (see 6.4.11), the focus for the first set of lessons was decimals and for the second set of lessons, fractions. I used a variety of tasks; some my own, some from other people and some published. Each task was intended to emphasise some structural element of the mathematics under consideration. For example, a task might be to show how properties between two instantiations were the same, or how a student could move from one instantiation to another. Details of each task and why they were selected are in section 6.3.

I found that, within the time scale of the intervention, students could change how they discussed magnitudes of fractional amounts. For example, James was able to place 0.25 on a number line when previously he had considered that $0.25 > 0.9$ (see section 6.4.9). However, the results of the delayed post tests showed that

many of the students had reverted to their original reasoning within four months of the intervention. The interviews that followed the post-test showed me that the students had retained mantras which supported the incorrect intuitions and reasonings. Most appeared to be imprecise, internal mantras; ones the students themselves could not pin down. None of the students said 'Mrs Jones said...', so it appeared that they might have come from their own sense making. For example: - Tasha's comment:

Tasha: because if there's more. If there's a bigger number, there's less. If there's a smaller number, there's like more.

I realised that I had underestimated the long-term effects of intuition and I reviewed literature to see if I could find out more. Authors such as Fischbein (1999) and Kidron (2011) investigated intuitions, how they arise, how they develop and how they can be replaced. For a full description see section 7.1.4. Many intuitions arise naturally, and are replaced by more sophisticated ones through maturing and life experience. However, some intuitions persist. There is a link between understanding and intuition. V. Byers and Herscovics (1977) expanded Skemp's (1976) list of types of understanding, to include both intuition and formal understanding. In their hierarchy of understanding Pirie and Kieren (1994, p. 167) call an intuitive response "primitive knowing". Sierpinska (1994) discusses being able to understand something when there is a sense of closure. Intuition gives students a sense of certainty; the intuitive response is something that they 'just know'. As a result of the observations made in the study, I suggest that the sense of ease caused by activating an intuition is similar, to a student, to the sense of ease that follows a thorough understanding of a concept. My first intervention and my previous experience had shown me that even careful teaching is not enough to replace these intuitions. Skemp (1976), V. Byers and Herscovics (1977) and Buxton (1978) all discuss how hard it is to shift an understanding when

a learner believes that they understand. Given that there is a the sense of certainty and cognitive ease that accompanies both understanding and intuition it appears reasonable that a student would not want to adjust their beliefs to reject an intuitive response. I suggest that, from the point of view of the learner, thorough understanding and intuition are so similar they cannot tell them apart. In turn this means that the role of the teacher when seeking to overcome intuitions requires them to adjust the student's understanding, taking the students out of their sense of certainty, through a feeling of cognitive discomfort and then back into an area of comfort. It seems that the job of a teacher includes noticing intuitive misunderstandings and helping the child to replace them with new ideas, for which they then have to hold a similar level of *just knowing* so that the previous intuitions can be replaced.

During the second cycle I tried to pre-empt the intuitive mantras. I used the three strategies suggested by the literature; teaching by analogy, teaching by cognitive conflict (Stavy & Tirosh, 2000) and by warning students about intuitive errors (Allaire-Duquette *et al.*, 2019; Babai *et al.*, 2015). One of the tasks that used cognitive conflict as a tool was the Maths4Life task (Department of Education, 2007). This was written by Swan, whose work on cognitive conflict (Swan, 1983) explores a comparison of teaching decimal place value by cognitive conflict with other positive means that did not use cognitive conflict. He finds that there is a greater gain in teaching by cognitive conflict. For teaching by analogy, I built up the teaching of the decimat and number line slowly, aiming to ensure that at each level the next step appeared obvious to the student. In the decimat worksheet I ensured that there was a careful build of complexity. I warned students about intuitions and asked them, for example, why a student might think that 0.24 was

greater than 0.3. As an example, in section 7.3.4 I reported Jenny saying that she would have made the misconception that I was warning about.

The other possibility would be to 'over practise', in other words to keep going over the ideas again and again, until they became second nature. This would require repetitions of the right structure so that it becomes habitual. However, I needed to be sure that the students were practising the correct answers, and not working with the incorrect intuitions, and so the approaches above were useful. As I only taught the class for a few days, I could not keep revisiting these areas of mathematics. It raises an interesting and often lamented dilemma faced by teachers about curriculum coverage versus curriculum depth. Although the students were reliably comparing magnitudes of fractional amounts in the lessons, they reverted to using similar mantras to those used in Cycle 1 in their interviews (several weeks after the intervention). For example, Charles' comment when asked why 0.24 was bigger than 0.235 'Because that's like going back in the wrong way'.

The Cycle 2 interviews showed me that the students could remember what they had learnt, but that they were answering questions following an intuition which produces a sense of cognitive ease. What was clear, was that some questions in the interview were answered because it was how things had been taught and that there were gaps in the basic knowledge of how place value is structured. Cycle 3 was a very short-term intervention, considering only the students I had interviewed in Cycle 2. What I wanted to discover was whether I could teach them to overcome their intuition and whether I could also address the issues around place value.

I decided to focus the third intervention on two aspects, one the vocabulary around decimals and place value, and second using the idea from Luculano and

Butterworth (2011) of the NP (number to place) task, tasking students to place numbers on the number line. I created an Excel tool allowing students to quickly represent numbers using the number line and the decimal and to check whether they were correct.

After the third cycle I asked the same students to complete a question over three separate points in time to see if they could still complete the decimal and place a decimal on the line. The students completed the decimal questions more accurately over time than they did the number line questions. This was, though, a very small number of students and a small number of questions.

9.5.2 Instantiations

I defined instantiations for this study in section 3.2. The purpose was to recognise that linked concepts share many properties and knowing this allows students to move from one instantiation to another or from one concept to another. It is also significant that instantiations have properties that do not hold between them. I quickly realised that what students perceive as instantiations might not be the same as what teachers perceive and teaching with a focus on structure requires teachers to consider the instantiations from the point of view of the learner.

Recently a colleague, Mike, told me about the following interaction between him and a student, Simon.

Simon: Your question is wrong, sir.

Mike: What do you mean?

Simon: $5x = 17$. It does not have a solution. You have got the question wrong.

Simon is the sort of student who engages well with his mathematics lessons and would often be labelled 'good at maths'. This comment, therefore, surprised Mike. I explained the idea of the instantiations to Mike and that, although he and I looked

at $5x = 17$ in the same way we looked at $5x = 15$, the same was not true for Simon. By being able to recognise this, teachers are empowered to help students improve their mathematical thinking and the range of questions that they will be able to approach and answer.

9.5.3 Through the lens of the learner's cognitive experience

As a teacher I have always felt that there was a strength in students using self-assessment. In the end, as a teacher, when assessing, I need to use all my experience to work out why a student has made a mistake. Where there are many steps to the problem, and the answer is incorrect, it can be very difficult, even as the expert in the room, to find that error. Often, I need to work through every step to find the mistake. In addition, if a student writes, for example $6 \times 7 = 44$, I have no way of knowing whether this was a one-off mistake or whether the student believes that $6 \times 7 = 44$. I have found that a more useful experience is to ask the students to mark their own work. Yes, there is the chance of students marking incorrectly or not participating, but for these students they are unlikely to have looked at the teacher's comments in depth. The students who are motivated to assess their own work can be heard to say 'Why?', 'Not again', 'You've marked this incorrectly' or 'Why is this wrong?'. When I was first a teacher, I would wonder why a child had answered a question a particular way, then I would start to see not just one child, but several, had made the same mistake. At this point, it appears that there is something systematic happening. If, when asked, 'what is the answer to $\frac{1}{2} + \frac{1}{3}$?', the student gives the answer $\frac{2}{5}$ then it appears likely that the student has added the numerators and added the denominators. To a seasoned mathematician this might seem absurd, but consider that $\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right) = \left(\frac{2}{5}\right)$ then it should be acknowledged that some of the issue must be with how mathematical ideas are represented.

Considering the above, it is important to recognise that often an answer is incorrect only because it does not follow all the expected rules and does not give the generally accepted correct answer. If the learner's answer is analysed, the listener can begin to build a picture of the student's experience of mathematics. Incorrect answers can follow a perfectly logical application of a concept, that is just not applicable in the particular set of circumstances. Throughout the study, I tried to ensure that I was investigating the student's viewpoint; trying to understand why the student gave a particular answer. It is not enough for the teacher or researcher to know that the student has the answer right or wrong; an investigation into their thinking is necessary. As part of this I decided to relook at how errors were classified and to reclassify them through the lens of the learner's cognitive experience. It is important to find out how the learner felt when they saw the error. Classifying errors through the lens of the student's experience makes it easier to know how to help the student correct the error. I used this approach to analyse answers to questions in the interviews.

My research has enabled me to ask questions of classes that I have taught since undertaking the research in order to uncover intuitions, to construct probing questions to reveal intuitions and also has made me more aware that I do not know what I do not know. Because I am more aware that I do not know all the misconceptions that students might hold I have questioned more carefully. I have discovered misconceptions that I was completely unaware of before, for example to see that in a class of previously high prior attaining Year 7 students there were eight students who held the belief that $0.6 < 0$.

Another example of this misconception is in a lesson with a different class – also looking at ordering decimals.

Student: 0.236 is bigger because it's smaller and with decimals smaller is bigger

Teacher: Can you explain that a bit more please

Student: 0.36 is closer to zero and when you are closer to zero the number is bigger.

Teacher: draws a number line marking 0 some way in from the left and 1 further up – the number line extends past both 0 and 1

Teacher: Can you please come and show me where 0.36 is on the number line?

Student draws 0.236 to the left of 0.

Teacher: is 0.236 bigger or smaller than 0?

Student: smaller.

This was the first time I had come across the misconception quoted in Stacey, Helme and Steinle (2001) where, potentially due to a misplaced mental image of a mirror, students consider decimals between 0 and 1 are less than zero.

During my teacher training in 2007, I had been taught to interrogate a student's thinking; not just to accept an answer and say right or wrong. However, in the classroom, this can lead to lengthy diversions or the concern of confusing others, and I think that I had only ever managed to do this on a surface level. Having taken the time during the research, in the interviews, to question deeply around student responses, I was more able to understand why students gave certain answers. I became more aware of intuitions and alternative conceptions. This has crossed over into my lessons, where I am more likely to try to delve to find out why a student is saying something than to merely check whether the answer is what I was expecting them to say. I am more aware of what am I attending to when I teach.

9.5.4 Classroom culture

It is hard work to shift the intuitions held by students. In essence, how can I ask a child to change their way of thinking to reflect the fact that 0.123 is smaller than 0.9, when the obverse appears blatantly obvious to them? As Skemp (1976), V. Byers and Herscovics (1977) and Buxton (1978) all discuss there is a resistance to such change.

My research and experience suggest that it is necessary to establish classroom norms that explicitly value perseverance, accuracy, and effort but which do not discourage wrong answers. Students need to feel valued for their contributions and attempts (Yackel & Cobb, 1996). It is possible to create this culture in the short term, whilst parachuting in a teacher, as I needed to do with these short interventions. As a school leader I frequently cover single lessons and I have found that I am, maybe because of novelty, maybe because of my status within the school, maybe because of my reputation, able to have a temporary relationship with a class; I find this takes a longer time to build when I am the regular class teacher. As a class teacher I need to build the culture in my class of what is expected and the ways of working. Sometimes, in doing this, I am having to undo the classroom culture that exists because of the previous experiences of a child. I frequently find myself saying phrases similar to 'Mr Smith might have done that, but I don't'. I frequently have to instil that I value bravery and perseverance over correct answers. Sometimes, I have to unpick some mathematics that the students believe they have been taught, but I have to spend a lot of time building different ways of working. My experience is that it is easier to build such a classroom atmosphere with Year 7 students than with older students. Older students often appear more cynical and the teacher has to put time into improving the classroom culture and the mathematical norms (Yackel & Cobb, 1996). For

Intervention 1, I worked with the students in September, then again in January and then interviewed them in March. The ways of working that I developed in each session would not have been adhered to between the sessions. In secondary school, students work with many different teachers and my experience shows me that most classes quickly fit into the way of working that they know is expected from each teacher.

In my study, all of the students participated fully and engaged fully with the tasks. This might have been due to my coming in as a different 'special' teacher; it could have been the work that their normal class teacher had done to ensure the mathematical norms in the classroom; or it could have been the age and experience of the class. Indeed, the last intervention and all the interviews were all voluntary, those not wishing to participate were not included. I had more students willing to take part than I had places.

9.5.5 Attending to every child

By attending to individual students and delving deeply into their intentions I have managed to uncover misconceptions and to gain access into the thinking of students I have worked with. Some of these misconceptions (e.g. $0.6 < 0$) I had not met in my teaching career before. I first saw this particular misconception referenced in Stacey *et al.* (2001). Knowing about it, I was very careful in my questioning to allow this misconception to surface. I am well aware that this is not always reproducible in a class of 32. However, it is worth noting that my work on this project has made me more careful in my class questioning and analysis of answers. Without the work here, I would not even have considered asking a class, particularly a class of high attaining students, which was larger 0 or 0.6. In my observations of other teachers and in my reflection of my own practice there are times when I ask a follow up question and assume that an answer means

something specific. This is particularly noticeable when I am observing others. I often see a student say something and the teacher respond, making an inference from what the student has said. I might not have made the same inference and I can see from the student's face that the teacher's interpretation was incorrect. I looked for an instance of this from my own teaching – as I am sure I do this too, but I would only be able to have an example of this if someone had observed it in my lesson as by its very nature, I would not see it. This is, perhaps, another example of commognitive conflict (Sfard, 2007) where both sides on a discussion are using different rules of discourse and do not realise that the other has a different viewpoint. It is difficult to spot this type of interaction from one's own practice because you would have to be aware of your lack of awareness in the moment. To illustrate this, I have a fictional interaction:

Jimmy: Miss, the numbers all make a pattern.

Teacher: Yes, Jimmy they are all odd numbers well done.

Jimmy: (looking perplexed)

Here the teacher, expecting that Jimmy has spotted the pattern that she has seen, has assumed what Jimmy means and has extended the answer. The teacher has not checked what pattern Jimmy is talking about. She has moved on before Jimmy has the chance to protest. In my role on the senior leadership, I spend much of my day visiting lessons, to support teachers, to help maintain a calm climate and as part of the school's quality assurance process. When I do go to a lesson I will often stay for a few minutes, and I see examples of interactions between students and teachers in a large proportion of lessons.

Within the project, however, I spotted that I did something similar (reported in section 7.3.5) when I asked a student to say how they knew a number was bigger. Their response was that the number was further to the number 1. Although the

answer was correct it was only correct because I was picturing a 0 to 1. If I had anticipated that a different number line could have been used, I could have extended the questioning to ensure that the student also was considering a 0 to 1 number line. The difference between this and the fictional interaction above is that both the student and the teacher were unaware of the missed opportunity where in the fictional interaction the student does not feel that the teacher has expressed the same idea that he has. In the interaction about the number line, I have assumed part of the answer, which I believe the student also assumed, however it would have been better to check.

A teacher has to decide in a classroom how much time can be afforded to unpicking the thoughts of each individual student. There is a balance between keeping asking the same student until you get to the answer that helps you understand their thinking, and involving other students. It might be that a conversation like this might flag the need for further one-to-one intervention with a student. It is a matter of considering resources, including time, because you cannot, in a classroom, spend the time exhaustively investigating the thoughts of each student. The project has allowed me time with students one-to-one which has allowed me to unpick their thoughts, but has also allowed me to anticipate misconceptions and difficulties of others which I have then been able to assess and address in a whole class setting.

9.5.6 Development of intuitions

Initially my work was not about intuitions or even how learners think. It was an investigation into whether or not a particular way of teaching could solve a problem. The problem was that, from what I saw in classrooms and textbooks, students were learning areas of mathematics in isolation siloed from other parts of mathematics even if they had an almost identical structure. Students were learning

how to answer questions and to pass exams. As I worked with small groups of students, I found that incorrect ideas kept surfacing; this was after the students appeared to have learned what I had taught. An example would be from the first intervention interviews where James, when asked how he knew that 0.24 was larger than 0.236 said.

James: Because there's three numbers after the decimal point in that one and there's only two in that one.

Here, James was equating size purely with the number of decimal places and displayed a shorter-is-larger misconception.

The incorrect ideas were repeating and were predictable. They were not just type 1 errors (errors where the student would immediately recognise their error; see section 7.1.5.4. for a full description of type 1 errors). The students could find the correct answer when prompted but there seemed to be a sense of cognitive certainty in these incorrect ideas.

After the first cycle, I researched more literature about intuitions and how they develop. There are developmental intuitions such as “more A more B” (Tirosh & Stavy, 1999). An example of this might be where young children seeing that a line of marbles is longer will assume that there are more marbles in the line. The step from this to 0.234 is bigger than 0.9 because 234 is bigger than 9 is perfectly feasible.

Intuitions develop (Tirosh & Stavy, 1999) so, for example, ‘more A more B’ is replaced by intuitions around conservation. Other than these intuitions there are tacit models (Fischbein, 1989, 2001). A tacit model is a model that a learner uses to understand an abstract concept. Graduate mathematics students are likely to have tacit models and visualisations of ideas that are so abstract that there is no physical model for them (Fischbein, 2001). Even though students may be aware of

the models and their limitation, they will still use them to analyse the abstract situation. Many of the issues around magnitudes of fractional amounts arise from a whole number bias whether this is genetic or learnt, students apply the logic that works with the instantiations for whole numbers to fractional amounts. This whole number bias is seen even in mathematically competent adults (Obersteiner *et al.*, 2016).

It seems likely that the students in my study are stuck with tacit models that they have not been able to replace. By giving students an explicit model, teachers are trying to define their tacit models. A student who uses the 100-square as their model for decimal representation is, by and large, limited to two decimal places and 0.25 becomes 25 blocks out of 100 this potentially reinforces whole number thinking. By sharing the model of the decimat it could be argued that this limits students to three decimal places. There are two counter arguments. The first is that there comes a point in teaching mathematics ‘the etc. moment’ by which I mean the moment when the teacher no longer needs to enumerate examples; when the student is able to continue the generalisation and as long as the examples have been curated well enough, the possibilities of incorrect overgeneralisation are limited. For example, when showing students that the interior angles of polygons can be found using the formula

$(\text{number of sides} - 2) \times 180^\circ$, showing that the interior angles of a triangle add to a 180° and that the interior angles of a quadrilateral add to 360° does not give enough information for the student to generalise. If they then considered the pentagon and hexagon, the student’s generalisation is likely to be sound. By the time students can consider any number of decimal places up to three decimal places they have left money thinking behind and are closer to being able to work fully with ragged decimals (Steinle, 2004b).

More importantly though, the decimat is *forward-looking*. Take, for example, the 100-square as a representation for decimals, representing three decimal places is hard and every subsequent decimal place becomes even harder. If, however, the smallest rectangle in the decimat is taken and split into 10, then four decimal places can now be represented, take the smallest rectangle now and split it into 10 and five decimal places can be represented and so on. By zooming in indefinitely, similar to the number line, any number of decimal places can be represented. Although this could also be done with the 100-square (splitting just one of the small squares into 10 for three decimal places) it is not built up quite the same way as with the decimat. I had never used the decimat before I started this study, and often used the 100-square as a model. The literature and my experiences during this study lead me to believe that the use of the decimat is preferable over the 100-square.

Mental models develop and change, and intuitions are undone sometimes to be replaced by others. Teaching, using the strategies listed in section 7.1.6, can help students overcome intuitions and tacit models, in their lessons and work with more accurate explicit models. However, explicit models can be overcome by the tacit models meaning that intuitions can return.

9.5.7 Misconceptions

Within the study students have displayed various misconceptions

- Longer-is-larger e.g. $0.235 > 0.24$ because 235 is bigger than 24
- Shorter-is-larger e.g. $0.3 > 0.45$ because 3 is has fewer decimal places than 45
- Whole number thinking e.g. $\frac{1}{4} > \frac{1}{3}$ because 4 is bigger than three
- Decimals are smaller than zero e.g. $0.6 < 0$

Misconceptions such as longer-is-larger and shorter-is-larger are widely referred to in the literature (e.g. Roche, 2005; Steinle, 2004; Swan, 1983). A whole or natural number bias is referred to by many sources (e.g. (Obersteiner *et al.*, 2013; Siegler *et al.*, 2011; Vamvakoussi *et al.*, 2012; Vamvakoussi & Vosniadou, 2010). These difficulties appear to be universal and therefore not related to the curriculum or culture. Many appear to arise from the natural intuition “more A more B” (Tirosh & Stavy, 1999).

The misconception that $0.6 < 0$ was one that I had not been aware of before the study. When I have discussed this with colleagues, they are surprised that this is a misconception that arises. When I first saw it in a classroom, I had refrained from drawing a number line, so that I did not lead the thinking of the students and one student offered me a number line with 0.6 to the left of 0, alongside the negative numbers. I have not seen this misconception referenced anywhere other than from Steinle and Stacey (2001). Being aware of it and altering my questioning so that this misconception could be a possible answer means that I have now been faced with three separate groups of students who have believed that decimals lie with negative numbers.

In addition, the second interviews showed me that many of the students had a very weak grasp on place value. This finding from the interviews helped me shape the third intervention.

Most of the difficulties and misconceptions that were apparent through the study were ones that several students showed and were similar to what might be expected from the literature.

9.5.8 Language

Language appears to play an important role in students' understanding of place value and fractional magnitude. The English language does not support a thorough understanding of magnitudes. This can cause learners difficulties that teachers need to be aware of. There are several situations which could be difficult for students. One example is that the words 'half' and 'quarter' have no link to the numbers two and four respectively. Another example is that the 'teen' numbers are constructed differently from all other two-digit numbers. Whereas two-digit numbers are named as tens then unit (e.g. twenty four) teens are named as unit then ten (e.g. fourteen). Most relevant for this study, though, is the naming convention for decimals. There is no reflection of the magnitude of the digits when naming decimals. Learners have to make a difficult shift when they move from language that names the magnitude (e.g. three thousand) to language that is based solely around the symbolic representation of a number (e.g. zero point zero zero three). Teachers need to be aware of their own choice of language as well as recognising the difficulties that the language might cause learners.

Some of the issue around language is likely to be cultural. In many Asian languages, all two-digit numbers have the same format and support understanding place value. In Dutch 0.14 is read as 'nul punt veertien' (zero point fourteen), with an understanding that fourteen here means fourteen hundredths. Students in England often arrive at secondary school saying 'zero point fourteen' and there appears to be a common understanding amongst mathematics teachers that this might promote misconceptions linked to whole number thinking.

9.5.9 Two-dimensional Imagery

Often fractions are introduced and taught with two-dimensional imagery, for example circles and squares cut up and shaded. Often fractions are taught in a

practical context with fractions of a pizza or a chocolate bar. When decimals are introduced, there are different models considered. Other than the number line, the other models are two-dimensional (for example the 100-square and the decimat). These two-dimensional models link decimals back to fractions and an understanding that, for example, $0.73 = \frac{73}{100}$. They support a one-dimensional image and a one-dimensional concept. In my study, the students displayed a clear preference for the decimat over the number line. This was seen, for example, in the Maths4Life task (see section 6.4.10), the Excel decimat task (see section 8.1.5) and the follow up questions (see section 8.2). It is not possible to know whether this preference was due to the tacit models of the students (Fischbein, 1989), or because of my influence.

Chapter 10

10 Reflections

This chapter provides my reflections on the study.

10.1 Structure, Understanding and Intuitions

Designing the intervention to be based on structure promoted many good conversations with the students about why things might be true in one setting and not another. The idea of using structure was not that students should necessarily be learning about structure, but more that teachers should be using structure, namely instantiations and properties, to design episodes where the students used these. Most of the tasks in my intervention used two or more instantiations and frequently asked the students to move between one and another. Immediately after the initial intervention, students were able to tell me about magnitudes and to order numbers, something that they could not do before.

Over the years I have been teaching it has been clear that understanding is a difficult concept and my review of the literature showed me that there are many layers to understanding from “primitive knowing” to “inventising” (Pirie & Kieren, 1994, p. 167). Though there are many different models of understanding they all have layers. It is difficult, therefore, for a student or their teacher to know whether the student has understood something. Some of the theories of understanding include an “intuitive understanding” (V. Byers & Herscovics, 1977, p. 26).

It was only after reviewing the interviews after intervention 1 that I realised that the students appeared to have reverted their understanding to an incorrect intuitive understanding; even though they still recalled and could discuss the lesson and what they had learnt. The intuition was strong and persistent. Because of this,

interventions 2 and 3 were designed to help overcome these intuitions.

Overcoming intuitions, though, takes time and a short intervention is unlikely to be enough by itself. The study does, however, give the teacher tools to work with to overcome the intuitions.

I found that when I designed the lessons using instantiations, I was more aware of the important aspect of the concepts that I wanted to ensure were taught. I think it is easy to sweep certain difficult ideas under the carpet when planning, knowing it is unlikely that it will be addressed in a lesson. If the lesson is designed with the instantiations and properties at the centre of the planning, then those difficult points can become of the focus of the lesson. An example might be that only integers have a successor. In lessons about decimals, it might happen that this question is considered, however in a lesson where the structure of natural numbers and rational numbers are compared this will be a central point. I have spoken to teachers who would suggest that mentioning this to a class might confuse them. I would argue that not mentioning it might confuse them more.

10.2 Curriculum

The students in my study were 11 and 12 year old students, all of whom had been taught decimals at primary school. Most arrived at secondary school with very confused ideas about fractional magnitude and place value. It would be easy as a secondary school teacher to assume that the students have been taught decimals and that at secondary school, we are just recapping the essentials. However, fractional magnitude and place value are central to much of number work that is the basis for the work in secondary school and it is imperative that it is not something that should be rushed. Students are not able to answer questions about decimal magnitude in Year 11. Consideration needs to be given to the pace of the curriculum when breadth is being valued over depth, but given some Year 11

students are not able to convert $\frac{1}{4}$ to a decimal. Teachers need to be aware of the difficulties in this basic area of number work that a large proportion of students hold and there needs to be training for teachers on how to overcome these problems.

10.3 Difficulties

My research design needed to be flexible and adaptable. In part, this was due to the qualitative and cyclical nature of my chosen research technique. Action research is intended to be iterative and to take account of the findings of previous cycles.

There was a greater need to be flexible because the intervention was in a school. The students changed from one intervention to another, some students had a high level of absence, some students gave consent and then retracted it. In addition, my changing role in the school meant that I had to adapt and be flexible. For example, I undertook the intervention with another teacher's class because I no longer had a Year 7 group of my own.

Challenges were also presented due to COVID. The students I wanted to work with were not in school when I wanted to interview them, and there were restrictions on how teachers were allowed to interact with students during the second and third intervention. The partial closure of the school due to COVID meant that I had to cancel most of the second interviews, however, by then I already had enough data to design the second intervention. My teaching in the second intervention was different from the way I would normally work. I was not able to use as much collaborative work as I would have liked, and I had to maintain a distance from students. I did, however, find ways around these problems, by using mini whiteboards, some specifically designed for the

intervention. I also ensured that there were enough resources so that students did not need to share.

10.4 Methodology

As I was researching my own practice, I was necessarily part of the study and I will have shaped it. I was aware of my possible impact all through the study, and although there will have been times where I lacked awareness in the moment, the fact that the project has taken place over a number of years means that in my reflections and my interpretation of events I am more able to be objective. The lens through which I have viewed the data became more objective as I interpreted the data as items of evidence and less connected to the memories and emotions that might have been part of my interpretation had it been closer in time to the events. Drake (2010) warns about the interpretations of data in qualitative research where the researcher is within the institute that is being researched. Her review of the data several years later gave her a more objective standpoint.

Qualitative research typically leads to a large amount of data for analysis (Cohen *et al.*, 2011). Because of this I chose the data I wanted to collect carefully. I decided not to use recording in lessons. Firstly, because this was a class I did not know, and I did not have the level of mutual trust that I would have had with my own class. I thought, therefore, that recording the class might lead to changes in behaviour that was less likely when not recording. I would also have had to decide whether to record myself and my interactions with students or whether to have several recording points around the room to record the collaborative utterances of the students. Either way, recording would have quickly amassed an overwhelming amount of data. On balance, I decided that with the photos and the journal I would have enough data to analyse. On reflection, there are a few conversations that I

had in the classroom that I would now like to be able to review verbatim, however, the interviews gave me the opportunity to question these points in detail.

By keeping a journal I was able to select key issues and focus progressively (Cohen *et al.*, 2011) responding to the misconceptions and problems that arose, adapting the intervention as it progressed. I was also able to justify the adaptations both within and between lessons. Due to the limitations of teaching and researching, I could not take notes during the lesson and did so as soon as I could after the lesson. This was usually in the evening. Completing the notes later required me to use my own memory which would might be “fallible, selective and over-interpreting a situation” (Cohen *et al.*, 2011, p. 541). Although this is a possible cause for concern, I triangulated the data with the photographs and I also used the interviews which were recorded and transcribed.

10.5 Positioning

In this project I was both teacher and researcher. As such, I had a different view of situations than either a teacher or a researcher would have had. As a teacher in the study school, I was aware of the pressures on staff to improve results, maintain behaviour, follow school policies and so on. I knew about the standard way of working within the school and was not able to deviate far from non-negotiables and standard ways of working, though my position on the senior leadership team gave me more flexibility than most teachers would have had. As a researcher, I was attending to different things than a teacher would have and therefore I was aware of things that I would not have noticed had I not also been a researcher. However, although I was both a teacher and a researcher, morally and ethically, I needed to ensure I was a teacher first. When Tasha from the first intervention said that 0.75 was smaller than 0.8 because when you cut a cake into 8 you get more of a cake than when you cut it into 75, I needed to revert to teacher

(see section 6.7). As an interested observer I could have recorded the interaction and moved on, but as a teacher I needed to talk to the student and try and unravel and overcome the misconception.

As a teacher, I have worked for several years in a reflective profession. The major difference between the reflection in my career and as a researcher is the depth of the reflection. As a researcher I have observed, questioned, inferred, built a working hypothesis, tested that hypothesis and concluded. As a teacher I questioned, assumed a result and worked from this point. As an observer in lessons, I have frequently seen a teacher assume that a response means a particular thing. The look of confusion on the student's face whilst the teacher has moved on tells me that the teacher was not always right. As a researcher I have checked myself and have slowed down, asking more questions to ensure my working hypothesis has more depth, tried to ensure that my questions are not leading and that I do not build up the answers that students give me to make them the answer I am expecting. For example, rather than drawing a number line, which by the nature of what I draw will restrict the answers that the student can give I have asked students to draw their own. When students have given me the correct answer, I have tried to remain neutral and asked for more information to ensure that the correct answer has been formulated from correct reasoning. I did realise that I frequently still jumped to a conclusion too quickly, however, I did attend to how I behaved in interactions and become much more aware of my actions.

I have found that in many conversations with teachers, sweeping statements are made; 'the students learned', 'we all discussed', 'they were able to'. These statements, 'teacher speak', lead to working theories and conclusions which are not always correct. This study has made me much more aware of 'teacher speak',

either when I hear it or when I am the speaker, and I am more wary of conclusions formed.

A teacher can test theories as an observer could not. The issue here might be that testing the theories could conflict with what is considered best practice in teaching or even in the school, leaving the teacher with a dilemma, this is the situation I faced in the interview with Tasha (see section 6.7).

10.6 Limitations

Returning to the research question

Can the mathematical understanding of students be modified by encouraging students to engage with underlying mathematical structures?

My work limited this question to a very narrow range of mathematical ideas, place value and the magnitude of fractional amounts. These ideas appear crucial to the mathematical development of secondary aged school students.

My study is the study of three small classes, in one school, with one teacher. I make no claims about the finding of my study being reproducible. The method and the testing are, though, reproducible. The work offers a guide to how teachers could interrogate the fractional magnitude understanding of the students they teach and may offer them insight into the difficulties that can arise in the classroom. It also shows how intuition can inhibit the mathematical development of students.

What I investigated was whether I could apply some theory on a small scale in a classroom. There are many questions that come out of the study that could be investigated in other classrooms. These are set out in section 10.8. As a teacher in the school, I knew the students and the processes in the school. I had had an input into writing the curriculum that the students were following with their class

teacher and was able to work with this in designing and implementing the intervention.

The work that I have done has been small scale. Most of the insights have been when I have worked with very small groups. The strength of this approach is the fact that working with small numbers of students I was able to work in detail, interrogating the mathematical thinking of school age children.

In addition, although the intervention was conceived when I would likely have been the class teacher and would have been considered a 'normal' teacher by most of the students in the school, the intervention has taken place whilst I have been a member of the senior leadership team. It is easier for me to have the students motivated and engaged than it would be for many teachers in the school. The counter to these advantages is that the students, seeing someone they might consider senior in the school might mean that the students are less likely to be open or allow themselves to appear vulnerable. Even with this, the findings around cognition are likely to be similar in any classroom and are of potential use to other teachers.

In terms of the study linking how students understand when the teaching focuses on structure, the study applies to a very small subset of mathematics. Where I had originally considered that I would teach the class for a year, I would have covered the basics of algebra and multiplicative reasoning as well as the magnitude of fractional amounts. Restricting the study to the magnitude of fractional amounts, whilst an important area of mathematics does limit the potential impact of any findings. However, it has led to some important insights into the persistence of intuitions and misconceptions and the value of close attention to one area of relevant mathematical knowledge.

In each cycle, the number of students involved was small. Because absences are high at the study school, and because students move in and out of the school all through the year, the number of students present for both the interventions and pre-test and post-test was small. However, this made it easier for me to home in to apparent contradictions in the knowledge of individuals as they undertook the tasks. This close attention led to me finding that students who appeared to have changed their understanding to match the correct mathematical concept, reverted to an intuitive response after a few weeks. The small number of students allowed me to track this development. It also allowed me to develop and adapt the tasks in a way that would not have been possible with a larger cohort. My work with the intervention groups was therefore more adaptive to the needs of the students than teaching them in a class of 32 would have allowed.

One limitation was only having one teacher. Because I did all of the research, I know that the work is uniformly applied, but I could have brought my own biases to the study. However, I maintained a questioning mode throughout, seeing their responses as data for new attempts at explanation. Consequently, my engagement with research literature throughout was to find explanations for their mathematical behaviour, rather than merely looking for confirmation of my initial hunches.

The pre- and post-tests were administered by other teachers, and whilst I did talk to some of the classes to explain the work, I still cannot say that the tests were administered exactly as I would have administered them. I did, however, mark all of the tests. As there was no teaching or teacher input the only real difference could have been that teachers offered covert help to the students. The tests were not used for quantitative data but were there to help me look for patterns or error, any extra help provided would only have hidden some errors.

Working with small groups on a focused topic area over a number of cycles allowed me to use the test instrument, the interventions and the interviews to interrogate deeply the thinking of the students involved. I was able to consider deeply how students perceive fractional amounts and how this can inhibit a good understanding of the magnitude of fractional amounts. I was able to design and adapt the intervention tasks in response to what I was seeing in lessons as well as by investigating the literature. I have been able to bring to the foreground the issues faced by class teachers trying to teach these areas of mathematics.

10.7 Recommendations

As a result of this study I make the following recommendations for future practice:

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- When planning teachers should consider the structure of the concepts to be taught. In particular, they should look at the different instantiations related to the concept and the essential and necessary properties linked to them.
- Teachers should use tasks that explicitly address the properties, so that students engage with where properties hold and where they do not.
- Given that instantiations should be considered from point of view of the learner, it is possible that the teacher would plan a series of lessons that will not include all of the instantiations. It is therefore important that the teacher is sensitive to the views of the students as the need for another instantiation might not arise until the lesson and the teacher would then need to consider how to incorporate that instantiation into the lesson. It is desirable for the teacher to work to merge some instantiations. For example, revisiting the example from earlier (see section 3.2), if students consider that solving the

linear equation $3x = 9$ is a different instantiation than solving the linear equation $3x = 10$, the teacher should aim to design tasks so that the students become aware that these could be the same instantiation. The study suggests that working with the students to encourage them to be more flexible with their engagement with instantiations will improve their mathematical achievement. When teaching the concepts, teachers should be open to recognising instantiations that the students are using that they themselves had not anticipated and working with those new instantiations.

- Teachers should anticipate existing incorrect intuitions and unhelpful mantras and design tasks that address these, by 1) cognitive conflict 2) teaching by analogy and by 3) explicitly warning students about the intuitions.
- Teachers should be very aware of their language around the magnitude of decimals and ensure that wherever possible they use language that supports the magnitudes, e.g. seven eighths as opposed to seven over eight.
- Teachers should use the decimat rather than the 100-square as a representation of a decimal.
- Teachers should avoid the use of mantras, where the mantra does not support future mathematical development.

These recommendations do not come in the form of an off-the-shelf package that could be given to schools to implement. As it is a way of viewing the mathematics to be taught and the students' reactions it would be best to disseminate the ideas via working with schools and via workshops. I presented interim findings at the British Society for Research in Learning Mathematics conference on 5th June 2021. This conference would be a suitable audience for the final findings. As I

work in a trust with three secondary schools, I will be able to present the ideas to local schools.

10.8 Questions for further study

Further work is needed to test the framework described in 9.1. I have applied it to a small setting and one context. The framework needs to be tested in a variety of contexts, across different aged students and different concepts to be taught.

Intuitions are hard to shift, and I have seen that, both in this study, and also in my teaching. I have noticed that the students who attained more highly at primary school rarely have the same level of confusion over fraction magnitude and are less likely to have some of the less helpful mantras. They also appear more ready to accept that mantras have been outgrown. I recognise that this study only worked with a small number of previously lower attaining students, however this study and my own experience of teaching students of all levels of attainment lead me to question whether there is a link between persistence of intuitions / mantras and difficulties experienced by learners in mathematics. Is it possible that mathematics difficulties are caused by the inability to shift intuitions or is that an inability to shift intuitions is the cause behind some mathematics difficulties? Investigating this link is an area for future study.

Research about tacit models and intuition (Fischbein, 2001) often discuss the difficulty in understanding the abstract nature of the infinite. For example, students have a tacit model of 0.999... approaching 1, but not a model that it is actually 1. Given the continuous nature of the number line, it seems possible that the difficulty in considering place value is linked with the density of numbers on the number line. All representations of decimal number are necessarily finite, however a thorough understanding of decimal place value requires an appreciation of the

infinite. Investigating whether there is a link between understanding place value and appreciating the infinite nature of the number line is an area for future study.

There is much in the current professional literature about practice and mastery. It is possible that 'over-teaching', i.e., revisiting the concepts of magnitudes of fractional frequently and practising working with them until procedures are fluently applied have allowed the students to reformulate their understanding and replace the intuitions with a more accurate model. This would require careful design of the teaching episodes so that students do not reinforce their existing but, inaccurate intuitions. The question for future study is whether incorrect intuitions can be overcome by frequent practice of procedures with correct structures.

The study, and my experience, suggest that there are mantras which the students have self-generated and uncovering these is an important step in understanding the mathematical thoughts of the students. Other mantras, however, have been offered to students earlier in their mathematical career to help them master some mathematical concepts. These mantras, alongside some overgeneralisations have developed into intuitions which then cause misconceptions and stop students from acquiring a thorough understanding of some concepts. It might also be the case that only a small proportion of students cling to the mantras that then cause difficulties. A research question is whether the mantras introduced to help the majority of students to acquire early mathematical concepts, then cause a minority of students to develop incorrect, persistent intuitions. This would require consideration of the difference between natural intuitions (more A more B) and intuitions which have arisen through teaching, either deliberately or from overgeneralising.

10.9 Conclusion

My initial focus in this study was on defining and using structure to design teaching episodes. There was a strength on viewing both structure and errors through the lens of the learner; allowing me access to the thinking of the student and in a sense allowing me to come alongside them; to work with them. Whilst this might not be feasible for a teacher teaching 32 students in a class, even having an awareness of the learner's viewpoints will direct what the teacher attends to in a lesson. By focusing on the instantiations and the properties, when planning lessons, teachers need to think carefully about what is important to teach and what language and tasks should be used. When classifying errors as I have done here the teacher needs to ask about the student's experience of the question. This will inform them about how to help the student further.

Although the literature about understanding referred to intuitive responses, I was surprised by the strength of these and how they disrupted new learning. This is perhaps an explanation as to why Year 11s still hold misconceptions that teachers feel they should have long outgrown. I think there is a significant piece of research that could be undertaken linking early learning with mantras, development of intuitions and the best way to prevent unnecessary intuitions and to overcome developmental ones.

Each of my interventions was short and I believe that using the ideas in the intervention sustained over a longer period of time might well have ensured that the new learning overcame the intuitions. An example here would be the continued use of the decimat. Students appeared to work well with this representation and were able to recall it much later, but needed to have worked with it more to be familiar with it. In addition, regular activities with the number line and connecting

that back to the decimal would help students to be able to work more accurately with fractional magnitudes.

One consideration is that this study has looked at students for whom mathematics was not successful at KS2. The difficulties that I have written about: not being able to use the number line; mantras building incorrect intuitive responses; the persistence of incorrect intuitions etc appear to have developed in those students who did not manage to access fully the KS2 curriculum. Because of the demographics of the study school, lower attaining students are over-represented within the school population. Because of this, shortfalls in knowledge at KS3 might be more noticeable and, indeed, without a significant number of students facing these difficulties I might not have noticed the effects of intuitions on the class, seeing it is just something affecting a few students. It also means that investigating these difficulties and ways to overcome them becomes very important to improve the mathematical achievement of the students in the school.

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Appendix 1 COVID general background.

On 18th March 2020 the government declared that schools were to be closed other than for vulnerable students and children of key workers from 20th March.

Wherever possible, people were required to maintain social distancing; this meant being 2m apart from each other. Anyone with symptoms of COVID or who was a contact of anyone with COVID was required to self-isolate. In May, it was announced that from early June, secondary schools should extend their opening to their year 10 and year 12 cohorts (aged 15 and 17), but only to have 25% of the cohort in school at any time and to ensure that the classrooms allowed social distancing. This was alongside continuing to care for and educate the vulnerable children and children of key workers. Through all this, the schools were expected to provide work for students at home.

At my school there were, on any given day between March and June between 2 and 30 children. (Up to about 8% of our roll). Our staff were split into 5 teams, each one headed by a member of the senior leadership team (SLT). Any teacher who was clinically vulnerable, had children who were clinically vulnerable or who did not have childcare, worked from home entirely. This made up two of the five teams. The other three teams came in on a rota basis, teaching the children in school. The school remained open over the Easter break and the bank holidays to support parents of key workers. Online work was provided for all our students, as well as paper-based work for those without internet. Any student who had SEND needs or was vulnerable but was not in school was contacted regularly. As the lockdown continued, the school's IT system was moved to a cloud-based system, including using Microsoft's office 365 for our email and Microsoft TEAMS for meetings and live lessons. This was the first point where the school was able to start providing live lessons, although with no training and many staff having to

juggle work and looking after their own children, live lessons were optional for staff.

Alongside this, the government provided laptops and mobile routers to schools to lend to students who did not have the capacity to access online learning at home.

In September 2020, schools reopened but had to maintain higher than normal levels of hygiene and maintained bubbles between which children did not mix. In secondary school, this was frequently that year groups did not mix. When any student tested positive for COVID-19, close contacts were identified amongst staff and students and bubbles were sent home for up to 10 days.

In classrooms, all the tables had to face the front. No equipment could be shared. Staff and students maintained a 2-metre distance. Staff were not able to circulate. If teachers spent more than a few moments with a student then the teacher could be identified as a close contact if the student tested positive. As part of the teaching strategies in school, staff increased the use of mini whiteboards so that they could interact with the students other than verbally. Everyone wore masks in communal areas. Students did not mix between year groups at any time they had separate areas at social times, separate areas on transport and would be wearing masks when they briefly crossed in corridors.

In September, one student tested positive for COVID 19. There was a delay before the case was officially confirmed and so year 10 were sent home for 2 days. In November, three siblings tested positive in three different years meaning that three year groups had to be sent home alongside six teachers who had not maintained sufficient distance. These six staff members included everyone in the mathematics department except for me. Once at home students had their normal lessons provided live.

In November, the government declared a four-week lockdown. This time, though, schools were not closed. Masks were now to be worn at all times in school.

Cases rose in December meaning the government had to scale back plans to allow families to spend Christmas together. On 3rd January, the government encouraged all parents to send their children to school on 4th January 2021 (the first day of term). Lateral flow testing was introduced into schools; the concept was that students would be regularly tested so that, if cases were identified, contacts could be tested daily rather than having to be sent home to self-isolate. On the evening of 4th January 2021, the prime minister announced a further lockdown and that schools were to be closed, other than to vulnerable students and children of key workers from the next day. The expectation was that all students would be provided with a remote education which could be a mix of work, pre-recorded and live lessons.

At my school between 30 and 60 students attended school through this lockdown daily. The students and staff were tested for COVID-19 at school through this period. The provision for these students was managed by SEND team and support staff who helped them to access the same live lessons that their classmates at home were accessing. These children along with those at home joined in live streamed lessons to their normal timetable.

Schools reopened on 8th March 2021 and the COVID precautions were the same as in December. All students were in bubbles, maintained a 2m distance from adults and wore masks. The school provided a lateral flow testing station, and each child was tested three times as they arrived back at school. From that point on students were required to test twice a week with home testing kits.

GCSE exams were cancelled in summer 2020 and in summer 2021. In summer 2020, grades were replaced by Centre Assessed Grades (CAGs). This required teachers and departments to predict grades for every student and to rank them from the highest to lowest grade. This work needed to be coordinated and quality assured by the school leadership. Similar procedures were required for BTEC qualifications. In summer 2021, however, schools were required to assess the students via portfolios of work for both GCSE and BTEC qualifications. From these assessments students were given Teacher Assessed Grades (TAGs). There were several layers of quality assurance that were required for these.

In July 2021 COVID restrictions were relaxed. By September 2021, when children returned to school, all the staff were doubly vaccinated and rules around self-isolation now stated that children and doubly vaccinated adults did not need to self-isolate if they were identified as close contacts. During the period up to September 2021 the school had had very few cases of COVID. The school had approximately 50 staff and 450 students. From September 2021 until the relaxing of all restrictions in February 2022 there were 50 days of staff absence due to COVID and 970 days of student absence, resulting from 186 students absent with COVID. However, the highest level of absence in the school from COVID was after February 2022, with 123 days of staff absence between February and July 2022. The school had to be closed to whole year groups in February and July because it was deemed that there were too few staff to keep the school open safely.

Appendix 2 Assessment Instrument

1

Add one tenth

(e) 4.254 →

(f) 2.9 →

Take away one hundred

(g) 583 →

(h) 30 000 →

2

Add ten

(a) 3597 →

(b) 0.15 →

Add one hundred

(c) 21 534 →

(d) 19 930 →

3

This meter counts the people
going into a football stand:

0	6	3	9	9

After **one** more person has gone in,
it will read:

4

For the answer to a maths question Dave got 4.9 and Lynn got 4.90

Is there any difference between them?

Why?

.....

.....

5

(a) Draw a ring round the **bigger** of the two numbers: 0.75 or 0.8

Why is it bigger?

.....

.....

(b) For each pair, draw a ring round the **bigger** number:

(i) 20 100 or 20 095

(ii) 7.55 or 7.5

(iii) 4.06 or 4.5

6

Write down any number between:

(a) 4000 and 5000

(b) 4100 and 4200

(c) 0.4 and 0.5

(d) 0.41 and 0.42

(e) **How many** different numbers **could** you write down which lie
between 0.41 and 0.42?

7

Write down the smallest and the largest of these: smallest largest

$n + 1,$ $n + 4,$ $n - 3,$ $n,$ $n - 7$

$2n > n+2$ Is this always, sometimes or never true

Explain

.....

.....

8

What can you say about a if $a + 5 = 8$

What can you say about b if $b + 2$ is equal to $2b$

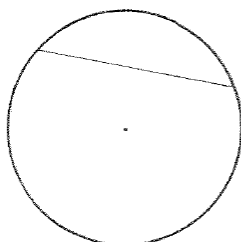
9

When are the following true – always, never, or sometimes?
Underline the correct answer:

$A + B + C = C + A + B$ Always Never Sometimes, when

$L + M + N = L + P + N$ Always Never Sometimes, when

10



(a) Has this circle been split into 2 halves?

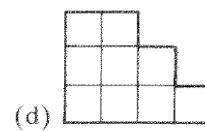
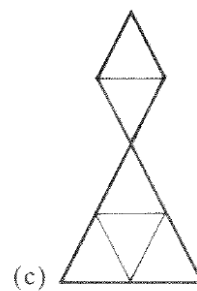
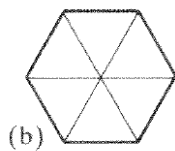
.....

(b) Why do you think this?

.....

.....

Shade in two-thirds of each of these shapes:



Appendix 3 Comparison Pre- and Post-test Raw

Results

Student	Gender	Service	SEN Status	FSM	K32 Eng	K32 Maths	post test question by question																				post test		pre test					
							1e	1f	1g	1h	2a	2b	2c	2d	3	4	5a	5b1	5b2	5b3	6a	6b	6c	6d	6e	10	11a	11b	11c	11d	y	n	y	n
1	F	y	N		91	95	n	n	y	n	y	n	y	n	y	n	y	n	y	n	y	n	y	n	y	y	n	y	n	9	9	11	7	
2	M	n	N		96	99	n	n	y	n	y	n	y	n	y	n	y	y	n	y	y	y	y	y		y	n	n	n	12	9	11	6	
3	F	y	N		101	100	n	y	n	n	y	n	y	y	y	y	y	y	y	y	y	y	n	n		y	y	y	y	17	6	15	9	
4	F	y	N		99	99	n	n	n	n	y	n	y	y	y	y	y	y	y	n	y	y	n	n	n	y	y	y	n	13	11	16	8	
5	F	y	N		103	116	n	n	y	y	y	n	n	n	y	n	y	y	y	y	y	y	n	y	y	y	y	y	y	17	7	22	2	
6	M	y	N		101	103	y	y	y	y	y	y	y	n	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	23	1	19	5	
7	F	n	K						n	n	n				n	y	n	n	y	n					n	y	y	n	n	n	4	11	0	6
8	F	n	N		100	91	n	y	y	n	y	n	n	y	y	y	y	y	y	y	y	y	y	n	y	y	n	n	y	17	7	1	0	
9	M	n	K		100	99	n	n	y	n	n	n	n	n		y	n	y	n	y	y				y	n	n	n	n	6	14	7	6	
10	F	y	N		88	96	n	y	y	n	n	n	y	y	y	y	y	n	y	n	y	y	y	y	n	y	y	y	y	n	16	8	12	8
11	M	n	K		100	89	y	n	y	n	y	n	y	y	y	n	n	y	y	n	y	y	n	n	n	y	y	n	y	n	13	11	9	15
12	F	y	N		96	105	n	n	n	n	y	n	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	n	17	7	20	4	
13	F	n	N		119	110	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	23	1	23	1	
14	F	y	N		110	109	y	y	y	y	y	n	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	22	2	23	1	
15	M	y	K		87	91	n	n	y	n	y	n	y	n	y	n	n	y	y	y	y	y	n	n	n	y	y	n	n	n	11	13	12	8
16	M	y	K		100	102	n	n		n	y	y	y	y	y		y	n	n	y	y	y	y	y		y	y	y	y	y	16	5	13	9
17	F	n	N		108	99	n	n	n	n	y	n	y	y	y	y	n	n	y	y	y	y	y		n	y	n	n	n	12	11	12	7	
18	M	n	K		91	91	n	n	y	y	y	n	y	n	y	n	n	y	n	y				n	y	y	n	n	y	10	10	6	4	
19	M	n	N		108	110	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	23	1	20	3	
20	M	y	N		101	108	y	y	y	y	y	n	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	23	1	18	1	
21	M	n	N		113	111	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	n	n	y	y	y	y	22	2	22	2	
22	F	y	N		118	103	y	n	n	n	y	n	y	y	y	y	y	y	y	y	y	y	n		n	y	y	y	y	17	6	16	8	
23	F	n	N		99	101	n	n	y	n	n	n	n	n	n	y	n	n	y	y	y	y	n	n	n	n	y	y	n	10	14	12	8	
24	F	n	N		104	104	y	y	y	y	y	n	n	n	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	20	4	17	7	
25	F	n	N		100	108	y	y	y	y	y	n	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	22	2	21	3	
26	F	y	N		91	96	n	n	y		n	n	y	n	n	y	y	y	n	y	y	y	y	n	y	y	n	n	n	12	11	7	5	
27	M	n	K		110	109	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	y	23	1	19	5	
28	M	y	K		107	104	n	n	y	y	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	n	19	4	20	4		
29	F	y	N		105	104	n	n	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	21	3	20	4	
30	F	n	N		100	97	y	n	y	n	y	n	y		y	y	y	n	y	y	y	y	y		y	y	y	n	y	17	5	14	10	
31	M	y	K		0	88																			y	n	n	y		2	2	1	6	
32	M	n	N	Y	107	103	y	n	y	n	y	n	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	y	20	4	11	7	
33	M						n	n	y	y	y	y	y	y	y	n	y	n	y	y	y	y	y	n	y	y	n	n	n	16	8	11	12	
34	F	n	N		103	99	y	y	y	y	y	n	y	y	y	y	y	n	y	y	y	y	y	n	y	y	y	y	n	20	4	19	5	
35	F	y	N		84	93			y	n	y	n	y	y	y	n	y	y	y	n	y	y	y	n	y	y	n	n	n	14	8	7	13	
36	M	n	N		107	110	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	23	1	22	2		
37	F	n	K		103	102			y	n	y	n	y	y	n	y	y	y	n	y	y	y	y	n	y	y	n	n	n	14	8	12	11	
38	M	n	N		113	100	n	n	y	y	y	n	y	y	y	n	y	y	y	y	y	y	n	n	n	y	n	n	n	13	11	11	13	
39	F	n	n		112	99	n	n	n	n	n	n	n	n	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	15	9	16	8	
40							y	y	y	y	y	y	n	y	y	y	y	n	y	y	y	y	y	y	y	y	y	y	y	22	2	19	5	
41	M	n	N		108	105	n	y	y	y	y	n	y	y	y	y	y	y	y	y	y	y	n	y	n	y	y	y	y	20	4	13	11	
42	F	n	N		108	111	n	n	n	n	y	n	y	y	y	y	y	y	y	y	y	y	y	n	n	y	y	y	y	17	7	15	9	
43	F	n	N		113	110	y	y	y	n	n	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	y	21	3	19	3	
44	M	n	N	Y	90	101	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	n	n	y	y	y	n	n	19	5	17	7	
45	M	n	N		116	104	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	n	22	2	20	4		
46	M	n	N		101	106	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	24	0	22	2	
47	M	n	N		103	104	y	y	y	y	n	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	n	y	21	3	20	4		
48	F	n	K	Y	96	89	y	y	y	y	n	n	y	n	n	y	y	n	y	y	y	n	n	n	n	y	y	n	n	n	13	11	3	5
49	F	n	N		110	103	n	n	n	n	y	y	y	y	y	y	y	y	y	y	y	y	n	n	n	y	y	y	y	17	7	16	6	
50	M	y	N		99	105	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	23	1	20	4		
51	M	n	K		91	97	n	n	y	n	y	y	y	y	y	n	n	n	n	y	y	n	n	n	n	y	y	y	y	14	10	5	8	
52	M	n	N		108	103	n	n	y	y	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	y	21	3	19	5	

Appendix 4 Intervention Pre and Post-test Raw Results⁴

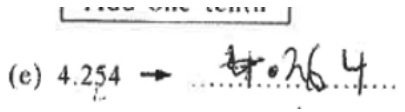
Student number	Gender	Service	FSM	SEN CODE	KS2 English	KS2 Mathematic	Group	Post test question by question																						Post test		Pre Test			
								1e	1f	1g	1h	2a	2b	2c	2d	3	4	5a	5b1	5b2	5b3	6a	6b	6c	6d	6e	10	11a	11b	11c	11d	y	n	y	n
1	M	Y	F		103	102	Neither	n	y	y	y	y	n	y	y	y	y	y	y	y	y	n	b	y	y	y	y	20	3	17	7				
2	M	N	F				Neither	n	n	y	y	y	n	y	y	y	y	y	y	y	b	y	y	y	y	y	19	4	19	5					
3	F	N	F	K	98	100	Neither	n	n	y	y	y	b	y	n	y	b	b	y	y	n	n	b	b	b	y	y	n	n	n	10	8	11	9	
4	M	N	F		93	96	Neither	n	b	y	b	y	n	y	y	y	n	y	y	n	y	n	n	y	n	n	n	12	10	12	8				
5	F	Y	T		99	99	Neither	n	n	y	n	y	y	y	y	y	y	y	n	y	n	n	y	n	y	y	y	17	7	15	6				
6	M	N	F				Neither	n	n	y	n	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	20	4	17	7				
7	M	N	T		90	99	Neither	n	n	n	y	y	n	y	y	y	y	y	y	y	y	n	n	y	y	n	n	15	9	14	6				
8	M	Y	F		98	99	Neither	n	n	y	n	y	y	y	y	y	y	y	y	y	y	n	y	y	y	n	n	17	7	21	3				
9	M	Y	F		104	101	Neither	y	b	y	y	y	y	n	y	y	y	b	y	y	y	y	y	y	y	y	y	18	4	13	6				
10	F	Y	F		92	98	Neither	n	n	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	y	y	21	3	16	6				
11	M	N	T	K			Neither	n	n	y	y	y	y	y	y	n	n	n	n	y	n	b	n	b	y	n	n	11	11	12	7				
12	F	Y	F		102	99	Neither	n	n	n	y	y	n	y	y	y	n	y	y	y	y	y	n	b	n	y	y	n	y	15	8	12	11		
13	M	Y	F		101	97	Neither	n	y	y	n	y	n	y	y	y	n	y	y	n	y	y	b	b	n	y	y	n	n	13	9	12	10		
14	M	N	T		99	101	Neither	n	y	y	n	y	n	y	y	y	y	n	y	y	y	y	n	b	n	y	n	n	13	10	7	13			
15	F	Y	F		104	100	Neither	n	n	y	n	n	n	n	y	n	y	y	y	y	n	y	b	b	b	y	y	n	n	11	10	9	8		
16	F	Y	F		108	101	Neither	n	n	y	y	y	y	y	y	y	y	y	y	y	y	y	n	y	y	y	b	y	20	3	19	5			
17	M	N	F		99	100	Neither	y	y	y	y	y	y	y	y	y	y	y	y	n	n	y	y	y	y	n	y	y	y	21	3	16	4		
Amber	F	N	F	K	96	86	Both																					0	0	5	9				
Anton	M	Y	F	K	93	97	One	n	n	n	n	n	n	n	n	y	y	y	y	y	y	n	n	n	b	y	y	n	n	n	9	14	7	8	
Billy	M	Y	F				One	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	b	n	y	n	n	19	4	21	3			
Camero	M	Y	F	K	95	95	One	y	y	y	b	y	y	y	n	y	b	b	y	n	y	y	y	b	b	b	b	y	n	n	12	5	8	11	
Chantell	F	Y	F		102	96	Both	n	n	n	n	n	y	n	n	b	y	b	n	y	n	n	y	y	y	b	b	y	y	n	n	8	12	10	13
Ciaran	M	N	F	K	90	93	Both	b	b	y	b	b	n	y	b	y	y	y	y	n	n	n	y	b	b	n	y	y	n	n	9	8	7	15	
Claire	F	N	T		100	94	Both																					0	0	9	11				
Fergus	M	N	T				One	y	y	n	y	y	y	y	y	n	y	y	y	y	y	y	y	y	y	n	y	y	y	y	21	3	23	1	
India	F	N	F	K	97	96	Both	y	y	y	n	y	y	y	y	y	n	y	n	y	n	n	n	n	n	y	y	n	n	13	11	12	6		
James	M	N	F		102	92	Both	n	y	y	n	y	y	y	y	n	y	y	y	y	n	y	n	n	n	b	b	y	y	n	y	13	9	11	9
Joseph	M	N	F		95	94	Both	n	y	y	y	y	y	y	y	y	y	b	y	y	y	y	y	y	y	n	y	y	b	n	n	18	4	17	5
Katie							One																					0	0	13	11				
Liz	F	Y	F		90	90	Both	n	n	n	n	n	n	n	n	n	y	n	y	n	n	n	n	n	n	n	y	y	n	n	n	5	19	4	18
Louisa	F	N	T		96	89	Both	n	n	y	n	n	n	y	y	y	n	y	y	n	y	n	n	n	n	n	b	y	n	n	n	8	15	10	8
Lucy	F	N	F	K	87	95	Both	b	b	y	n	n	n	b	b	b	b	b	b	b	b	y	y	b	b	b	b	y	n	n	n	4	6	6	7
Nuri							One																					0	0	21	3				
Rachel	F	N	F		99	96	Both	n	y	n	n	n	n	n	n	y	y	y	y	y	n	y	y	y	b	n	y	y	n	n	n	11	12	13	10
Ruth	F	Y	F		87	83	Both	n	n	n	b	b	b	b	b	b	b	b	b	b	b	b	b	b	b	b	b	b	b	0	3	4	18		
Tasha	F	N	F		93	92	Both	n	n	n	n	y	n	y	n	y	y	y	n	y	y	y	n	n	n	y	y	n	n	11	13	8	16		
Tayla	F	N	F	K	93	86	Both	n	n	n	n	n	n	n	n	n	n	n	n	n	y	n	y	b	b	n	y	n	y	y	7	15	3	21	

Appendix 5 Example of Interview Analysis

Student A						
Student	Initial	Final	Learning		SEN	
A	shorter-larger	apparent - expert	learning retained but clear		None	
Stage management	Teacher	Student	Work	Structure - intuition	Thoughts	Reflection
Student A						
Warning/instructions						
Working till 3.47			1 Add one tenth (c) $4.254 \rightarrow 4.264$ (f) $2.9 \rightarrow 2.99$ 2 Add ten (a) $3997 \rightarrow 3607$ (b) $0.15 \rightarrow 0.25$ 3 This meter counts the people going into a football stand: After one more person has gone in, it will read: 0 6 3 9 9 5 6 4 0 0 4 For the answer to a maths question Dave got 4.9 and I, you got 4.90 Is there any difference between them? <i>the place holder</i> Why? <i>Same</i>			
	T So if we go back to the first question – you've got 4.264 as your answer? Ok so how did you get that? Can you tell me what you did?		1 2 (c) $4.254 \rightarrow 4.264$	Indication of an error in structure of decimal place or of not understanding the question	This was common error in the pretest as there are three places it is not clear whether the students are viewing place value from left to right or right to left.	
		A I added 1 to the 5			This tells me that if A has understood the question she considers the hundredths column to be tenths column, but does not tell me why this is so. Does A consider the 2nd column from the left or second column from the right to be tenths?	
	T Ok so why did you add. Can you explain why you've done that?					

⁴ This includes the students who were in the intervention groups – whether or not they sat both the pre and the post test

This is a photograph of one of the pages of the interview analysis. On the next page there is an example of three line of the analysis

Stage management				
Teacher	T So if we go back to the first question – you’ve got 4.264 as your answer? Ok so how did you get that? Can you tell me what you did?		T Ok so why did you add. Can you explain why you’ve done that?	
Student		A I added 1 to the 5		A It says add one tenth. And one tenth is gonna be 5
Work				
Structure - intuition	Indication of an error in structure of decimal place or of not understanding the question			

Thoughts	This was common error in the pretest as there are three places it is not clear whether the students are viewing place value from left to right or right to left.	This tells me that if A has understood the question she considers the hundredths column to be tenths column, but does not tell me why this is so. Does A consider the 2nd column from the left or second column from the right to be tenths?		
Reflection				

Appendix 6 Example of an entry from the reflective journal

Entry	Classification	Reflections
We were looking at fractions today. I started the lesson by having a number line on the board just like I did in the September intervention. I asked students to come to the front with their white board pens and fill in $\frac{1}{10}$, $\frac{2}{10}$ etc.	Descriptive	
Three students filled in $\frac{1}{10}$, $\frac{2}{10}$ and $\frac{3}{10}$	Student interaction	
Anton filled in $\frac{1}{2}$	Student interaction	Aren had not been in the class in September. I had expected $\frac{5}{10}$ but this gave me the opportunity to discuss equivlances
Lucy came to the board, but seemed not to know what to do.	Student Interaction	
I asked her to read out what was on the board. She then filled out $\frac{6}{10}$	Student Interaction	
Another student filled in $\frac{9}{10}$ then another filled $\frac{7}{10}$ and $\frac{8}{10}$.	Student Interaction	
I asked the student about Anton's answer	Descriptive	

Entry	Classification	Reflections
One student said that he was wrong and that it should be $\frac{5}{10}$.	Student Interaction	
Another student appeared conflicted and said that he agreed it should be $\frac{5}{10}$ but he also saw that it should be $\frac{1}{2}$.	Student interaction	
I managed a discussion around the equivalence of $\frac{5}{10}$ and $\frac{1}{2}$ incorporating the number line from September where we also would have considered 0.5		
I asked the students to complete the fraction comparison test	Student interaction	This task is based on the DCT (Steinle and Stacey) and aims to see what misconceptions
The FCT is based on the Steinle and Stacey DCT so it starts off with some questions such as the smaller of $\frac{1}{7}$ and $\frac{2}{7}$ then it moves on to ones where only the denominators change and then it's one where the denominators change where the top is not 1 so its 5/9 and 5/7 and then it's questions where both the denominators and numerators are different and the students did that quite quickly actually	Descriptive	

Entry	Classification	Reflections
The students completed the task quickly. There was no intention to go over this with the students but looking at it briefly before carrying on with the rest of the lesson almost all of them had made the same mistake, the whole number thinking, the seven is smaller than the nine so $\frac{1}{7}$ is smaller than $\frac{1}{9}$ and I so the aim therefore makes me very aware that something we are going to have to delve into quite deeply is that thing that you are thinking of them as two numbers.	Reflective	Most of the students appear to have a whole number thinking misconception. This will be analysed to find out for each student what their misconceptions are. Expected misconception
I asked the question from Nieme – what is a fraction and why does it have two numbers	Description	
I asked students to write down their answer	Student interaction	The answers (reviewed after the lesson) were varied – some were based on notation. Some gave the specific of one half.
I asked students to share their answers with each other and then to share with the class	Description	
Child A told me that the top number was a numerator Child B said that the number was a denominator	Student interaction	correct

Entry	Classification	Reflections
No student was able to tell me what a fraction was.	Student interaction	This could have been because it was the first lesson. Potentially there is still a lack of trust and the students are not happy to answer questions where they are not sure.
I showed the students a picture of one square out of 4 squares	Description	
One student said that is $\frac{1}{4}$	Student interaction	correct
I asked how they knew and another student said that it was one square out of four	Student interaction	Correct.
I missed the opportunity to check back with the students that this should have been one out of four equal parts.	Reflection	

Entry	Classification	Reflections
The next think the students did was the paper folding fraction activity. I gave each child 4 sheets of different coloured paper. I wrote 1 on one sheet of paper and asked the students to do the same. I demonstrated folding the paper in half. I asked the students how much each folded piece represented	Description	
I asked the students to fold the paper into 2, I demonstrated folding the paper in half. I asked the students how much each folded piece represented		
One student said $\frac{1}{2}$	Student interaction	correct
When I asked why the student answered because it was one out of two	Student interaction	
I wrote $\frac{1}{2}$ on each half of the folded sheet.	Description	
The students all folded and cut their second sheet, and wrote $\frac{1}{2}$ on both halves	Student interaction	
I asked the students to place the two halves on their whole piece of paper to illustrate that $\frac{1}{2} + \frac{1}{2} = 1$	Description	
I took the next piece of paper and demonstrated folding the paper in half and then half again. I asked the students how much each folded piece represented	Description	
One student said $\frac{1}{4}$	Student interaction	

Entry	Classification	Reflections
I wrote $\frac{1}{4}$ on each of the four pieces of the folded sheet.	Description	
The students all folded their second sheet, and wrote $\frac{1}{4}$ on each part	Student interaction	
I took the final piece of paper, folded it in two, then again and again. I asked the students what each area represented	Description	
One student said $\frac{1}{8}$	Student interaction	
I wrote $\frac{1}{8}$ on each of the eight pieces of the folded sheet.	Description	
The students all folded their second sheet, and wrote $\frac{1}{8}$ on each part	Student interaction	
Each student now had several pieces of paper in front of them with representations of 1, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. I asked the students to use the pieces of paper to work out what $\frac{1}{8} + \frac{1}{8}$ is.		
One student is able to show that the answer is $\frac{1}{4}$ by placing the bits of paper together	Student interaction	
I asked the students to work out other sums and write these down.	Description	

Entry	Classification	Reflections
The students, working in pairs, were able to 'make' sums using the bits of paper.	Student interaction	
To finish off the lesson I asked the students to play the fraction game. The aim is to cover the piece of paper with the 1 on it with the fractional bits of paper. Students roll a dice. If the student rolls a one two or three they can put down the piece of paper for $\frac{1}{8}$. If they roll a four or five they can put down $\frac{1}{4}$. If they roll a six they can put down $\frac{1}{2}$. There has to be room for the piece of paper to go down, The winner is the child who first covers their own piece of paper.	Description	
The aim of this task is that the students are engaging with fractional amounts and enjoying their mathematics.	Reflection	As this was my first lesson with the class in this intervention, I wanted to ensure that the students had an opportunity to see that the lessons would be varying and engaging
There was nothing from this lesson that means that I need to alter the plans for next lesson. When I have marked the FCT there may be adjustments – however, I have already planned for whole number thinking which looks like it is the most likely misconception from a quick review of the answers.		

Appendix 7 DCT Classification

Marking Instructions for the 30 item Decimal Comparison Test

Look at the accompanying version of the test, where the different groups of items have been colour coded. First mark Red and Blue groups (both out of 5) and assign a Primary Classification (Longer-is-larger, Shorter-is-larger, Apparent-expert).. Results other than those listed (scores of 2 or 3 or the Low/Low combination) are labelled as Unclassified. Many students will be able to be further classified using their responses to the remaining four groups of coloured items. (Note that Reciprocal/Negative thinkers cannot be separated with this test.)

Red Group	Blue Group	Primary Classification	Yellow Group	Azure Group	Green Group	Pink Group	Finer Classification
Low	High	Longer-Larger	Low	High	High	High	whole number
			High	High	High	High	column overflow
			Low	High	High	Low	reverse
High	Low	Shorter-Larger	High	Low	High	High	denominator focussed
			High	Low	Low	Low	reciprocal/negative
High	High	Apparent-Expert	High	Low	High	High	money
			High	High	High	High	taskexpert
Low = 0 or 1 High = 4 or 5			Low = 0 or 1 High = 3 or 4		Low = 0 or 1 High = 2 or 3		

The table above is published on the website 'Teaching and Learning about Decimals' (Steinle *et al.*, 2006)

Appendix 8 Letters

Second Intervention Permission Letters

18th October 2020

Dear Parent / Carer,

As part of my commitment to teaching I am undertaking further study in teaching. As part of my studies I am undertaking an action research project, with the aim of improving mathematics teaching. Specifically, I would like to investigate how linking different areas of mathematics together can improve children's understanding and their progress.

I will be teaching your child for a few lessons in November. This will form part of their normal maths lessons and integral to the curriculum. I will be basing part of my project on what happens in those lessons. I will be collecting evidence from the work of the children. I will make sure that no child can be identified from the work before it is used in my project.

The results of the final project will be shared with the university and also with staff at school. They may also be used in papers written for academic journals. It will not be possible to identify your child or their work from any written or verbal report.

The aim of the research is to improve teaching and learning in mathematics for your child.

Please contact me if you have any questions or concerns. If you do not wish your child's work to be included in this research then please fill in the form below and return to me as soon as possible.

Regards,

Fiona Richards

18th October 2020

Dear Student

As part of my commitment to teaching I am undertaking further study in teaching. As part of my studies I am undertaking a project, with the aim of improving mathematics teaching. I would like to investigate how linking different areas of mathematics together can improve your progress in maths.

I will be teaching you for a few lessons in November. I will be taking note of things that happen in the lesson and taking photographs of work. This will only be used by me for my project. The findings from the final project will be shared with staff at school and at my university. They might be used in papers written for journals. Before anyone sees them I will ensure you cannot be identified.

Regards,

Fiona Richards

Email address and school phone number

14th December 2020

Dear

As part of my commitment to teaching I am undertaking further study in teaching. As part of my studies I am undertaking a project, with the aim of improving mathematics teaching. I would like to investigate how linking different areas of mathematics together can improve your progress in maths.

As part of this investigation I would like to interview you with a small group of students. The purpose of the interview is to find out your views on your mathematics lessons this year. Your participation is totally voluntary. I would like to record the interview so that I can easily remember what is said. Only I will listen to the recording and it will be destroyed after I have completed my project. If you do not want me to record the interview you can say no.

The results of these tests will only be used by me for my project. The findings from the final project will be shared with staff at school and at my university. They might be used in papers written for journals. Before anyone sees them I will ensure you cannot be identified.

If you have any questions please ask me. If you are happy to participate then please fill in and sign the attached consent form.

Regards,

Fiona Richards

Please sign and return to Mrs Richards

Name

- ☐ Yes. I am happy to take part in the interview.

Signature

Date

4th December 2020

Dear Parent / Carer

As part of my commitment to teaching I am undertaking further study in teaching. As part of my studies I am undertaking an action research project, with the aim of improving mathematics teaching. Specifically, I would like to investigate how linking different areas of mathematics together can improve children's understanding and their progress.

As part of this I would like to interview your child with a small group of other students to find out more about their understanding of their mathematics lessons within the project. Your child will only be involved if I have both your and their consent

I would like to record the interview so that I can easily remember what is said. Only I will listen to the recording and it will be destroyed after I have completed the project. I will only record the interview if I have both your and your child's consent.

The results of the final project will be shared with the university and also with staff at the school. They may also be used in papers written for academic journals. It will not be possible to identify your child or the results of their testing from any written or verbal report.

Please contact me if you have any questions or concerns. Please fill in the form below and return to me as soon as possible.

Regards,

Fiona Richards

Please sign and have your child return this slip

Name of child

(tick as appropriate)

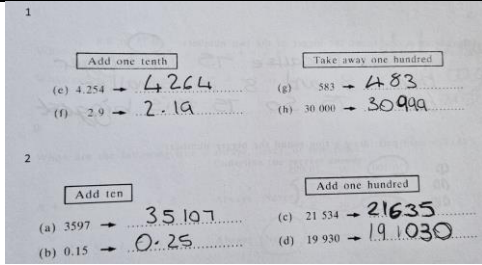
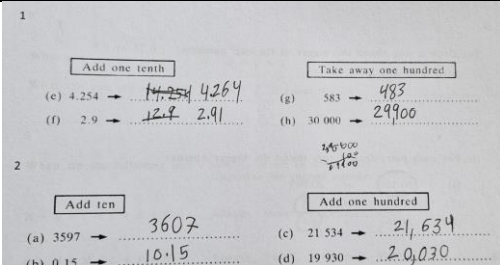
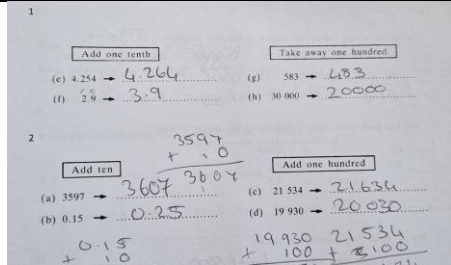
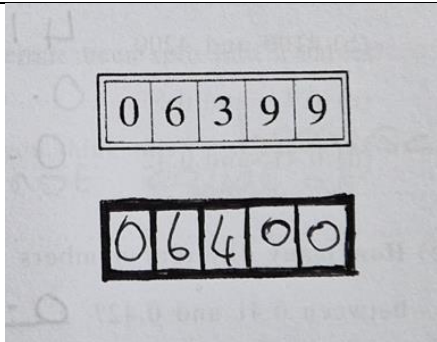
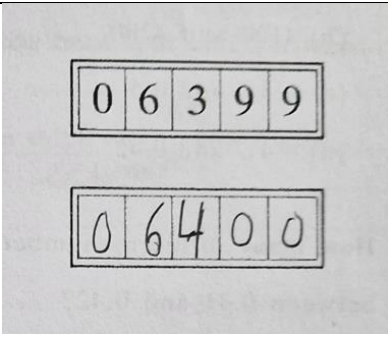
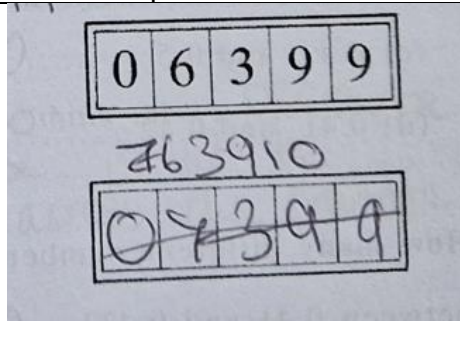
Yes. I give permission for my child to take part in the interview.

- ☐ No, I do not wish my child to take part in the interview

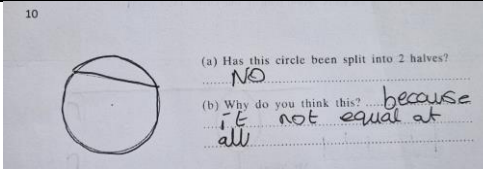
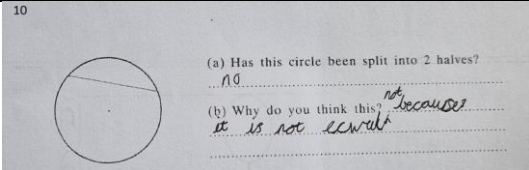
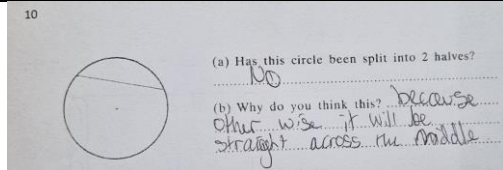
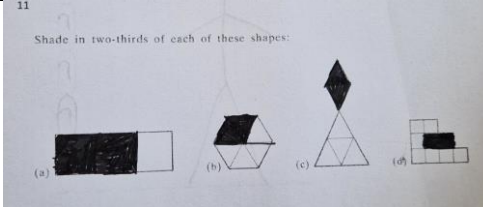
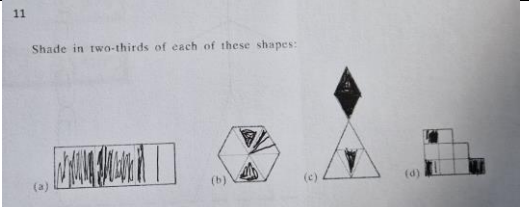
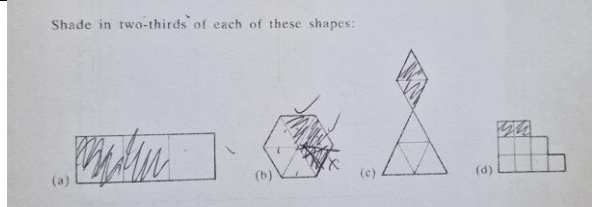
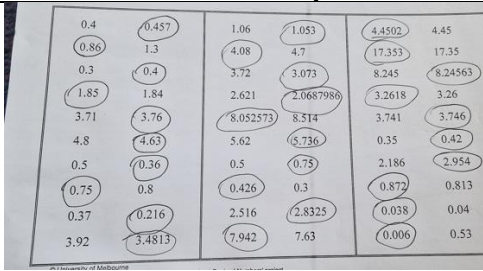
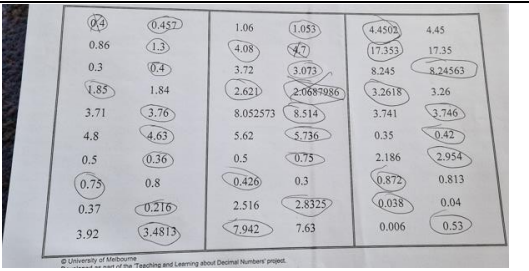
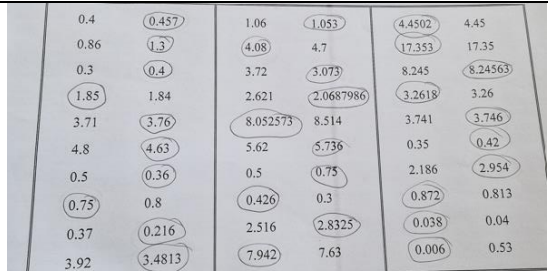
Signature of Parent / Carer

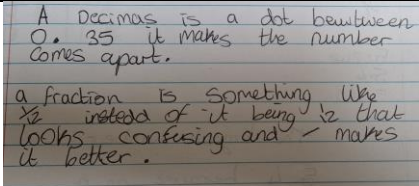
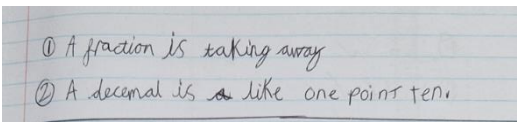
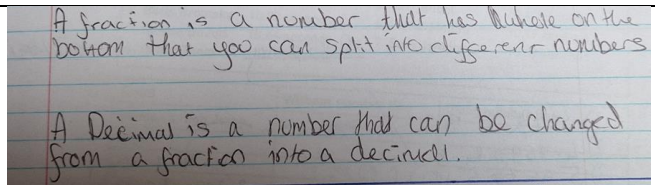
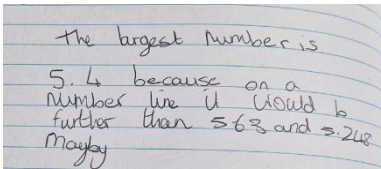
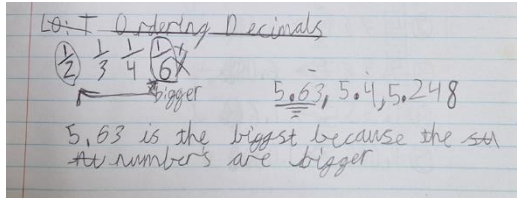
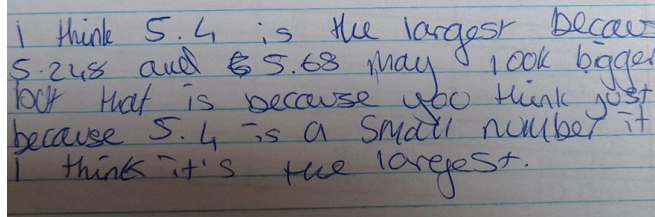
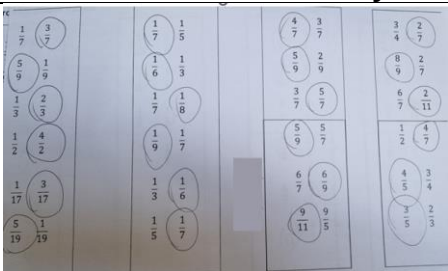
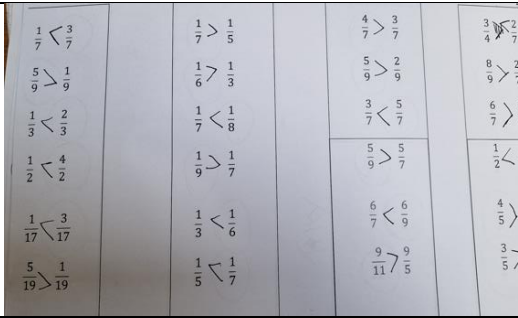
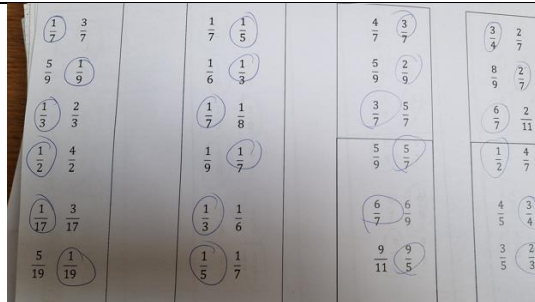
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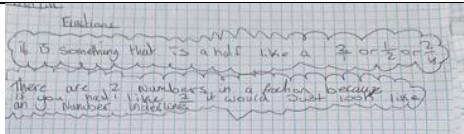
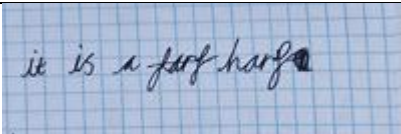
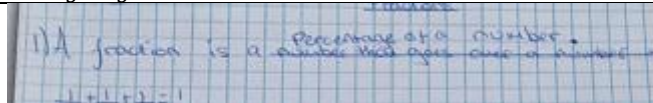
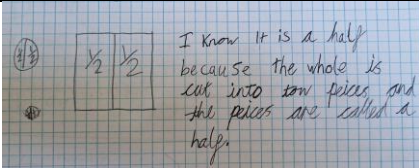
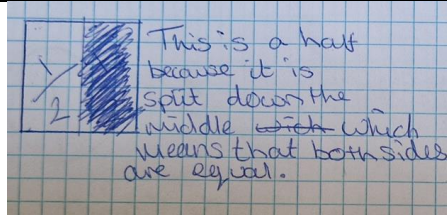
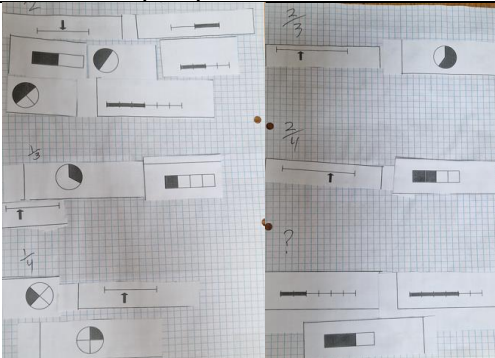
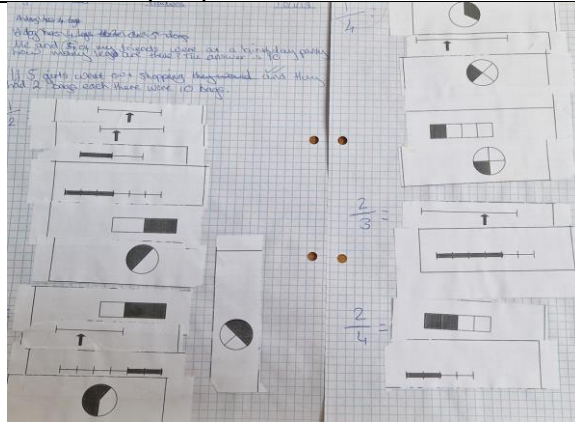
Appendix 9 Illustrative Cases

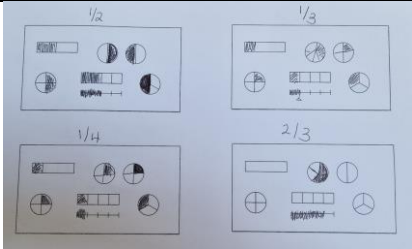
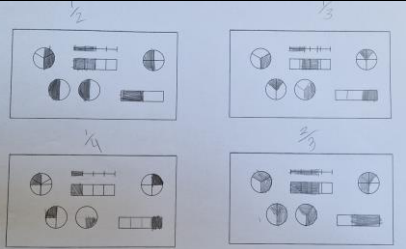
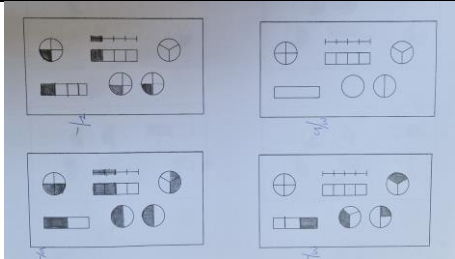
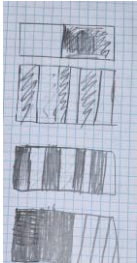
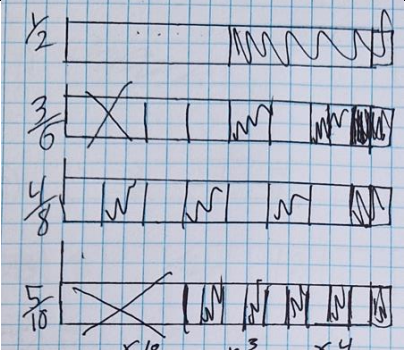
Student	Tasha	India	Rachel
	Key Stage 2 scaled score (80 to 120 – national average 105)		
KS2 maths	92	96	96
	Pre-test		
Pretest	8 (out of 24)	12 (out of 24)	13 (out of 24)
Q1/2			
	Tasha displays an insecure application of place value.	India displays a more secure application of place value. However, there are still mistakes when adding one tenth	Rachel has used algorithms to calculate the answers rather than applying the properties of place value. Even using the algorithms there are errors in applying place value. Adding 10 to 0.15 is an example of this
			

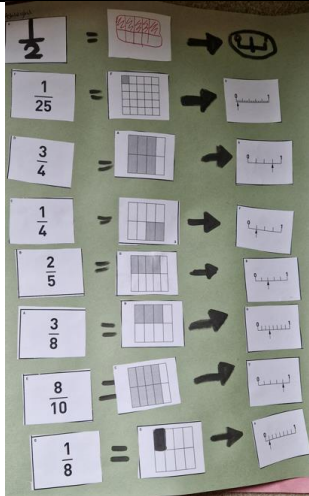
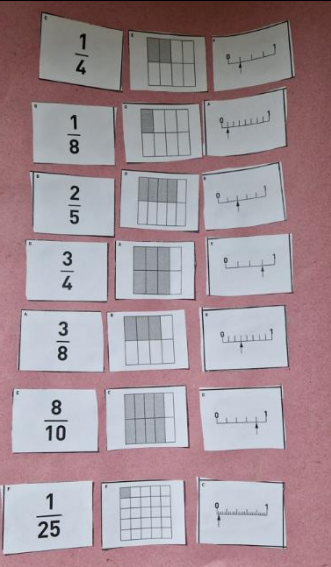
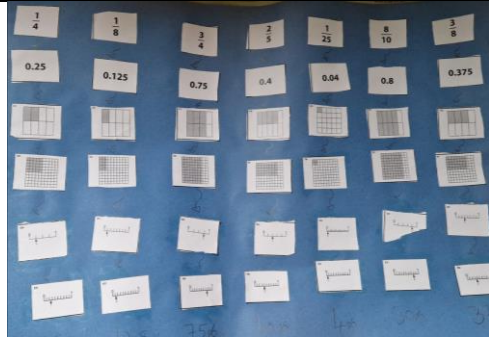
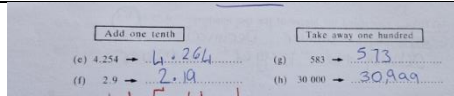
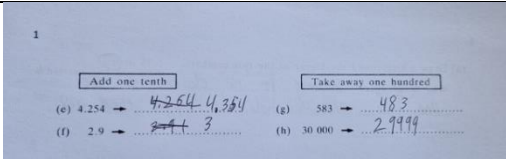
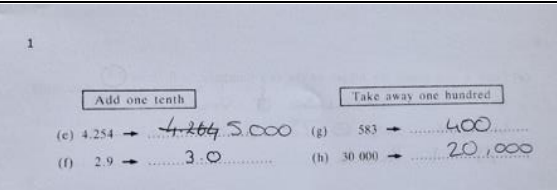
			Rachel seems unsure how to tackle this at all. It appears that she has placed a 10 in the units column.
5	(a) Draw a ring round the bigger of the two numbers: 0.75 or 0.8 Why is it bigger? Because 75 is bigger than 8 and 8 is smaller than 75 so 75 is biggest	(a) Draw a ring round the bigger of the two numbers: 0.75 or 0.8 Why is it bigger? ?	5 (a) Draw a ring round the bigger of the two numbers: 0.75 or 0.8 Why is it bigger? It is bigger because in a percentage it will 75% and 8% and 75% is bigger than 8% Rachel has a strategy here to convert to percentages, something she repeats in lesson. Unfortunately, she converts 0.8 to 8% and therefore her answer displays a longer is larger misconception.
	(b) For each pair, draw a ring round the bigger number: (i) 20 100 or 20 095 (ii) 7.55 or 7.5 (iii) 4.06 or 4.5	(b) For each pair, draw a ring round the bigger number: (i) 20 100 or 20 095 (ii) 7.55 or 7.5 (iii) 4.06 or 4.5	(i) 20 100 or 20 095 (ii) 7.55 or 7.5 (iii) 4.06 or 4.5 6 Rachel appears to have a longer is larger misconception where 4.06 is larger than 4.5 potentially due to the greater number of digits in the first.
	Write down any number between: (a) 4000 and 5000 4137 (b) 4100 and 4200 4125 (c) 0.4 and 0.5 0.04 (d) 0.41 and 0.42 0.42	Write down any number between: (a) 4000 and 5000 ? (b) 4100 and 4200 (c) 0.4 and 0.5 (d) 0.41 and 0.42	6 Write down any number between: (a) 4000 and 5000 404500 (b) 4100 and 4200 4125 (c) 0.4 and 0.5 0.45 (d) 0.41 and 0.42 0.42 Rachel has made a good attempt here. If the question had been more specific and ensured that the answer for d could not be 0.41 or 0.42, it might have been more informative.

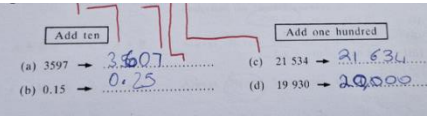
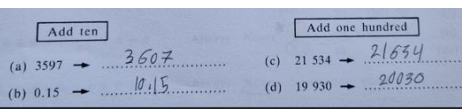
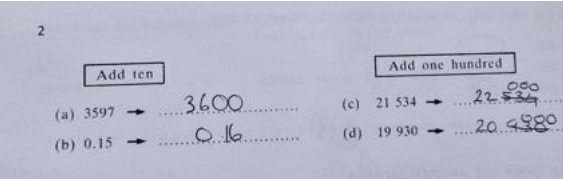
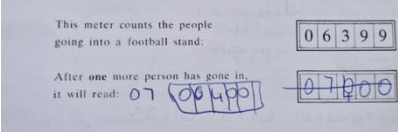
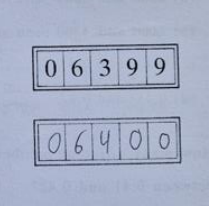
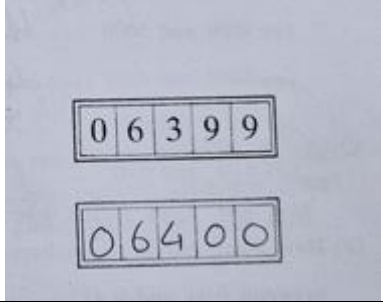
	 Tasha can express the concept of a half	 India can express the concept of a half	 Rachel can express the concept of a half
	 Although A is correct, Tasha has coloured two for each shape, not considering the denominator	 It is not clear what India intends here.	 Although A is correct, Rachel has coloured two for each shape, not considering the denominator
First Intervention – September Tasks			
			
DCT	Longer is larger misconception (whole number bias)	Longer is larger misconception	Longer is larger misconception (whole number bias)
Analysis of pre-test and DCT	At the beginning of the intervention, Tasha shows a longer is larger misconception. She does not display a secure understanding of place value or fractions	At the beginning of the intervention, India shows a longer is larger misconception. She does not display a secure understanding of place value or fractions	At the beginning of the intervention, Rachel shows a longer is larger misconception. She does not display a secure understanding of place value or fractions. Rachel appears to have some algorithms which she applies to help her answer these questions.
	Tasks		

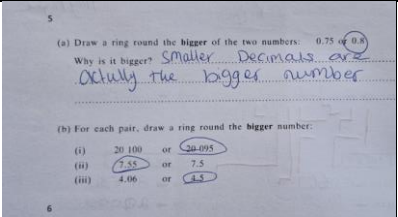
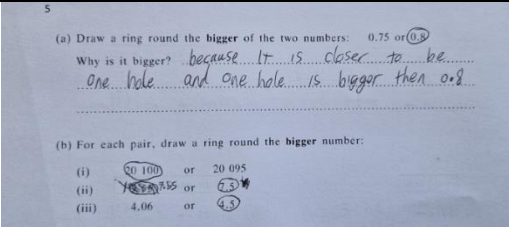
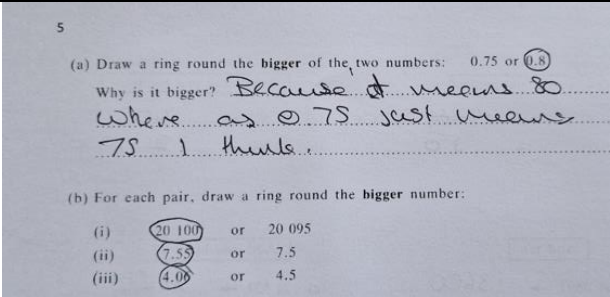
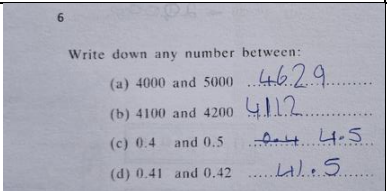
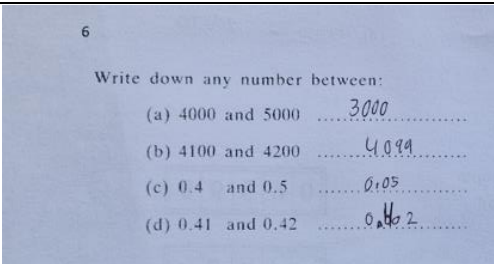
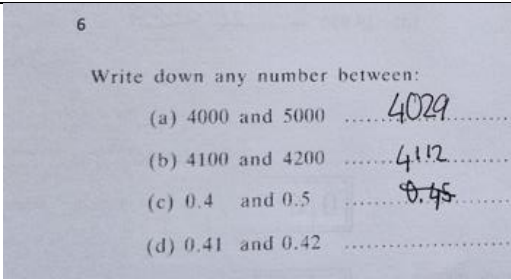
What is a decimal / fraction			
	Tasha is not able to clearly explain what a fraction or decimal are.	India does not explain what a fraction is but gives an example of a decimal	Rachel appears to have some understanding of what a fraction and a decimal is but does not express it clearly
Ordering 5.4, 5.63 and 5.248			
	Tasha displays a shorter-is-larger misconception here	India correctly identifies the largest number.	Rachel appears to select 5.4 as the largest number because it feels it should be the smallest. It appears that her 'gut' feeling is that longer-is-larger but she is choosing the opposite to her 'gut feeling'.
First Intervention – January tasks			
FCT section 6.4.15			
FCT results	6,0,3,0,3 Able to compare fractions with the same denominator but not where the denominators are different.	6,0,3,0,4 Able to compare fractions with the same denominator but not where the denominators	0,6,0,3,3 It seems likely that Rachel has found the smaller in each pair. If this is the case her results are the same as Tasha and India.

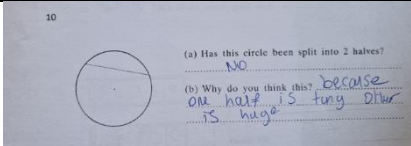
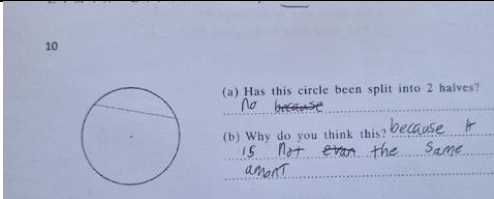
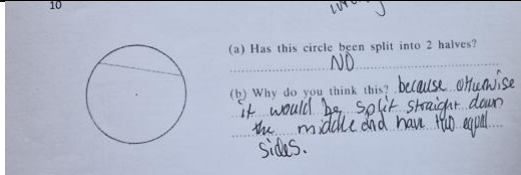
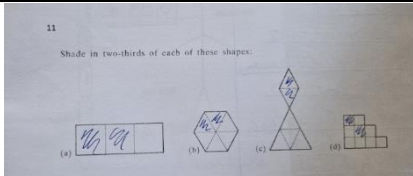
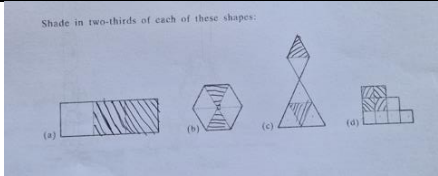
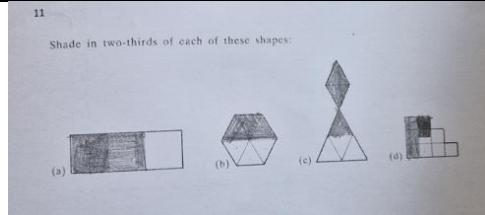
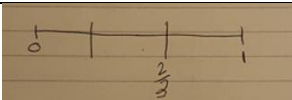
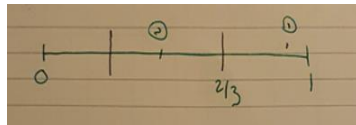
	This suggests a whole number bias ($3 < 4$ implies $\frac{1}{3} < \frac{1}{4}$)	are different. This suggests a whole number bias ($3 < 4$ implies $\frac{1}{3} < \frac{1}{4}$)	If not, then she is able to see that $\frac{1}{5} > \frac{1}{3}$ but not that $\frac{2}{3} > \frac{1}{3}$
91 What is a fraction			
Draw a half			
		India explains the concept of a half but does not mention equal parts.	Rachel explains the concept of a half and does mention equal parts.
			
		India has identified fractions in their continuous, discrete, shape and number line forms. She has not identified three of them.	Rachel has identified fractions in their continuous, discrete, shape and number line forms. She has not identified three of them.

Fraction s with distract ors			
	Tasha has sketched representations of fractions with distractors	India has sketched representations of fractions with distractors	Rachel has sketched representations of fractions with distractors
Equival ent fraction s			
	Asked to draw fractions equivalent to a half, Tasha has found some equivalent fractions	Asked to draw fractions equivalent to a half, India has found some equivalent fractions	Asked to draw fractions equivalent to a half, Rachel has found some equivalent fractions

Matching Fractions Task			
	Tasha has matched fractions to decimals to a number line	India has matched fractions to decimals to a number line	Rachel has matched fractions to decimals to a number line
Post-test			
Post test	11 (out of 24) (8 pretest)	13 (out of 24) (12 pretest)	11 (out of 24) (13 pretest)
Post test Q1	 This is similar to the pretest, with only g different. In the pre-test Tasha wrote 483 (correct)	 For e, the answer that India has crossed out is the answer she gave in the pretest. Her answer in the post-test is correct. For f India had written 2.91 in the pre-test. For h India had the correct answer in the pre-test	 Rachel did not get e correct in either the pre-test or the post -test. The answer she has crossed out matches the answer from the pretest. f is correct and was incorrect in the pre-test. G was correct in the pre-test. Rachel has answered h the same way for both tests.

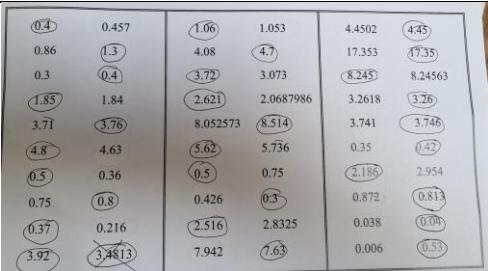
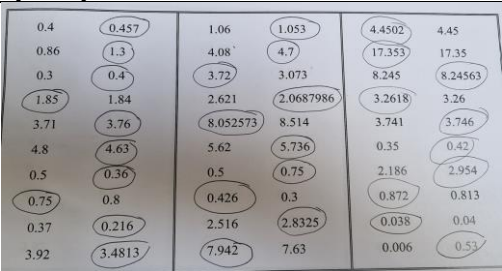
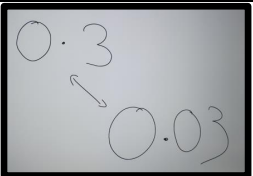
	 This is correct in the post-test but the pre-test showed that Tasha was not competent with place value. In a she had written 35107, indicating that the 'tens' column could contain 10 rather than the maximum of 9. She did the same on d with an answer of 19 1030. B and c were answered in a similar manner both pre- and post-test.	 India answered all four questions the same pre- and post- test.	 pre-test Rachel used a column addition algorithm to answer these questions. In the post-test it appears that Rachel is not answering the question and is perhaps answering using rounding methods. There is no clear answer. Rachel in lesson often relied on what appeared to be complete algorithms, but does not appear to have used one here.
	 Tasha answered this correctly in the pre-test.		

	 In the pre-test Tasha displayed a longer is larger – where here Tasha appears to be displaying a smaller is larger misconception in part a, though she has answered part b correctly for decimals (though not part b i)	 India could not answer part a in the pre-test. She has answered part b in the same way as she did for the pre-test	 Here Rachel describes the annexe zero algorithm that she tended to rely on in lessons. She has not applied it correctly to b iii. In the pre-test she explained part a by converting to percentages.
	 Tasha has correctly answered a,b and c. d appears to have been considered as a whole number.	 None of these is correctly answered. This question was not attempted in the pre-test. d could possibly read 0.41.2 meaning that India appears to have tried to find a number between 0.41 and 0.42	 Rachel has answered a, b and c successfully, though then crossing out c. She has not attempted d. On the pre-test she selected a correct answer for all four questions.

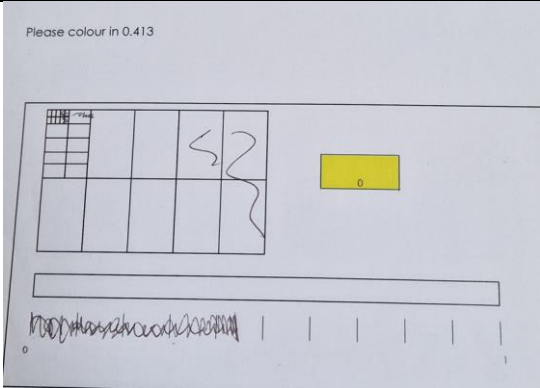
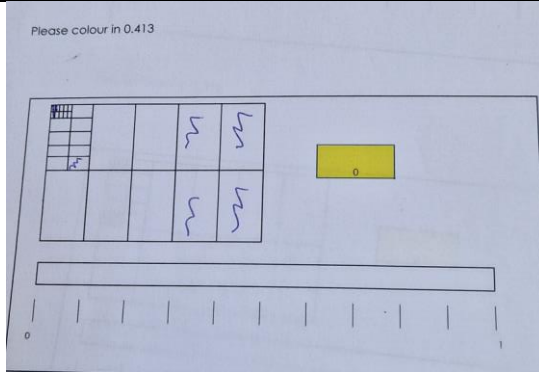
			
	 This is a similar response to the pre-test	 India's answer in the pre-test was very unclear. Here she has coloured in two for each shape (other than d) whatever the denominator is.	 For the pretest Rachel coloured in two for each shape. In the post-test she has coloured in $\frac{2}{3}$ for a and $\frac{1}{2}$ for the other shapes. This perhaps shows that she has an understanding of equivalent fractions but has found one-half of each shape rather than two-thirds.
	Interview		
What is a fraction	a one and a line underneath and a two	A fraction is like one and a half, or one over four or one over two, that's half. One over three that's a third	Something where you simplify a number to make it easier. To understand.. so if you had like a box of 10 and it was split into 5 and only one box was coloured in it would be $\frac{1}{5}$
Place $\frac{2}{3}$ on a number line			First indicated $\frac{2}{3}$ was at the point numbered 1 

Using the decimal	Remembered the decimal but needed reminding on how to use it. Used it properly once reminded	Remembered the decimal but needed reminding on how to use it. Used it properly once reminded	Remembered the decimal but needed reminding on how to use it. Used it properly once reminded
Ordering Decimals	Tasha: because it is 24 instead of 23. .	India: because if... 4 is bigger than 3 so that's 0.04 and 0.03 – and because it's 3 and 6 isn't a whole number yet it will be less. So, it means 0.24 will be bigger – it's got more pieces.	Rachel: because I think that if you take like (pause). Because that's 0.236 and that's 0.24 and if you added a zero onto the end of 0.24 it would be 0.240 Used an algorithm accurately. (This was the same algorithm Rachel used in lessons)
Ordering fractions	Tasha: because 4 is bigger than three.	India: because if you have it in a circle a quarter will be 4 pieces but a third will be three and the three pieces will be bigger than a quarter	Rachel: because you would colour in more for one third than for one quarter.
Follow – up Interview			
Second Interview	Tasha displayed a shorter is larger misconception in the interview. She also stated that 0.8 was greater than 0.75 because if you shared a pizza in 8 you would have more than if you shared it between 75.		
Summary			
	In the pre-test, Tasha appears to display an insecure grasp of place value and a longer is larger misconception. The DCT showed that Tasha is likely to have held a longer-is-larger misconception with a whole number bias. In lessons, Tasha was able to order decimals, fractions and represent both. In the post-test Tasha has displayed a mixture of misconceptions. In the interview, Tasha could recall and	In the pre-test India displayed a shorter-is-larger misconception. This was on a small number of questions. The DCT, however, showed that she was likely to hold a longer is larger misconception. India was able to order three decimals towards the start of the intervention, stating that the “numbers were bigger”. In lessons, India was able to order fractions and decimals and match the different instantiations. In the post-test, India has reverted to the same answers she gave for the pre-test. In the interview, India appeared to be	In the pretest, Rachel displayed some difficulties with place value and magnitude of fractional amounts. She uses methods in the pre-test rather than relying on the structure of the decimal system. These methods were not complete and led to errors. Where Rachel did not appear to rely on a method she appeared to have a longer is larger misconception. Rachel was not able to represent fractions equivalent to $\frac{2}{3}$. The DCT showed that Rachel was likely to have a longer is larger misconception with a whole

	apply what had been learned in lessons. She was able to order decimals, by noting that 4 is bigger than 3 when considering 0.24 and 0.236. This is indicative of a full left-to-right comparison which, used properly, is a full algorithm. I interviewed Tasha again almost a year after the first interview. At this stage Tasha displayed a shorter is larger misconception. Her understanding of decimals appeared to be affected by fractions as she described comparing 0.8 and 0.75 by splitting a pizza into 8 pieced and 75 pieces.	using a left to right comparison to compare decimals. She correctly explained why 0. 24 was bigger than 0.236 by first noting that 4 was bigger than 3. She was also able t compare unit fractions and explain why $\frac{1}{3}$ is larger than $\frac{1}{4}$.	number bias. All but one of her responses would be selected by considering the decimal part as a whole number (eg 0.63 > 0.8 because 63 > 8). At the start of the intervention, when she didn't use the annexe-zero algorithm, Rachel selected the shortest decimal as the largest. At the start of the intervention, Rachel was able to order fractions with the same denominator but not with different denominators. In lessons, Rachel was quick to complete tasks. Given a choice, she used the annexe-zero algorithm to compare decimals rather than relying on other representations or instantiations. Rachel matched different instantiations and representations of decimals and fractions and ordered them. In the post-test, Rachel, displayed a longer is larger misconception. She was less reliant on algorithms, though her answers did not seem to answer the questions asked. In the interview, Rachel again relied on an algorithm and applied it accurately. Before the intervention Rachel had a longer is larger misconception. During the intervention she was able to order decimals and fractions both with and without an annexe zero algorithm. In the post-test and the interview she reverted to a longer is larger misconception, although she answered some questions correctly using the algorithm.

	Anya	Jenny
	Decimal Comparison Test (DCT)	
DCT initial (before the intervention)	 18/20 classified as shorter-is-larger	 21/30 classified as longer-is-larger
	Second Intervention	
Class discussions	Anya was very keen to ensure she answered questions and appeared disappointed when she was not correct.	Jenny stated that a decimal was 0.7. With support she realised this was 0.6.
		Jenny was aware of why someone might believe 0.234 was greater than 0.3. She stated that this is a mistake she would make
Final DCT	29/30 classified as apparent-expert	Did not complete – however, for the 10 questions she did complete the likely classification is shorter is larger
		 When asked what 0.3 was Jenny asked if it was 0.03.
	Interview	
Interview	The interview indicates that the learning has been retained and Anya appears to still be an apparent expert but there was also clear evidence of some insecurities with decimal places.	In the interview Jenny was able to recall and apply what had been taught in lessons but she still displayed a longer is larger misconception.
	Third Intervention	

NP Task – mark 0.23 on a number line	Anya Asked to place 0.23 on the number line – first with unmarked then with marked pips, Anya has managed to mark it correctly. There were no comments about the original placements, the students were just asked if they wanted to change after the numbers were added to the diagram.	Jenny This activity shows little evidence that Jenny is able to use the number line or that she has a clear understanding of the size of decimals
	Post-Interview task Anya has not attempted the number line. She has correctly coloured in the decimat to represent 0.312 Jenny has not attempted the number line. She has correctly coloured in the decimat to represent 0.312	

	 Please colour in 0.413 Anya has incorrectly coloured in the decimal. It is not clear whether this is 0.314 or 0.313 that has been coloured in. This time Anya has attempted the number line and it is relatively close to a representation of 0.413	 Please colour in 0.413 Jenny has not attempted the number line. She has correctly coloured in the decimal to represent 0.413
Summary		
	Anya moved from a shorter is larger misconception before the intervention to be classified as an apparent expert at the end. In lessons, Anya was able to order decimals and match different decimal representations. Although the interview showed that Anya still exhibited some difficulties with fluency in decimal place she appeared to have retained what she had learned in the lessons and was able to apply it. In the third intervention Anya was able to place decimals on a number line and succeeded in representing numbers in a decimal and on a number line.	Jenny displayed a longer-is-larger misconception before the intervention. During the lessons she was able to talk about intuitive responses and recognise that she might be susceptible to them. In lessons, Jenny often made mistakes which appeared to be type 1 or intuitive errors. They appeared to be intuitive errors rather than type 1 or slips because they recurred and needed support from the teacher to see that it was an error. In the interview, Jenny was able to recall the lessons and what had been learned but she still displayed a longer is larger misconception. In the third intervention, Jenny displayed an insecure concept of place value, thinking that 0.3 and 0.03 were the same and placing 0.23 near 0.8 on the number line. Using the Excel decimal, Jenny was able to represent a decimal in the form of a decimal and on a number line.